Collapse Dynamics of a Self-Gravitating Brownian Gas

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Introduction

General Background on self-gravitating gas

- Competition between kinetic energy (temperature) and gravitation
- Occurrence of a collapse phase below T_c or E_c
- The long-range nature of gravitation is crucial
- Relevance of the thermodynamical ensemble (CE *vs* MCE)
- MF approximation is claimed to be exact in the limit $N \to \infty$, $G \to 0$

Dynamics of a self-gravitating Brownian gas

Instead of treating the dynamics of the actual Newtonian gas of particles, we assume the existence of a large friction ξ and associated random force (inert gas, of effective dynamical origin...).

$$\frac{d^2\mathbf{x}_i}{dt^2} = -\xi\frac{d\mathbf{x}_i}{dt} - \nabla\Phi + \sqrt{2D\xi}\,\eta_i$$

For $\xi \to +\infty$, the problem is reduced to the dynamics of self-gravitating Brownian particles. We consider the general case $D = T\rho^{1/n}$.

Schmoluchowski-Poisson equation (SPE) reads :

$$\frac{\partial\rho}{\partial t} = \nabla\left[\frac{1}{\xi}(D\nabla\rho + \rho\nabla\Phi)\right], \qquad \Delta\Phi(\mathbf{r}) = GS_d\rho(\mathbf{r}).$$

The constraints are:

- Constant total mass M in the box of radius R
- Constant and uniform temperature T (canonical ensemble)

From now : $G = M = R = \xi = 1$

Analogy with chemotaxis

$$\frac{\partial \rho}{\partial t} = D\Delta\rho - \chi\nabla(\rho\nabla c),$$

$$D_c^{-1}\frac{\partial c}{\partial t} = \Delta c + \lambda\rho \approx 0,$$

where ρ is the concentration of a bacterial population, c the concentration of the substance secreted and χ measures the strength of the chemotactic drift.

Identify $\Phi \leftrightarrow -\frac{4\pi G}{\lambda}c$, $T \leftrightarrow \frac{4\pi G D}{\lambda\chi}$, $\xi \leftrightarrow \frac{4\pi G}{\lambda\chi}$.

Introducing the mass $M = \int \rho d^3\mathbf{r}$ of the system and the radius R of the domain, we can show that the static problem depends on the single dimensionless parameter

$$\eta = \beta GM/R \leftrightarrow \frac{\lambda\chi}{4\pi DR}.$$

Therefore, a large value of η corresponds to a small temperature T or a large mass M.

Collapse Dynamics in the Canonical Ensemble

Scale invariant collapse within SPE for $d > 2$

For $T < T_c$, we look for radial solution of SPE of the form

$$\rho(\mathbf{r}, t) = \rho_0(t) f[r/r_0(t)].$$

We find a scaling solution after introducing the King radius
$r_0(t) = \sqrt{T/\rho_0(t)}$, leading to $\rho_0(t) = \frac{1}{2}(t_{coll} - t)^{-1}$.

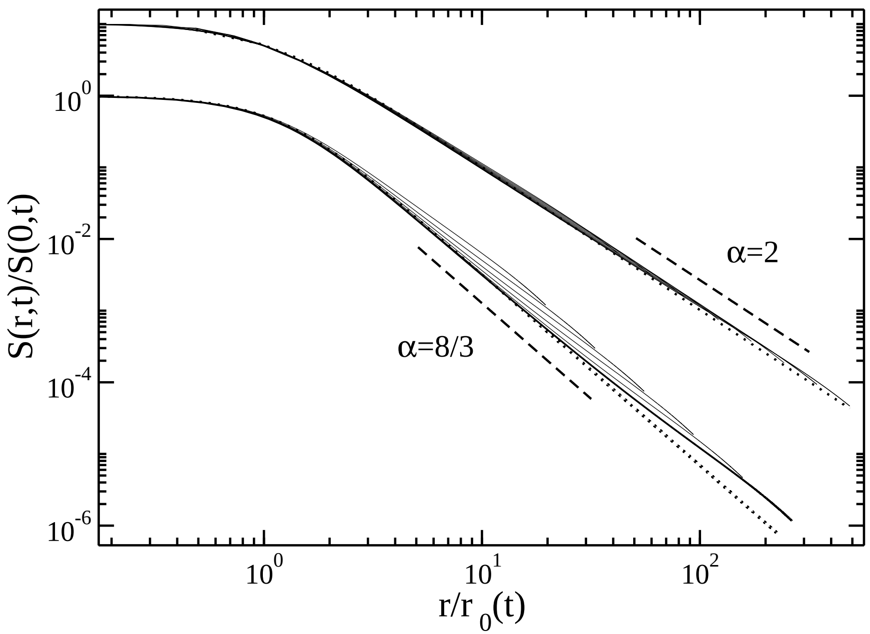
The central density diverges in a finite time t_{coll}.

Results

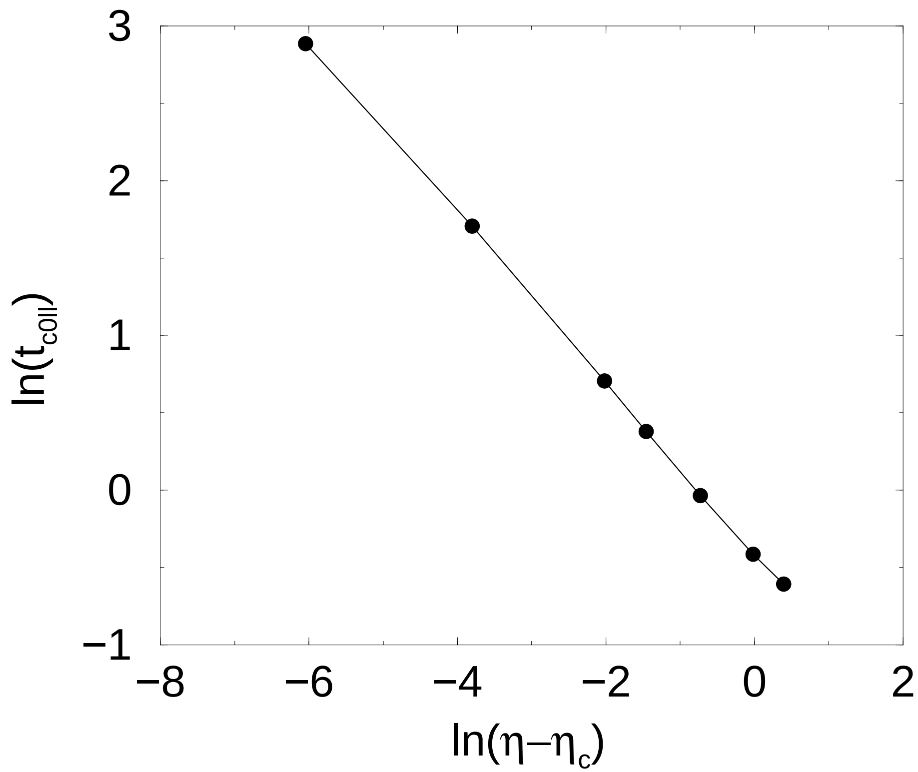
- The scaling function is given by ($n = \infty$)

$$f(x) = \frac{4(d-2)}{S_d} \frac{d + x^2}{\left(d - 2 + x^2\right)^2}.$$

- For large x, $f(x) \sim x^{-\alpha}$, with $\alpha = \frac{2n}{n-1}$.
- Near T_c, we find $t_{coll} \sim c_d(T_c - T)^{-1/2}$, and the width of the scaling regime is $\delta t \sim (T_c - T)^{1/2}$. Above T_c, the equilibration time is $\tau \sim (T - T_c)^{-1/2}$.
- Special treatment of the $d = 2$ case ($n = \infty$), and in general, of the case $n = n_* = d/(d-2)$.
- We estimated analytically and quantitatively the corrections to scaling (in $d = 3$), due to the existence of a finite confining box.
- We have computed the first instability modes.



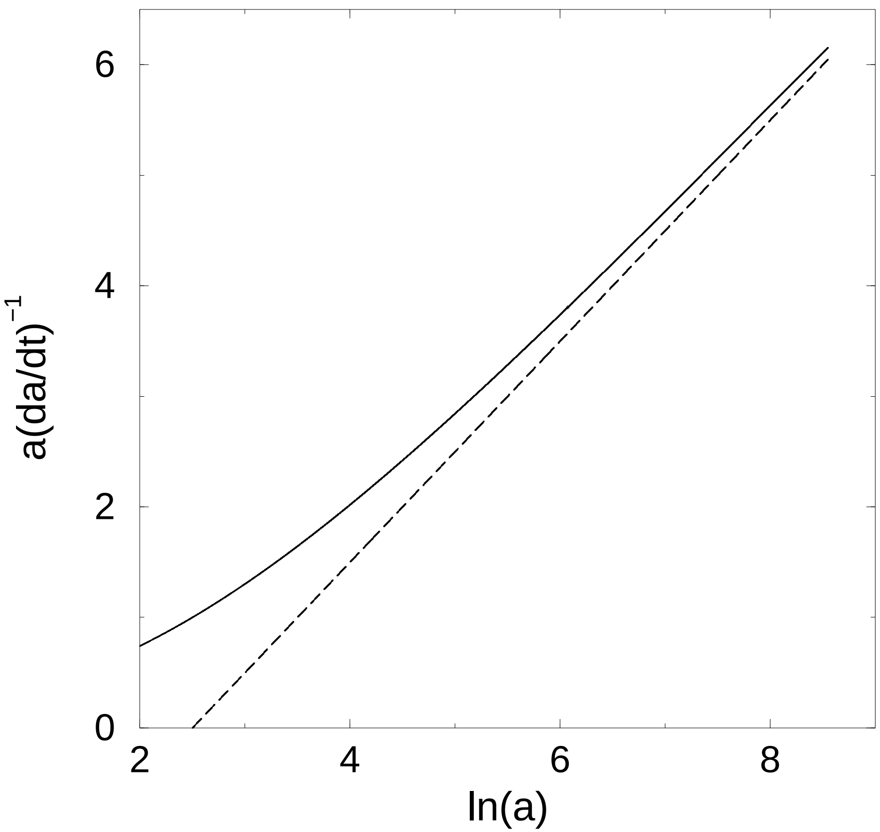
We plot $\rho(r,t)/\rho(0,t)$ as a function of $r/r_0(t)$ for different times (density range $10^2 - 10^7$) for $\alpha = 2$ and $\alpha = 8/3$ ($D \sim T\rho^{1/n}$, $\alpha = \frac{2n}{n-1}$, for $n = 4$), and compare the numerics to the analytical scaling solution.



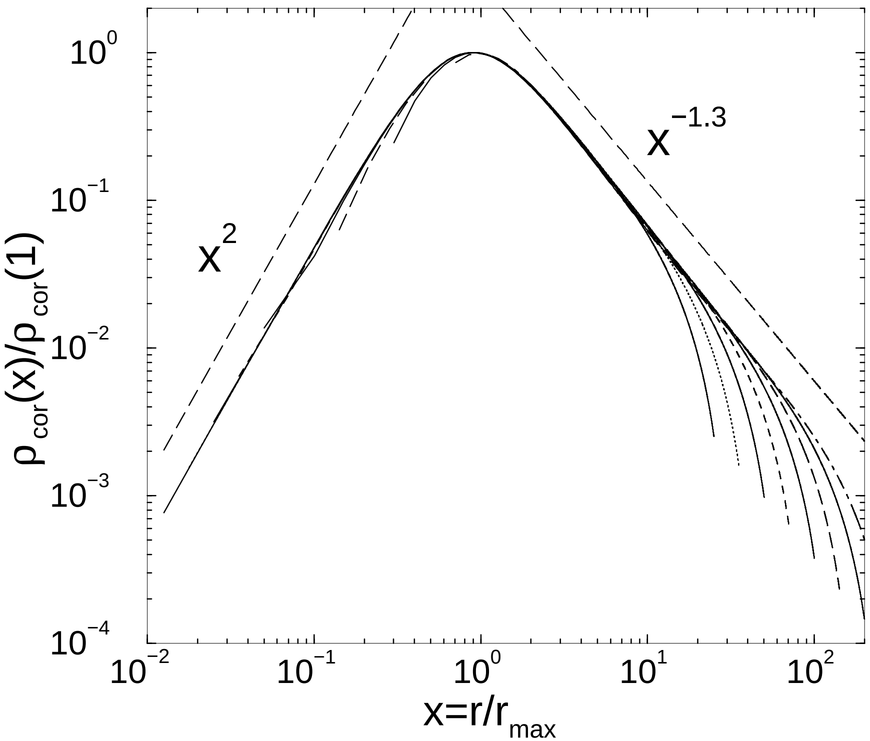
We plot t_{coll} as a function of $T_c - T$; the log-log slope is close to the theoretical result $-1/2$. (coefficient exactly known)

Statics and Collapse in $d = 2$ $(n = \infty = n_* = d/(d-2))$

- Above T_c, we find the equilibrium density profile $\rho(r) = \frac{4\rho_0}{\pi}\frac{1}{(1+(r/r_0)^2)^2}$, with $r_0(T) = \sqrt{T/T_c - 1}$, and $\rho_0 r_0^2 = T$.
- $T_c = 1/4$ and $r_0(T) \sim \sqrt{T/T_c - 1}$ are in perfect agreement with the exact solution of the problem using conformal invariance (Abdalla *et al.*).
- Contrary to the $d > 2$ case, collapse occurs at $T = T_c$, and $\rho(r,t) = \frac{4\rho_0(t)}{\pi}\frac{1}{(1+(r/r_0(t))^2)^2}$ $(\alpha = 4)$, with $\rho_0(t) r_0^2(t) = T_c$, and $\rho_0(t) = \frac{1}{4}\exp\left(\frac{5}{2} + \sqrt{2t}\right)\left[1 + O(t^{-1/2}\ln t)\right]$.
- Below T_c, the density is a sum of the scaling solution at $T = T_c$ (with weight T/T_c), plus a correction term obeying an apparent effective scaling with a slow varying exponent $\alpha(t) = 2 - \varepsilon(t)$, with $\varepsilon(t) = \sqrt{\frac{2\ln\ln\rho_0(t)}{\ln\rho_0(t)}}\left(1 + O([\ln\ln\rho_0(t)]^{-1})\right)$, and $\dot{\rho}_0(t) \sim \rho_0(t)^{1+\alpha(t)/2}$.



For $T = T_c$, we plot $a(da/dt)^{-1}$ ($a(t) = \pi\rho(0,t)$) as a function of $\ln a$, which is predicted to behave as $a(da/dt)^{-1} \sim \ln a - 5/2 + O([\ln a]^{-1})$ (dashed line).



We plot the residual density apparent scaling in a density (time) regime where the effective value of $\alpha \approx 1.3$, varies very little. To this the scaling contribution at $T = T_c$ (with weight T/T_c) must be added to get the total density profile.

Post-collapse Dynamics in the Canonical Ensemble

The scaling solution at $t = t_{coll}$ is **NOT** a stationary solution!

So, what happens for $t > t_{coll}$???

The exact solution at $T = 0$ suggests that a Dirac peak of mass $N_0(t)$ develops at $r = 0$, and that the residual density obeys a backward scaling relation $\rho(\mathbf{r}, t) = \rho_0(t) f[r/r_0(t)]$, where $\rho_0(t)$ decreases with time and $r_0(t)$ increases.

At $T = 0$, we find

$$N_0(t) \sim (t - t_{coll})^{\frac{d}{2}}, \quad \rho_0(t) = \frac{d}{2}(t - t_{coll})^{-1}, \quad r_0(t) = \left(\frac{2}{d}\right)^{\frac{d+2}{2d}} (t - t_{coll})^{\frac{d+2}{2d}},$$

and f is analytically known.

Post-collapse scaling equations for $0 < T < T_c$

$M(r,t)$ being the total mass within the shell of radius r, we define
$s(r,t) = \frac{M(r,t)-N_0(t)}{r^d} = \rho_0(t) S(\frac{r}{r_0(t)})$. Imposing scaling, we obtain for $t > t_{coll}$:

$$\frac{dN_0}{dt} = \rho_0 N_0, \qquad N_0(t) = \mu \rho_0 r_0^d = \mu \left(\frac{2}{d-2}\right)^{d/2-1} T^{d/2} \left(t - t_{coll}\right)^{d/2-1},$$

where $\rho_0(t)$ and $r_0(t)$ are given by

$$\rho_0(t) = \left(\frac{d}{2} - 1\right) \left(t - t_{coll}\right)^{-1}, \qquad r_0(t) = \left(\frac{T}{\rho_0(t)}\right)^{1/2}.$$

The resulting scaling equation is

$$\frac{1}{d-2}\left(2S + xS'\right) + S'' + \frac{d+1}{x} S' + S(dS + xS') + \mu x^{-d}(dS + xS' - 1) = 0,$$

where μ is an eigenvalue ensuring compatibility with pre-collapse for $r \gg r_0(t)$.

Remarks

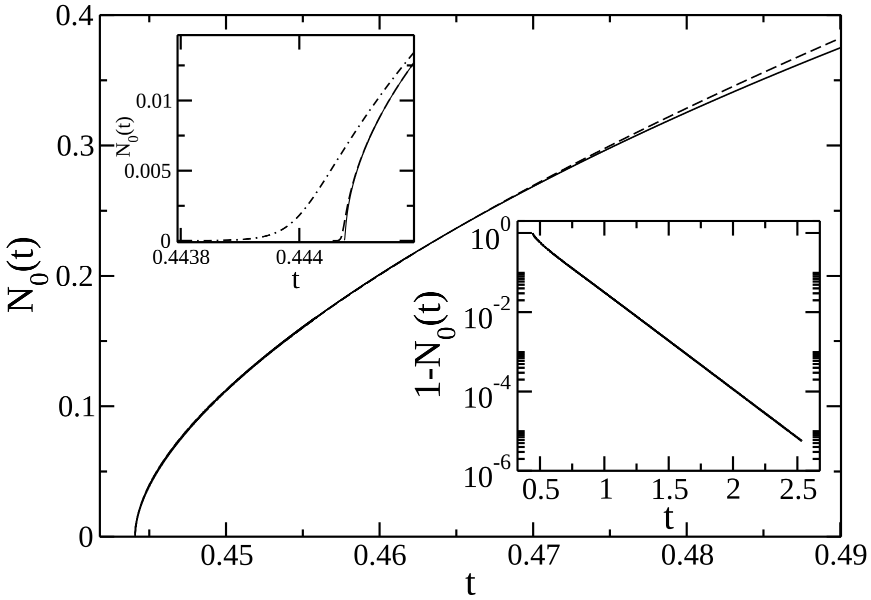
- The post-collapse scaling function is flatter near $x = 0$, as $S(x) - S(0) \sim x^d$ instead of $S(x) - S(0) \sim x^2$, below t_{coll}.
- The scaling holds only for short time after t_{coll}. For large time, $\rho(r,t) \sim \exp[-\lambda(T)t]\psi(r,T)$, and $1 - N_0(t) \sim \exp[-\lambda(T)t]$. We found that for small T, $\lambda(T) = \frac{1}{4T} + \frac{c_d}{T^{1/3}} + \ldots$, and derived analytical estimates for $\psi(r,T)$ (analogy with semiclassical methods: $T \leftrightarrow \hbar$).
- We introduce a numerical scheme in order to "cross the singularity" : $\frac{dN_0}{dt} = \rho_0 N_0$ is a first order differential equation starting from $N_0(t_{coll}^-) = 0$ (but $\rho_0(t_{coll}) = +\infty$!) :

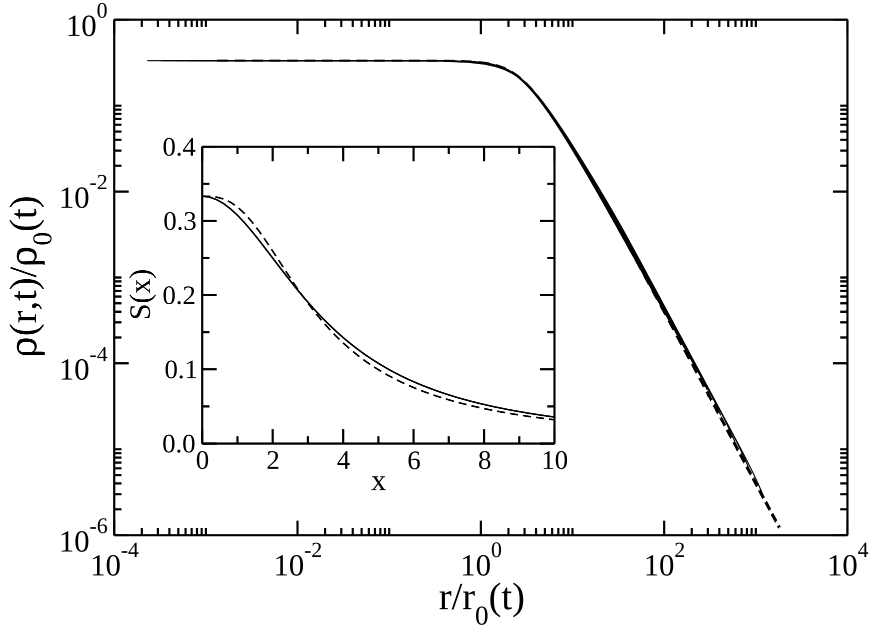
$$\frac{dN_0}{dt} = \rho_0^{fit} N_0^{fit}.$$

N_0^{fit} and ρ_0^{fit} are extracted from a fit of $M(r,t)$ to the functional form

$$M(r,t) \approx N_0^{fit}(t) + \frac{\rho_0^{fit}(t)}{d} r^d + a_{pre}(t) r^{d+2} + a_{post}(t) r^{2d},$$

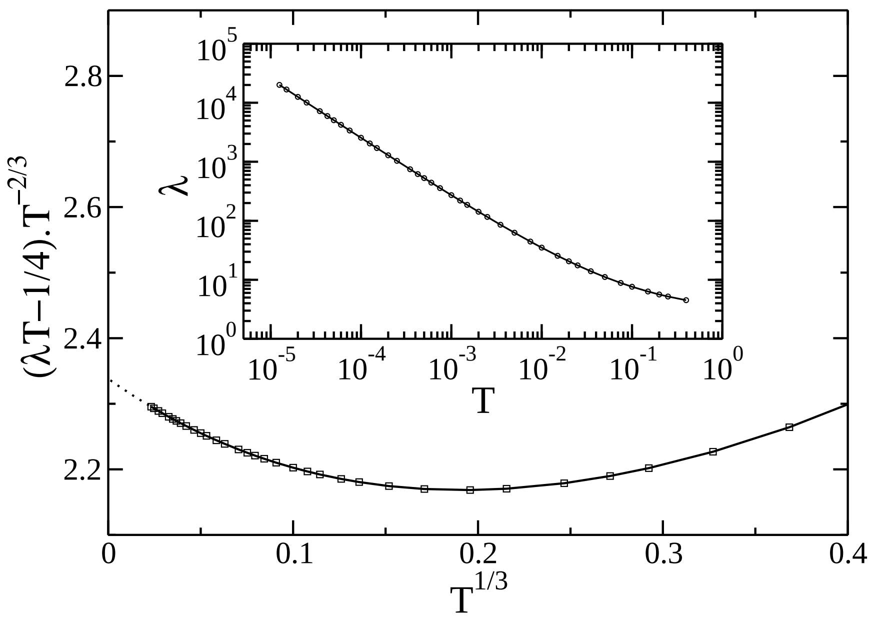
in a region of a few dr, excluding of course $r = 0$.





We plot $N_0(t)$ after t_{coll} (full line). This is compared to $N_0(t)^{\text{Theory}}$. The bottom insert illustrates the exponential decay of $1 - N_0(t) \sim e^{-\lambda t}$. Finally, the top inset illustrates the sensitivity of $N_0(t)$ to the spacial discretization.

In the post-collapse regime, we plot $\rho(r,t)/\rho_0(t)$ as a function $x = r/r_0(t)$. The insert shows the comparison between this post-collapse scaling function (dashed line) and the scaling function below t_{coll}.



We plot $\lambda(T)$ as a function of T (insert). The main plot represents $\left(\lambda(T)T - \frac{1}{4}\right).T^{-2/3}$ as a function of $T^{1/3}$ (line and squares), which should converge to $c_{d=3} = 2.33810741\ldots$.

Analogy with Bose-Einstein Condensation (Canonical Ensemble)

- We introduce a **semi-classical canonical ensemble dynamics** in momentum space, reproducing the well-known equilibrium state:

$$\frac{d\mathbf{k_i}}{dt} = -\frac{\mathbf{k_i}}{\xi}(1+\rho(\mathbf{k_i},t)) + \eta_{\mathbf{i}}(t), \tag{1}$$

$$\frac{\partial \rho}{\partial t} = \frac{1}{\xi}\nabla_{\mathbf{k}}\left[T\nabla_{\mathbf{k}}\rho + \rho(1+\rho)\mathbf{k}\right]. \tag{2}$$

- The equations for the integrated density are **strikingly similar**:

$$\frac{\partial M}{\partial t} = T\left(\frac{\partial^2 M}{\partial k^2} - \frac{d-1}{k}\frac{\partial M}{\partial k}\right) + k\frac{\partial M}{\partial k}\left(\frac{1}{k^{d-1}}\frac{\partial M}{\partial k} + 1\right), \tag{3}$$

$$\frac{\partial M}{\partial t} = T\left(\frac{\partial^2 M}{\partial r^2} - \frac{d-1}{r}\frac{\partial M}{\partial r}\right) + \frac{M}{r^{d-1}}\frac{\partial M}{\partial r} \quad (\textit{Gravitation}). \tag{4}$$

Analogy with Bose-Einstein Condensation (Canonical Ensemble)

- We have obtained analytical results for the **finite-time "collapse"** dynamics (scale invariant momentum density profile, residual density...) in $d = 3$.
- We have treated analytically the **"post-collapse"** (actual Bose-Einstein condensation), leading to a partially condensed delta peak at $\mathbf{k} = 0$ and to a residual Bose-Einstein distribution associated to zero chemical potential (at Finite $T < T_c$),
- We have computed the **typical duration of the collapse dynamics**, $\tau \sim \ln\left(\frac{T_c}{T-T_c}\right)$.

Conclusion

- Analytical and numerical study of the scaling theory of the collapse dynamics of a self-gravitating Brownian gas for a general diffusion coefficient $D = T\rho^{1/n}$.
- Understanding of the universal post-collapse scaling properties, as well as the very large time asymptotic regime.
- Extensive analysis of the static properties in all d and for all n, as well as for generalized entropy functional (Tsallis). Importance of the critical index $n_* = \frac{d}{d-2}$.
- Many other results: evaporation dynamics without a confining box, multi-component gas, analogy to Bose-Einstein condensation and chemotaxis...

Papers can be found on http://xxx.lanl.gov/archive/cond-mat, and are generally published in *Physical Review E* and *Physica A*.

Index	**Temperature**	**Bounded domain**	**Unbounded domain**
$n = \infty$	$T > T_c$	Metastable equilibrium state (local minimum of free energy): box-confined isothermal sphere	• Evaporation : asymptotically free normal diffusion (gravity negligible) • Collapse: pre-collapse and post-collapse as in a bounded domain
	$T < T_c$	Self-similar collapse with $\alpha = 2$ and self-similar post-collapse leading to a a Dirac peak of mass M	
$0 < n < n_*$	$T > T_c$	Equilibrium state: box-confined (incomplete) polytrope	Equilibrium state: complete polytrope (compact support)
	$T < T_c$	Equilibrium state: complete polytrope (compact support)	
$n_* < n < \infty$	$T > T_c$	Metastable equilibrium state (local minimum of free energy): box-confined polytropic sphere	• Evaporation: asymptotically free anomalous diffusion (gravity negligible) • Collapse: pre-collapse and post-collapse as in a bounded domain
	$T < T_c$	Self-similar collapse with $\alpha = 2n/(n-1)$ and post-collapse leading to a Dirac peak of mass M [N]	
$n = n_*$	$T > T_c$	Equilibrium state: box-confined (incomplete) polytrope	Self-similar evaporation modified by self-gravity
	$T < T_c$	Pseudo self-similar collapse leading to a Dirac peak of mass $(T/T_c)^{d/2}M$ + halo. This is followed by a post-collapse leading to a Dirac peak of mass M	Collapse
	$T = T_c$	Infinite family of steady states	Infinite family of steady states

Other results

- Exhaustive study of static properties in all dimensions.
- Analytical solution at $T = 0$ ($\alpha = \frac{2d}{d+2}$).
- Generalization of this study in all dimensions using Tsallis q-entropy ($S_q = -\frac{1}{q-1}\int(\rho^q - \rho)\, d^d r$), leading to a modified SPE.
  Full study of static (occurrence of confined polytropic states for certain values of q) and dynamical properties (collapse).
  When collapse occurs, the density scaling function decays as $x^{-\alpha}$, with $\alpha = \frac{2n}{n-1}$ and $n = d/2 + 1/(q-1)$ [link to anomalous diffusive Langevin walkers, chemotaxis, may be relevant for certain stellar systems...].
- Extension to degenerate systems (Fermions,...).
- Many-components system: the heaviest particles collapse as before ($\alpha = 2$) while the lightest has a scaling function decaying more slowly, with an exponent $\alpha(\mu = m_1/m_2 > 1, d) < 2$.

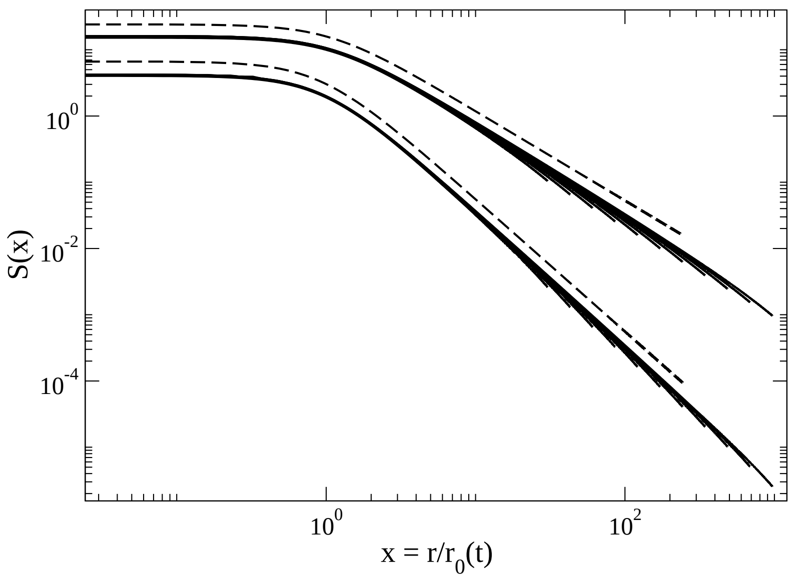
$$\rho_1(r,t) = \rho_0 f(r/r_0), \quad \rho_2(r,t) = \rho_0^{\alpha/2} f_\alpha(r/r_0), \quad f_\alpha(x) \sim x^{-\alpha}$$

- Large d expansion:

$$\alpha(\mu, d \gg 1) = \frac{4}{\mu+1}\left[1 - \frac{2(\mu-1)}{(\mu+1)^3} d^{-1} + O(d^{-2})\right]$$

- Expansion around $\mu = m1/m2$ close to 1:

$$\alpha(\mu = 1+\varepsilon, d) = 2 - \varepsilon 2(d-2) \frac{\int_0^{+\infty} y^{d+1} \frac{y^2+d+2}{(y^2+d-2)^2} \mathrm{e}^{-y^2/2}\, dy}{\int_0^{+\infty} y^{d+1} \mathrm{e}^{-y^2/2}\, dy} + O(\varepsilon^2)$$



Scaling for $\mu = 2$, for which
$\alpha(\mu = 2, d = 3) = 1.351914...$
Note that the large d result leads to
$\alpha(\mu = 2, d = 3) = 4/3$

Collapse Dynamics in the Microcanonical Ensemble

$T(t)$ **is still uniform but varies with time in order to conserve energy.**
We make the same ansatz as before: $\rho(\mathbf{r},t) = \rho_0(t) f[r/r_0(t)]$, with $\rho_0(t) r_0^2(t) = T(t)$, and assume $\rho_0(t) r_0(t)^\alpha = cst$.
Then $T(t) \sim \rho_0(t)^{1-2/\alpha}$, with $\alpha \geq 2$.

We now obtain the modified scaling equation :

$$\alpha S + xS' = S'' + \frac{d+1}{x} S' + S(xS' + dS), \qquad x = r/r_0(t),$$

which has a physical solution for any $\alpha \in [2; \alpha_{\max}]$ (in the limit of large d, we found $\alpha_{\max} = 2 + \frac{1}{2}\, d^{-1} + \frac{11}{16}\, d^{-2} + O(d^{-3})$, and gave perturbative expressions for $S(x)$; $\alpha_{\max}$=2.209733... in $d = 3$, in striking agreement with another treatment in the microcanical ensemble).

In principle, $\alpha_{\max}$ should be dynamically selected, as it leads to the maximum entropy production rate.