Weak propagation of chaos for Lévy-driven McKean-Vlasov SDEs

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Lévy processes and Poisson random measure

Lévy process

A $\mathbb{R}^d$-valued process $Z = (Z_t)_{t\geq 0}$ is a Lévy process if its paths are a.s. càdlàg starting from 0 and if it has stationary and indep. increments.

Lévy measure

A positive measure ν on $\mathbb{R}^d$ is a Lévy measure if

- $\nu(\{0\}) = 0$.
- $\int_{\mathbb{R}^d} \min(1, |z|^2)\, d\nu(z) < +\infty$.

Poisson random measure

A random measure $\mathcal{N}$ on $\mathbb{R}^+ \times \mathbb{R}^d$ is a Poisson random measure with intensity $dt \otimes \nu$ if

- $\mathcal{N}([0,t] \times A)$ is a Poisson r.v. with parameter $t\nu(A)$ for all measurable set A.
- For all $(A_i)_i$ measurable disjoints, $(\mathcal{N}(A_i))_i$ are indep.

Compensated Poisson random measure

If A is bounded from below: $\widetilde{\mathcal{N}}([0,t] \times A) := \mathcal{N}([0,t] \times A) - t\nu(A)$. $\widetilde{\mathcal{N}}$ is the compensated Poisson random measure $\rightarrow$ **martingale property**.

Lévy processes and Poisson random measure

- Lévy-Itô decomposition: General form of Z

$$Z_t = tb + \sigma B_t + \int_0^t \int_{D_1} z\widetilde{\mathcal{N}}(ds,dz) + \int_0^t \int_{D_1^c} z\mathcal{N}(ds,dz),$$

where $b \in \mathbb{R}^d$, σ is a positive matrix, B a Brownian motion indep. of $\mathcal{N}$ and $D_1 = \{z \in \mathbb{R}^d,\, |z| < 1\}$.

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α-stable process

A Lévy process Z is a α-stable process with $\alpha \in (0, 2]$ if for all $c > 0$, $(Z_{ct})_t$ has the same distribution as $(c^{\frac{1}{\alpha}} Z_t)_t$. **(Scaling property)**

- $\alpha = 2 \longrightarrow$ Brownian motion.
- Lévy measure writes in polar coordinates $\nu(dz) = \lambda(d\theta)\frac{dr}{r^{1+\alpha}}$, where λ is a non-zero finite measure on the sphere $\mathbb{S}^{d-1}$.

Rotationally invariant stable process

Rotationally invariant α-stable process

A α-stable process Z is rotationally invariant if $\nu(dz) = C\frac{dz}{|z|^{d+\alpha}}$.

- Moments: $\forall t$, $\mathbb{E}|Z_t|^\beta < +\infty \Leftrightarrow \int_{D_1^c} |z|^\beta \, d\nu(z) < +\infty \Leftrightarrow \beta \in [0, \alpha)$ and one has

$$\mathbb{E}|Z_t|^\beta = t^{\frac{\beta}{\alpha}} \mathbb{E}|Z_1|^\beta. \quad \textbf{(Scaling property)}$$

- Lévy-Itô decomposition: When $\alpha > 1$, $Z_t = \int_0^t \int_{\mathbb{R}^d} z \, \widetilde{\mathcal{N}}(ds, dz) \quad \longrightarrow$ **centered martingale.**

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- Lévy-Itô decomposition: When $\alpha > 1$, $Z_t = \int_0^t \int_{\mathbb{R}^d} z\,\widetilde{\mathcal{N}}(ds,dz) \quad \longrightarrow$ **centered martingale.**
- Strong Markov process: **Generator** $\mathcal{L}^\alpha$ given by

$$\mathcal{L}^\alpha f(x) := \int_{\mathbb{R}^d} [f(x+z) - f(x) - \nabla f(x)\cdot z]\, d\nu(z).$$

$\rightarrow$ non-local operator **(fractional Laplacian)** defined for $f \in \mathcal{C}_b^{1+\gamma}(\mathbb{R}^d)$ with $\gamma \in (\alpha-1, 1]$ i.e. $f \in \mathcal{C}_b^1(\mathbb{R}^d)$ and ∇f is γ-Hölder since

$$\int_{D_1} |z|^{1+\gamma}\, d\nu(z) < +\infty \Leftrightarrow \gamma \in (\alpha-1, 1].$$

McKean-Vlasov SDEs

Fix $T > 0$.

$$\begin{cases} dX_t = b(t, X_t, [X_t])\, dt + dZ_t, \quad \forall t \in [0, T], \\ X_0 = \xi, \quad [\xi] = \mu \in \mathcal{P}(\mathbb{R}^d), \end{cases} \tag{McKean-Vlasov}$$

where Z is a rotationally invariant α-stable process with $\alpha > 1$ and $[\xi]$ is the distribution of the r.v. ξ.

- Dynamics depends on the **position** (space variable) and on the **marginal distribution** (measure variable).
- Also called **non-linear SDEs**.
- Standard case of convolution interaction:

$$b(t, x, \mu) = b_0(t, x) + \int_{\mathbb{R}^d} b_1(t, x - y)\, d\mu(y) = b_0(t, x) + b_1(t, \cdot) * \mu(x).$$

Why study this kind of SDEs ?

Interacting particle system

Consider the following mean-field interacting particle system

$$\begin{cases} dX_t^{i,N} = b(t, X_t^{i,N}, \overline{\mu}_t^N)\, dt + dZ_t^i, \quad \forall t \in [0,T], \forall i \leq N, \\ \overline{\mu}_t^N := \frac{1}{N}\sum_{j=1}^N \delta_{X_t^{j,N}}, \quad \textbf{(=empirical measure)} \\ X_0^{i,N} = X_0^i, \end{cases} \tag{Particle system}$$

where $(Z^i)_i$ and $(X_0^i)_i$ are i.i.d with the same distributions as Z and ξ.

- Standard (linear) SDE on $(\mathbb{R}^d)^N$.
- Each particle feels all the others through **mean-field interaction (empirical measure)**.
- The particles have the **same distribution but are not indep.**

Interacting particle system

Consider the following mean-field interacting particle system

$$\begin{cases} dX_t^{i,N} = b(t, X_t^{i,N}, \overline{\mu}_t^N)\, dt + dZ_t^i, \quad \forall t \in [0,T], \forall i \leq N, \\ \overline{\mu}_t^N := \frac{1}{N}\sum_{j=1}^{N} \delta_{X_t^{j,N}}, \quad \textbf{(=empirical measure)} \\ X_0^{i,N} = X_0^i, \end{cases} \tag{Particle system}$$

where $(Z^i)_i$ and $(X_0^i)_i$ are i.i.d with the same distributions as Z and ξ.

- Standard (linear) SDE on $(\mathbb{R}^d)^N$.
- Each particle feels all the others through **mean-field interaction (empirical measure)**.
- The particles have the **same distribution but are not indep.**

Mean-field limit and propagation of chaos

It is expected that when $N \to +\infty$:

- the dynamics of $X^{1,N}$ is described by (McKean-Vlasov),
- for any k, $(X_t^{1,N}, \ldots, X_t^{k,N})$ become indep.

Semi-group, generator and PDE: linear case

Standard linear stable-driven SDE:

$$\begin{cases} dX_t^{s,x} = b(t, X_t^{s,x})\, dt + dZ_t, & \forall t \in [s, T], \\ X_s^{s,x} = x, \end{cases}$$

- Semi-group: For $f : \mathbb{R}^d \to \mathbb{R}$, $T_{s,t} f(x) := \mathbb{E}(f(X_t^{s,x})) \to T_{s,t} = T_{s,\tau} \circ T_{\tau,t}$. **(Markov property)**
- Generator: $\mathcal{L}_s f(t,x) := b(s,x) \cdot \nabla f(t,x) + \mathcal{L}^\alpha f(t,\cdot)(x)$. **(Itô's formula)**
- Backward Kolmogorov PDE: If b regular enough, $(s,x) \in [0,t] \times \mathbb{R}^d \mapsto T_{s,t} f(x)$ is the unique solution of the parabolic problem

$$\begin{cases} \partial_s u(s,x) + \mathcal{L}_s u(s,\cdot)(x) = 0, & \forall (s,x) \in [0,t) \times \mathbb{R}^d, \\ u(t,\cdot) = f. \end{cases} \tag{1}$$

- Transition density: $p(s,t,x,\cdot)$ is the fundamental solution of (1) and $T_{s,t} f(x) = \int_{\mathbb{R}^d} f(y) p(s,t,x,y)\, dy \to$ **Smoothing properties for** (1).

Semi-group, generator and PDE: non-linear case

What happens for (McKean-Vlasov)**?**

McKean-Vlasov SDE:

$$\begin{cases} dX_t^{s,\xi} = b(t, X_t^{s,\xi}, [X_t^{s,\xi}])\, dt + dZ_t, \quad \forall t \in [s, T], \\ X_s^{s,\xi} = \xi, \quad [\xi] = \mu \in \mathcal{P}(\mathbb{R}^d), \end{cases} \tag{McKean-Vlasov}$$

- **If** (McKean-Vlasov) **well-posed**, $[X_t^{s,\xi}] =: [X_t^{s,\mu}]$ depends on ξ only through its distribution μ.
- Semi-group: For $\phi : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$, $T_{s,t}\phi(\mu) := \phi([X_t^{s,\mu}]) \to T_{s,t} = T_{s,\tau} \circ T_{\tau,t}$.
- Generator: $\mathscr{L}_s$ should be a PDE operator acting on functions defined on the space of measures $\to$ **differentiation of a map w.r.t. the measure + Itô's formula along a flow of measures.**
- Backward Kolmogorov PDE and transition density: **Smoothing properties ?**

Here $\mathcal{P}(\mathbb{R}^d)$ is **infinite-dimensional** while the noise Z is finite-dimensional.
$\hookrightarrow$ **More subtle smoothing properties of the PDE**.

Questions

- **Well-posedness of** (McKean-Vlasov) **?** $\rightarrow$ Hölder regularity of b.
- **Study of the transition density: which regularity & bounds ?**
- **Form of the generator** ? $\rightarrow$ Differential calculus and Itô's formula along a flow of measures.
- **Smoothing properties of the associated PDE ?**
- **Mean-field limit of the particle system and propagation of chaos ?**

$\rightarrow$ When Z is a Brownian motion, this has been done by P.-E. Chaudru de Raynal and N. Frikha.

$\rightarrow$ Since $\mathbb{E}|Z_t|^2 = +\infty$, we cannot work in L^2 here and benefit from previous works (Mean-field games, PDE's on the Wasserstein space $\mathcal{P}_2(\mathbb{R}^d)$, Itô's formula...) as in the Brownian case.

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Spaces of measures

- $\mathcal{P}(\mathbb{R}^d)$ is the space of probability measures on $\mathbb{R}^d$.
  $\to$ Topology of weak-convergence.
  $\to$ **Total variation metric:**

$$d_{TV}(\mu,\nu) := \frac{1}{2} \sup_{\|f\|_\infty \leq 1} \left| \int_{\mathbb{R}^d} f \, d(\mu-\nu) \right| \underbrace{= \frac{1}{2} \left\| \frac{d\mu}{dx} - \frac{d\nu}{dx} \right\|_{L^1}}_{\text{when } \mu \text{ and } \nu \text{ have densities}} .$$

- $\mathcal{P}_\beta(\mathbb{R}^d)$ is the space of probability measures μ s.t. $\displaystyle\int_{\mathbb{R}^d} |x|^\beta \, d\mu(x) < +\infty$, for $\beta > 0$.
  $\to$ **Wasserstein metric:**

$$W_\beta(\mu,\nu) := \left(\inf_{\pi \in \Pi(\mu,\nu)} \int_{\mathbb{R}^d} |x-y|^\beta \, d\pi(x,y) \right)^{\frac{1}{\beta} \wedge 1},$$

  where $\Pi(\mu,\nu)$ is the set of measures $\pi \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ having μ and ν as marginal distributions and $\wedge$ is the minimum.
- $\mathcal{P}_0(\mathbb{R}^d) := \mathcal{P}(\mathbb{R}^d)$.

Differential calculus: linear derivative

Linear derivative

A function $u : \mathcal{P}_\beta(\mathbb{R}^d) \to \mathbb{R}$ has a linear derivative if there exists $\frac{\delta}{\delta m} u \in \mathcal{C}^0(\mathcal{P}_\beta(\mathbb{R}^d) \times \mathbb{R}^d; \mathbb{R})$ s.t.

- For all compact $\mathcal{K} \subset \mathcal{P}_\beta(\mathbb{R}^d)$, there exists $C_\mathcal{K} > 0$ s.t.

$$\forall \mu \in \mathcal{K}, \forall v \in \mathbb{R}^d, \left| \frac{\delta}{\delta m} u(\mu)(v) \right| \leq C_\mathcal{K}(1 + |v|^\beta).$$

- For all $\mu, \nu \in \mathcal{P}_\beta(\mathbb{R}^d)$

$$u(\mu) - u(\nu) = \int_0^1 \int_{\mathbb{R}^d} \frac{\delta}{\delta m} u(\lambda\mu + (1-\lambda)\nu)(v) \, d(\mu - \nu)(v) \, d\lambda. \tag{2}$$

- (2) is equivalent to

$$\lim_{h \to 0} \frac{u(\mu + h(\nu - \mu)) - u(\mu)}{h} = \int_{\mathbb{R}^d} \frac{\delta}{\delta m} u(\mu)(v) \, d(\mu - \nu)(v).$$

- Gateaux derivative in the space of signed measures.

Properties and examples

- $\frac{\delta}{\delta m} u(\mu)$ defined up to the addition of a constant C_μ.
- If $\frac{\delta}{\delta m} u$ is globally bounded $\to$ u is Lipschitz w.r.t. d_{TV}.

$$|u(\mu) - u(\nu)| \leq \left| \int_0^1 \int_{\mathbb{R}^d} \frac{\delta}{\delta m} u(\lambda\mu + (1-\lambda)\nu)(v)\, d(\mu - \nu)(v)\, d\lambda \right| \leq C d_{TV}(\mu, \nu).$$

Examples:

- Linear functions: $u(\mu) = \int_{\mathbb{R}^d} \phi \, d\mu$ with ϕ continuous and $|\phi(x)| \leq C(1 + |x|^\beta)$ then

$$u(\mu) - u(\nu) = \int_0^1 \int_{\mathbb{R}^d} \phi(v)\, d(\mu - \nu)(v)\, d\lambda \Rightarrow \frac{\delta}{\delta m} u(\mu)(v) = \phi(v).$$

- $u(\mu) = F\left(\int_{\mathbb{R}^d} \phi \, d\mu\right)$ with $F \in \mathcal{C}^1$ $\to$ $\frac{\delta}{\delta m} u(\mu)(v) = \nabla F\left(\int_{\mathbb{R}^d} \phi \, d\mu\right) \phi(v)$.

Empirical projection

Let $u : \mathcal{P}_\beta(\mathbb{R}^d) \to \mathbb{R}$ having a linear derivative $\frac{\delta}{\delta m} u$.

Empirical projection

The empirical projection of u is $u^N : \boldsymbol{x} = (x_1, \cdots, x_N) \in (\mathbb{R}^d)^N \mapsto u(\overline{\mu}^N_{\boldsymbol{x}})$, where $\overline{\mu}^N_{\boldsymbol{x}} = \dfrac{1}{N} \displaystyle\sum_{k=1}^N \delta_{x_k}$

If $\frac{\delta}{\delta m} u(\mu)(\cdot)$ is $\mathcal{C}^1$ for all μ with $\partial_v \frac{\delta}{\delta m} u$ continuous on $\mathcal{P}_\beta(\mathbb{R}^d) \times \mathbb{R}^d$, then u^N is $\mathcal{C}^1$ on $(\mathbb{R}^d)^N$ and

$$\partial_{x_k} u(\overline{\mu}^N_{\boldsymbol{x}}) = \frac{1}{N} \partial_v \frac{\delta}{\delta m} u(\overline{\mu}^N_{\boldsymbol{x}})(x_k).$$

Itô's formula along a flow of measures

General jump process:

$$\forall t \in [0, T],\ X_t := X_0 + \int_0^t b_s\, ds + \int_0^t \int_{D_1} H(s, z)\, \widetilde{\mathcal{N}}(ds, dz) + \int_0^t \int_{D_1^c} K(s, z)\, \mathcal{N}(ds, dz),$$

with $\mu_t := [X_t]$.

Itô's formula: Description of the dynamics of $t \mapsto u(\mu_t)$ for some function $u : \mathcal{P}_\beta(\mathbb{R}^d) \to \mathbb{R}$.

Itô's formula along a flow of measures

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Itô's formula: Description of the dynamics of $t \mapsto u(\mu_t)$ for some function $u : \mathcal{P}_\beta(\mathbb{R}^d) \to \mathbb{R}$.

Assumption: For fixed $\beta \in (0,2]$ and $\gamma \in (0,1]$ s.t. $\beta \leq 1+\gamma$, one has

- $\mathbb{E}|X_0|^\beta + \mathbb{E}\int_0^T |b_s|^{\beta \vee 1}\, ds + \mathbb{E}\int_0^T \int_{D_1^c} |K(s,z)|^\beta\, d\nu(z)\, ds < +\infty,$ **(Moments/large jumps)**

  $\hookrightarrow$ The flow $t \in [0,T] \mapsto \mu_t := [X_t] \in \mathcal{P}_\beta(\mathbb{R}^d)$ is **continuous**.

- $\mathbb{E}\int_0^T \int_{D_1} |H(s,z)|^{1+\gamma}\, d\nu(z)\, ds < +\infty.$ **(Small jumps)**

$\to$ In the α-stable case with $H(s,z) = K(s,z) = z$, the constraint is $\boldsymbol{\beta < \alpha < 1+\gamma}$.

Heuristic derivation of Itô's formula

For $t \in [0, T)$, $h > 0$, setting $m_\lambda := \lambda\mu_{t+h} + (1-\lambda)\mu_t$, we have

$$\begin{aligned} u(\mu_{t+h}) - u(\mu_t) &= \int_0^1 \int_{\mathbb{R}^d} \frac{\delta}{\delta m} u(m_\lambda)(v)\, d(\mu_{t+h} - \mu_t)(v)\, d\lambda \\ &= \int_0^1 \mathbb{E}\left(\frac{\delta}{\delta m} u(m_\lambda)(X_{t+h}) - \frac{\delta}{\delta m} u(m_\lambda)(X_t) \right) d\lambda. \end{aligned}$$

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By the standard Itô's formula for $F := \frac{\delta}{\delta m} u(m_\lambda)(\cdot)$

$$\begin{aligned} F(X_{t+h}) - F(X_t) &= \int_t^{t+h} \nabla F(X_s) \cdot b_s\, ds \\ &+ \int_t^{t+h} \int_{D_1^c} [F(X_{s-} + K(s,z)) - F(X_{s-})]\, \mathcal{N}(ds,dz) \\ &+ \int_t^{t+h} \int_{D_1} [F(X_{s-} + H(s,z)) - F(X_{s-})]\, \widetilde{\mathcal{N}}(ds,dz) \\ &+ \int_t^{t+h} \int_{D_1} [F(X_{s-} + H(s,z)) - F(X_{s-}) - H(s,z) \cdot \nabla F(X_{s-})]\, d\nu(z)\, ds. \end{aligned}$$

Heuristic derivation of Itô's formula

By taking expectation:

$$
\begin{aligned}
&u(\mu_{t+h}) - u(\mu_t) \\
&= \int_0^1 \int_t^{t+h} \mathbb{E}\left(\partial_v \frac{\delta}{\delta m} u(m_\lambda)(X_s) \cdot b_s\right) ds\, d\lambda \\
&\quad + \int_0^1 \int_t^{t+h} \int_{D_1^c} \mathbb{E}\left(\frac{\delta}{\delta m} u(m_\lambda)(X_{s-} + K(s,z)) - \frac{\delta}{\delta m} u(m_\lambda)(X_{s-})\right) d\nu(z)\, ds\, d\lambda \\
&\quad + \int_0^1 \int_t^{t+h} \int_{D_1} \mathbb{E}\left(\frac{\delta}{\delta m} u(m_\lambda)(X_{s-} + H(s,z)) - \frac{\delta}{\delta m} u(m_\lambda)(X_{s-})\right. \\
&\qquad\qquad\qquad \left. - \partial_v \frac{\delta}{\delta m} u(m_\lambda)(X_{s-}) \cdot H(s,z)\right) d\nu(z)\, ds\, d\lambda.
\end{aligned}
$$

Heuristic derivation of Itô's formula

Dividing by h and $h \to 0$ yields

$$\begin{aligned}\frac{d}{dt}u(\mu_t) = \mathbb{E}\left(\partial_v \frac{\delta}{\delta m}u(\mu_t)(X_t)\cdot b_t\right) \\ &+ \int_{D_1^c} \mathbb{E}\left(\frac{\delta}{\delta m}u(\mu_t)(X_t + K(t,z)) - \frac{\delta}{\delta m}u(\mu_t)(X_t)\right)\, d\nu(z) \\ &+ \int_{D_1} \mathbb{E}\left(\frac{\delta}{\delta m}u(\mu_t)(X_t + H(t,z)) - \frac{\delta}{\delta m}u(\mu_t)(X_t) - \partial_v \frac{\delta}{\delta m}u(\mu_t)(X_t)\cdot H(t,z)\right)\, d\nu(z).\end{aligned}$$

Itô's formula for (McKean-Vlasov)

In the case where $X_t = X_0 + \int_0^t b(s, X_s, \mu_s)\, ds + Z_t$, it takes the form

$$u(\mu_t) - u(\mu_0) = \int_0^t \mathscr{L}_s u(\mu_s)\, ds,$$

where the **generator** $\mathscr{L}_s$ is given by

$$\begin{aligned}\mathscr{L}_s u(\mu) := &\int_{\mathbb{R}^d} \partial_v \frac{\delta}{\delta m} u(\mu)(v) \cdot b(s, v, \mu)\, d\mu(v) \\ &+ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left[\frac{\delta}{\delta m} u(\mu)(v+z) - \frac{\delta}{\delta m} u(\mu)(v) - z \cdot \partial_v \frac{\delta}{\delta m} u(\mu)(v) \right] \frac{dz}{|z|^{d+\alpha}}\, d\mu(v).\end{aligned}$$

- **Crucial assumption:** $\partial_v \frac{\delta}{\delta m} u(\mu)(\cdot)$ γ-**Hölder** unif. in μ with $\gamma > \alpha - 1$.
- Non local PDE operator acting on functions of measures.
- Denoting by $\mathcal{L}_s^{\mu}$ the generator of the corresponding linear SDE where μ is fixed

$$\mathscr{L}_s u(\mu) = \int_{\mathbb{R}^d} \mathcal{L}_s^{\mu} \frac{\delta}{\delta m} u(\mu)(v)\, d\mu(v).$$

Regularization

- **Study the semi-group** $\rightarrow$ need to compose $\phi : \mathcal{P}_\beta(\mathbb{R}^d) \rightarrow \mathbb{R}$ with the flow of measures given by (McKean-Vlasov)
- **Generator** $\rightarrow$ PDE.

What can we hope in terms of regularization for the semi-group/the associated PDE?

Regularization

- **Study the semi-group** $\to$ need to compose $\phi : \mathcal{P}_\beta(\mathbb{R}^d) \to \mathbb{R}$ with the flow of measures given by (McKean-Vlasov)
- **Generator** $\to$ PDE.

What can we hope in terms of regularization for the semi-group/the associated PDE?

Take ϕ admitting a **bounded** linear derivative. If $\boldsymbol{b} = \mathbf{0}$, then $X_t^{0,\xi} = \xi + Z_t$ and $[X_t^{0,\mu}]$ has a density given by $q_t * \mu$, where q transition density of Z.

$$\phi([X_t^{0,\mu}]) - \phi([X_t^{0,\nu}]) = \int_0^1 \int_{\mathbb{R}^d} \frac{\delta}{\delta m}\phi(q_t * (\lambda\mu + (1-\lambda)\nu))(v)\, d(q_t * \mu - q_t * \nu)(v)\, d\lambda$$

$$= \int_0^1 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\delta}{\delta m}\phi(q_t * (\lambda\mu + (1-\lambda)\nu))(y) q_t(y-v)\, dy\, d(\mu - \nu)(v)\, d\lambda,$$

$$\Rightarrow \frac{\delta}{\delta m}\left[\phi([X_t^{0,\mu}])\right](v) = \frac{\delta}{\delta m}\phi(q_t * \mu)(\cdot) * q_t(v).$$

Regularization

Spatial regularization: $\frac{\delta}{\delta m}\left[\phi([X_t^{0,\mu}])\right](\cdot)$ is differentiable with

$$\partial_v \frac{\delta}{\delta m}\left[\phi([X_t^{0,\mu}])\right](v) = \int_{\mathbb{R}^d} \frac{\delta}{\delta m}\phi(q_t * \mu)(y)\nabla q_t(y-v)\,dy.$$

$\to \left|\partial_v \frac{\delta}{\delta m}\left[\phi([X_t^{0,\mu}])\right](v)\right| \leq Ct^{-\frac{1}{\alpha}}$. **(Gradient estimate on q_t)**

Regularization

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$\rightarrow \left|\partial_v \frac{\delta}{\delta m}\left[\phi([X_t^{0,\mu}])\right](v)\right| \leq Ct^{-\frac{1}{\alpha}}$. **(Gradient estimate on q_t)**

Regularization w.r.t. the measure:

- $\mu \in \mathcal{P}_\beta(\mathbb{R}^d) \mapsto \phi(\mu)$ is **Lipschitz w.r.t.** d_{TV} since $\frac{\delta}{\delta m}\phi$ is **bounded**.
- $\mu \in \mathcal{P}_\beta(\mathbb{R}^d) \mapsto \phi([X_t^{0,\mu}])$ is **Lipschitz w.r.t.** $\boldsymbol{W_1}$ since its linear derivative is **unif. Lipschitz continuous**.

$\hookrightarrow$ **Regularization through a change of the topology.**

If $\mathcal{K}$ is compact and $\mu, \nu \in \mathcal{P}_1(\mathcal{K})$, then $W_1(\mu,\nu) \leq \mathrm{Diam}(\mathcal{K}) d_{TV}(\mu,\nu)$.

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Cauchy-Lipschitz theory

$$\begin{cases} dX_t = b(t, X_t, [X_t])\, dt + dZ_t, \quad \forall t \in [0, T], \\ X_0 = \xi, \quad [\xi] = \mu \in \mathcal{P}(\mathbb{R}^d). \end{cases} \tag{McKean-Vlasov}$$

$\rightarrow$ **Cauchy-Lipschitz theory:**

- If $b(t, \cdot, \cdot)$ unif. **Lipschitz** on $\mathbb{R}^d \times \mathcal{P}_\beta(\mathbb{R}^d)$ with $\beta \geq 1 \rightarrow \exists!$ strong solution through fixed point (see C. Graham for $\beta = 1$).
- If $b(t, \cdot, \cdot)$ unif. **Lipschitz** on $\mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d)$ (w.r.t. a Wasserstein-type metric) and **bounded** w.r.t. $\mu \rightarrow \exists$ a weak solution through martingale problem (see B. Jourdain - S. Méléard - W. Woyczynski) $\rightarrow$ uniqueness not established.

Cauchy-Lipschitz theory

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→ **Cauchy-Lipschitz theory:**

- If $b(t,\cdot,\cdot)$ unif. **Lipschitz** on $\mathbb{R}^d \times \mathcal{P}_\beta(\mathbb{R}^d)$ with $\beta \geq 1 \to \exists!$ strong solution through fixed point (see C. Graham for $\beta = 1$).
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→ **Difficult to go beyond Lipschitz assumptions.**

- **Counter-example:** $dX_t = \mathbb{E}(b(X_t))\,dt$ with b **locally Lipschitz bounded** → may be ill-posed (M. Scheutzow).

Going beyond Lipschitz assumptions: regularization by noise

$\rightarrow$ **Other well-posedness results:**

- Under **Hölder** assumptions w.r.t. to $(x, \mu) \in \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d)$ on $b \rightarrow \exists!$ weak or strong solution (see N. Frikha - V. Konakov - S. Menozzi).
- For **convolution interaction** $b(t, x, \mu) = b_1(t, \cdot) * \mu(x)$ with $b_1(t, \cdot)$ in some **Besov space** $\rightarrow \exists!$ weak or strong solution (see P.-E. Chaudru de Raynal - J.-F. Jabir - S. Menozzi).

$\hookrightarrow$ **Regularization by noise.**

Going beyond Lipschitz assumptions: regularization by noise

$\rightarrow$ **Other well-posedness results:**

- Under **Hölder** assumptions w.r.t. to $(x, \mu) \in \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d)$ on $b \rightarrow \exists!$ weak or strong solution (see N. Frikha - V. Konakov - S. Menozzi).
- For **convolution interaction** $b(t, x, \mu) = b_1(t, \cdot) * \mu(x)$ with $b_1(t, \cdot)$ in some **Besov space** $\rightarrow \exists!$ weak or strong solution (see P.-E. Chaudru de Raynal - J.-F. Jabir - S. Menozzi).

  $\hookrightarrow$ **Regularization by noise.**

$\rightarrow$ **Difficult to avoid some Lipschitz assumption w.r.t. the measure since the noise is only finite dimensional.**

- **Counter-example:** $dX_t = b([X_t])\,dt + dZ_t$ where $b(\mu) = \tilde{b}\left(\int_{\mathbb{R}^d} x\,d\mu(x)\right)$ with $\tilde{b}$ Hölder bounded. ***b* is Hölder w.r.t. $\boldsymbol{W_1}$** $\rightarrow$ may be ill posed (F. Delarue).

Indeed, $t \mapsto \mathbb{E}(X_t)$ solves $y' = \tilde{b}(y)$ which can have **several solutions**.

Weak well-posedness and Picard approximation

Assumptions:

- b is measurable and **bounded** on $[0,T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d)$.
- $b(t,\cdot,\mu)$ is η-**Hölder** unif. in (t,μ) with $\eta \in (0,1]$.
- $b(t,x,\cdot)$ is **Lipschitz** w.r.t. d_{TV} unif. in (t,x).

Theorem

The non-linear martingale problem related to (McKean-Vlasov) *is well-posed, thus there exists a unique weak-solution* $([X_t^\mu])_t$. *Moreover, if* $P \in \mathcal{C}^0([s,T];\mathcal{P}(\mathbb{R}^d))$ *with* $P_0 = \mu$*, we define recursively* $\overline{X}^{(m)}$ *by*

$$\begin{cases} d\overline{X}_t^{(m)} = b(t, \overline{X}_t^m, [\overline{X}_t^{(m-1)}])\, dt + dZ_t, \quad \forall t \in [s,T], \\ \overline{X}_s^{(m)} = \xi, \quad [\xi] = \mu, \end{cases}$$

with $([\overline{X}_t^{(0)}])_{t\in[s,T]} = P$. *Then*

$$\sup_{t\in[0,T]} d_{TV}([X_t^\mu], [\overline{X}_t^{(m)}]) \underset{m\to+\infty}{\longrightarrow} 0.$$

Idea of proof

Idea of proof:

- Set $\mathcal{A} := \left\{ P \in \mathcal{C}^0([0,T]; \mathcal{P}(\mathbb{R}^d)),\ P_0 = \mu \right\}$ and consider for $P \in \mathcal{A}$

$$\begin{cases} d\overline{X}_t^{\xi,P} = b(t, \overline{X}_t^{\xi,P}, P_t)\, dt + dZ_t, & \forall t \in [0,T], \\ \overline{X}_s^{\xi,P} = \xi, \quad [\xi] = \mu. \end{cases}$$

- Show that the map $T : P \in \mathcal{A} \mapsto ([X_t^{\xi,P}])_t \in \mathcal{A}$ is s.t. T^n is a contraction for some n w.r.t. $d(P,Q) = \sup_{t\in[0,T]} d_{TV}(P_t, Q_t)$ $\rightarrow$ **Banach fixed point theorem.**

  $\hookrightarrow$ Total variation $\sim$ control the difference of the densities $\hookrightarrow$ **track the dependence w.r.t.** P **of the density of** $[X_t^{\xi,P}]$ **using the** $\underbrace{\textbf{parametrix expansion.}}_{\text{what is this ?}}$

Parametrix expansion - linear case

Standard stable-driven SDE:

$$\begin{cases} dX_t^{s,x} = b(t, X_t^{s,x})\, dt + dZ_t, & \forall t \in [s, T], \\ X_s^{s,x} = x. \end{cases}$$

The transition density $p(s, t, x, \cdot)$ solves

$$\begin{cases} \partial_s p(s,t,x,y) + \mathcal{L}_s p(s,t,\cdot,y)(x) = 0, & \forall (s,x) \in [0,t) \times \mathbb{R}^d, \\ p(s,t,x,\cdot) \underset{s \to t^-}{\longrightarrow} \delta_x, & \text{in the weak sense,} \end{cases}$$

where $\mathcal{L}_s f(t,x) := b(s,x) \cdot \nabla f(t,x) + \mathcal{L}^\alpha f(t,\cdot)(x) \rightarrow$ **fundamental solution**.

Parametrix expansion - linear case

Rewrite the PDE as:

$$\begin{cases} \partial_s p(s,t,x,y) + \mathcal{L}^\alpha p(s,t,\cdot,y)(x) = (\mathcal{L}^\alpha - \mathcal{L}_s) p(s,t,\cdot,y)(x), & \forall (s,x) \in [0,t) \times \mathbb{R}^d, \\ p(s,t,x,\cdot) \underset{s \to t^-}{\longrightarrow} \delta_x, & \text{in the weak sense.} \end{cases}$$

$\to$ Fundamental solution of $\partial_s + \mathcal{L}^\alpha$ given by the transition density $\widehat{p}(s,t,x,y)$ of Z yields

$$p(s,t,x,y) = \widehat{p}(s,t,x,y) + \widehat{p} \otimes \left[(\mathcal{L}_s - \mathcal{L}^\alpha) p \right](s,t,x,y),$$

where $\otimes$ = space-time convolution operator.

Parametrix expansion - linear case

Rewrite the PDE as:

$$\begin{cases} \partial_s p(s,t,x,y) + \mathcal{L}^\alpha p(s,t,\cdot,y)(x) = (\mathcal{L}^\alpha - \mathcal{L}_s)p(s,t,\cdot,y)(x), \quad \forall (s,x) \in [0,t) \times \mathbb{R}^d, \\ p(s,t,x,\cdot) \underset{s \to t^-}{\longrightarrow} \delta_x, \quad \text{in the weak sense.} \end{cases}$$

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where $\otimes$ = space-time convolution operator. By iteration:

$$\begin{aligned} p(s,t,x,y) &= \widehat{p}(s,t,x,y) + \sum_{k=1}^{\infty} \widehat{p} \otimes \mathcal{H}^k(s,t,x,y) \\ &= \widehat{p}(s,t,x,y) + p \otimes \mathcal{H}(s,t,x,y), \quad \text{(implicit formulation)} \end{aligned}$$

where

$$\begin{cases} \widehat{p}(r,t,x,y) := \text{transition density of } Z, \quad \textbf{(Proxy)} \\ \mathcal{H}(s,t,x,y) := b(s,x) \cdot \partial_x \widehat{p}(s,t,x,y). \quad \textbf{(Parametrix kernel)} \end{cases}$$

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Assumptions and example

Assumptions:

- b jointly continuous and globally **bounded** on $[0,T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d)$.
- For all t, μ, $b(t,\cdot,\mu)$ is η-**Hölder** on $\mathbb{R}^d$, for some $\eta \in (0,1]$, unif. in t, μ.
- For all t, x, the map $b(t,x,\cdot)$ has a linear derivative s.t. $\frac{\delta}{\delta m} b(t,x,\mu)(\cdot)$ is η-**Hölder** on $\mathbb{R}^d$ unif. in t, x, μ and $\frac{\delta}{\delta m} b$ is **bounded**.
- For any (t,x,v), the map $\frac{\delta}{\delta m} b(t,x,\cdot)(v)$ has a linear derivative s.t. $\frac{\delta^2}{\delta m^2} b(t,x,\mu)(v,\cdot)$ is η-**Hölder** continuous unif. in t, x, μ, v and $\frac{\delta^2}{\delta m^2} b$ is **bounded**.

Example: Polynomial interaction

$$b(t,x,\mu) = \int_{\mathbb{R}^d} \tilde{b}(t,x,y_1,\dots,y_l)\, d\mu(y_1)\dots d\mu(y_l),$$

with $\tilde{b}$ η-**Hölder** w.r.t. to $(x, y_1, \dots, y_l)$ unif. in t and **bounded**.

Fix $s \in [0, T)$, $\beta \in (1, \alpha)$ and consider

$$\begin{cases} dX_t^{s,\xi} = b(t, X_t^{s,\xi}, [X_t^{s,\xi}])\, dt + dZ_t, \quad \forall t \in [s, T], \\ X_s^{s,\xi} = \xi, \quad [\xi] = \mu \in \mathcal{P}_\beta(\mathbb{R}^d), \end{cases} \tag{McKean-Vlasov}$$

$\to [X_t^{s,\xi}] =: [X_t^{s,\mu}]$.

We introduce, for $x \in \mathbb{R}^d$, the **decoupled stochastic flow** associated to (McKean-Vlasov)

$$\begin{cases} dX_t^{s,x,\mu} = b(t, X_t^{s,x,\mu}, [X_t^{s,\mu}])\, dt + dZ_t, \quad \forall t \in [s, T], \\ X_s^{s,x,\mu} = x \in \mathbb{R}^d. \end{cases}$$

- $[X_t^{s,x,\mu}]$ has a density $p(\mu, s, t, x, \cdot)$.
- $[X_t^{s,\mu}]$ has a density $p(\mu, s, t, \cdot)$.
- **Key relation:** $p(\mu, s, t, y) = \int_{\mathbb{R}^d} p(\mu, s, t, x, y)\, d\mu(x)$.

How to study the regularity w.r.t. μ ?

Iteration

Picard iteration:

$$\begin{cases} dX_t^{s,\xi,(m+1)} = b(t, X_t^{s,\xi,(m+1)}, [X_t^{s,\mu,(m)}])\, dt + dZ_t, \quad \forall t \in [s,T], \\ X_s^{s,\xi,(m+1)} = \xi, \quad [\xi] = \mu \in \mathcal{P}_\beta(\mathbb{R}^d), \end{cases}$$

with $[X_t^{s,\xi,(1)}] = \nu \in \mathcal{P}_\beta(\mathbb{R}^d)$ and

$$\begin{cases} dX_t^{s,x,\mu,(m+1)} = b(t, X_t^{s,x,\mu,(m+1)}, [X_t^{s,\mu,(m)}])\, dt + dZ_t, \quad \forall t \in [s,T], \\ X_s^{s,x,\mu,(m+1)} = x \in \mathbb{R}^d. \end{cases}$$

$\hookrightarrow$ density $p_m(\mu, s, t, x, \cdot)$.

Iteration

Picard iteration:

$$\begin{cases} dX_t^{s,\xi,(m+1)} = b(t, X_t^{s,\xi,(m+1)}, [X_t^{s,\mu,(m)}])\, dt + dZ_t, & \forall t \in [s, T], \\ X_s^{s,\xi,(m+1)} = \xi, \quad [\xi] = \mu \in \mathcal{P}_\beta(\mathbb{R}^d), \end{cases}$$

with $[X_t^{s,\xi,(1)}] = \nu \in \mathcal{P}_\beta(\mathbb{R}^d)$ and

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$\hookrightarrow$ density $p_m(\mu, s, t, x, \cdot)$.

Proxy and parametrix kernel: For $0 \leq s \leq r < t \leq T$

$$\begin{cases} \widehat{p}(r, t, x, y) := q(t - r, y - x), & \text{(transition density of } Z\text{)} \\ \mathcal{H}_m(\mu, s, r, t, x, y) := b(r, x, [X_r^{s,\mu,(m-1)}]) \cdot \partial_x \widehat{p}(r, t, x, y). \end{cases}$$

Parametrix expansion:

$$\begin{aligned} p_m(\mu, s, t, x, y) &= \widehat{p}(s, t, x, y) + p_m \otimes \mathcal{H}_m(\mu, s, s, y, x, y) \quad \text{(implicit formulation)} \\ &= \widehat{p}(s, t, x, y) + \sum_{k=1}^{\infty} \widehat{p} \otimes \mathcal{H}_m^k(\mu, s, s, t, x, y). \end{aligned}$$

Induction and passage to the limit

Induction $\Rightarrow$ Regularity of p_m w.r.t. $(\mu, s, x) \in \mathcal{P}_\beta(\mathbb{R}^d) \times [0, t) \times \mathbb{R}^d$ + bounds + Hölder controls **uniform in m**.

How to deduce the corresponding regularity and estimates on p ?

Induction and passage to the limit

Induction $\Rightarrow$ Regularity of p_m w.r.t. $(\mu, s, x) \in \mathcal{P}_\beta(\mathbb{R}^d) \times [0, t) \times \mathbb{R}^d$ + bounds + Hölder controls **uniform in m**.

How to deduce the corresponding regularity and estimates on p ?

- Since $\sup_{t\in[s,T]} d_{TV}([X_t^{s,\mu,(m)}], [X_t^{s,\mu}]) \underset{m\to 0}{\longrightarrow} 0 \Rightarrow (p_m)_m$ converges point-wise to p **(parametrix expansion)**.
- **Bounds + equi-continuity** $\Rightarrow$ Extraction of converging subsequences of the derivatives of p_m through compactness argument **(Arzéla-Ascoli theorem)**.

  $\hookrightarrow p(\cdot, \cdot, t, \cdot, y) \in \mathcal{C}^1(\mathcal{P}_\beta(\mathbb{R}^d) \times [0, t) \times \mathbb{R}^d)$.
- Deduce the corresponding bounds + Hölder controls on the derivatives of p by taking the limit in those on p_m.

Backward Kolmogorov PDE for the transition density

Markov property stemming from the weak well-posedness + **Itô's formula** $\Rightarrow$ p solves

$$\begin{cases} \partial_s p(\mu, s, t, x, y) + L_s p(\cdot, s, t, \cdot, y)(\mu, x) = 0, \quad \forall (\mu, s, x) \in \mathcal{P}_\beta(\mathbb{R}^d) \times [0, t) \times \mathbb{R}^d, \\ p(\mu, s, t, x, \cdot) \underset{s \to t-}{\longrightarrow} \delta_x, \quad \text{in the weak sense,} \end{cases}$$

where

$$\begin{aligned} L_s h(\mu, x) &:= b(s, x, \mu) \cdot \partial_x h(\mu, x) + \int_{\mathbb{R}^d} [h(\mu, x+z) - h(\mu, x) - z \cdot \partial_x h(\mu, x)] \, \frac{dz}{|z|^{d+\alpha}} \\ &\quad + \int_{\mathbb{R}^d} b(s, v, \mu) \cdot \partial_v \frac{\delta}{\delta m} h(\mu, x)(v) \, d\mu(v) \\ &\quad + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left[\frac{\delta}{\delta m} h(\mu, x)(v+z) - \frac{\delta}{\delta m} h(\mu, x)(v) - z \cdot \partial_v \frac{\delta}{\delta m} h(\mu, x)(v) \right] \frac{dz}{|z|^{d+\alpha}} \, d\mu(v). \end{aligned}$$

Backward Kolmogorov PDE for the transition density

Markov property stemming from the weak well-posedness + **Itô's formula** $\Rightarrow$ p solves

$$\begin{cases} \partial_s p(\mu, s, t, x, y) + L_s p(\cdot, s, t, \cdot, y)(\mu, x) = 0, \quad \forall (\mu, s, x) \in \mathcal{P}_\beta(\mathbb{R}^d) \times [0, t) \times \mathbb{R}^d, \\ p(\mu, s, t, x, \cdot) \underset{s \to t-}{\longrightarrow} \delta_x, \quad \text{in the weak sense,} \end{cases}$$

where

$$\begin{aligned} L_s h(\mu, x) := {} & b(s, x, \mu) \cdot \partial_x h(\mu, x) + \int_{\mathbb{R}^d} [h(\mu, x+z) - h(\mu, x) - z \cdot \partial_x h(\mu, x)] \, \frac{dz}{|z|^{d+\alpha}} \\ & + \int_{\mathbb{R}^d} b(s, v, \mu) \cdot \partial_v \frac{\delta}{\delta m} h(\mu, x)(v) \, d\mu(v) \\ & + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left[\frac{\delta}{\delta m} h(\mu, x)(v+z) - \frac{\delta}{\delta m} h(\mu, x)(v) - z \cdot \partial_v \frac{\delta}{\delta m} h(\mu, x)(v) \right] \frac{dz}{|z|^{d+\alpha}} \, d\mu(v). \end{aligned}$$

Hölder space

For $\delta \in [0, 1]$, $\mathcal{C}^\delta(\mathcal{P}_\beta(\mathbb{R}^d))$ is the set of functions $\phi : \mathcal{P}_\beta(\mathbb{R}^d) \to \mathbb{R}$ having a linear derivative s.t.

$$\left| \frac{\delta}{\delta m} \phi(\mu)(v_1) - \frac{\delta}{\delta m} \phi(\mu)(v_2) \right| \leq C |v_1 - v_2|^\delta.$$

Solution to the backward Kolmogorov PDE with terminal condition

Theorem

For $\phi \in \mathcal{C}^\delta(\mathcal{P}_\beta(\mathbb{R}^d))$, we define $U : (t,\mu) \in [0,T] \times \mathcal{P}_\beta(\mathbb{R}^d) \mapsto \phi([X_T^{t,\mu}])$. Then, U belongs to $\mathcal{C}^0([0,T] \times \mathcal{P}_\beta(\mathbb{R}^d)) \cap \mathcal{C}^1([0,T) \times \mathcal{P}_\beta(\mathbb{R}^d))$ and satisfies:

$$\left|\frac{\delta}{\delta m} U(t,\mu)(v)\right| \leq C(1+|v|^\delta), \quad \left|\partial_v \frac{\delta}{\delta m} U(t,\mu)(v)\right| \leq C(T-t)^{\frac{\delta-1}{\alpha}},$$

and for $\gamma \in (0,1] \cap (0,(2\alpha-2)\wedge(\eta+\alpha-1))$

$$\left|\partial_v \frac{\delta}{\delta m} U(t,\mu)(v_1) - \partial_v \frac{\delta}{\delta m} U(t,\mu)(v_2)\right| \leq C(T-t)^{\frac{\delta-1-\gamma}{\alpha}} |v_1-v_2|^\gamma.$$

Moreover, U is the (unique) classical solution to

$$\begin{cases} \partial_t U(t,\mu) + \mathscr{L}_t U(t,\cdot)(\mu) = 0, & \forall (t,\mu) \in [0,T) \times \mathcal{P}_\beta(\mathbb{R}^d), \\ U(T,\mu) = \phi(\mu), & \forall \mu \in \mathcal{P}_\beta(\mathbb{R}^d), \end{cases}$$

where

$$\begin{aligned} \mathscr{L}_s h(\mu) := &\int_{\mathbb{R}^d} b(s,v,\mu)\cdot \partial_v \frac{\delta}{\delta m} h(\mu)(v)\, d\mu(v) \\ &+ \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left[\frac{\delta}{\delta m} h(\mu)(v+z) - \frac{\delta}{\delta m} h(\mu)(v) - z\cdot \partial_v \frac{\delta}{\delta m} h(\mu)(v)\right] \frac{dz}{|z|^{d+\alpha}}\, d\mu(v). \end{aligned}$$

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Interacting particle system

Dynamics:

$$\begin{cases} dX_t^{i,N} = b(t, X_t^{i,N}, \overline{\mu}_t^N)\,dt + dZ_t^i, \quad \forall t \in [0,T], \forall i \leq N, \\ \overline{\mu}_t^N := \dfrac{1}{N}\displaystyle\sum_{j=1}^N \delta_{X_t^{j,N}}, \quad \textbf{(= empirical measure)} \\ X_0^{i,N} = X_0^i, \end{cases}$$

where (X_0^i) i.i.d. with distribution $\mu_0 \in \mathcal{P}_\beta(\mathbb{R}^d)$ and $(Z^i)_i$ i.i.d. with same distribution as Z.

Vectorial form: We set for $t \in [0,T]$ and $\boldsymbol{x} = (x_1, \cdots, x_N) \in (\mathbb{R}^d)^N$

$$\boldsymbol{X}_t^N = \begin{pmatrix} X_t^{1,N} \\ \vdots \\ X_t^{N,N} \end{pmatrix}, \quad \boldsymbol{Z}_t^N = \begin{pmatrix} Z_t^1 \\ \vdots \\ Z_t^N \end{pmatrix} \quad \text{and} \quad \boldsymbol{b}^N(t, \boldsymbol{x}) := \begin{pmatrix} b(t, x_1, \overline{\mu}_{\boldsymbol{x}}^N) \\ \vdots \\ b(t, x_N, \overline{\mu}_{\boldsymbol{x}}^N) \end{pmatrix} \in (\mathbb{R}^d)^N$$

$$\begin{cases} d\boldsymbol{X}_t^N = \boldsymbol{b}^N(t, \boldsymbol{X}_t^N)\,dt + d\boldsymbol{Z}_t^N, \quad \forall t \in [0,T], \\ \boldsymbol{X}_0^N = \begin{pmatrix} X_0^1 \\ \vdots \\ X_0^N \end{pmatrix}. \end{cases}$$

Quantitative weak propagation of chaos (at the level of semi-group)

Test functions

For $\delta \in (0,1]$ and $L > 0$, $\mathcal{C}_L^{\delta,\delta}(\mathcal{P}_\beta(\mathbb{R}^d))$ is the set of functions $\phi : \mathcal{P}_\beta(\mathbb{R}^d) \to \mathbb{R}$ admitting two linear derivatives $\frac{\delta}{\delta m}\phi$ and $\frac{\delta^2}{\delta m^2}\phi$ s.t.

$$\left|\frac{\delta}{\delta m}\phi(\mu)(v_1) - \frac{\delta}{\delta m}\phi(\mu)(v_2)\right| + \left|\frac{\delta^2}{\delta m^2}\phi(\mu)(v_1, v_1') - \frac{\delta^2}{\delta m^2}\phi(\mu)(v_2, v_2')\right| \leq L(|v_1 - v_2|^\delta + |v_1' - v_2'|^\delta).$$

Theorem

Let us fix $\gamma \in (0,1] \cap (0, (\delta + \alpha - 1) \wedge (2\alpha - 2) \wedge (\eta + \alpha - 1))$. There exists a positive constant $C_T = C(d, T, \alpha, \beta, \eta, \gamma, \delta, L)$ non-decreasing in T such that for all $\phi \in \mathcal{C}_L^{\delta,\delta}(\mathcal{P}_\beta(\mathbb{R}^d))$, it holds

$$\mathbb{E}|\phi(\overline{\mu}_T^N) - \phi(\mu_T)| \leq C_T \mathbb{E} W_1(\overline{\mu}_0^N, \mu_0) + \frac{C_T}{N^{1-\frac{1}{\beta}}},$$

and

$$|\mathbb{E}(\phi(\overline{\mu}_T^N) - \phi(\mu_T))| \leq \begin{cases} C_T \mathbb{E} W_1(\overline{\mu}_0^N, \mu_0) + \dfrac{C_T}{N^\gamma}, \\ \dfrac{C_T}{N^\gamma}, \quad \text{if } \mu_0 \in \mathcal{P}_{2\delta}(\mathbb{R}^d). \end{cases}$$

Using the result of Fournier-Guillin, we have

$$\mathbb{E}W_1(\overline{\mu}_0^N, \mu_0) \leq C \begin{cases} N^{-\frac{1}{2}} + N^{-\frac{\beta-1}{\beta}}, & \text{if } d = 1 \text{ and } \beta \neq 2, \\ N^{-\frac{1}{2}} \ln(1+N) + N^{-\frac{\beta-1}{\beta}}, & \text{if } d = 2 \text{ and } \beta \neq 2, \\ N^{-\frac{1}{d}} + N^{-\frac{\beta-1}{\beta}}, & \text{if } d \geq 3 \text{ and } \beta \neq \frac{d}{d-1}. \end{cases}$$

Remark: For all $\varphi : \mathbb{R}^d \to \mathbb{R}$ s.t. $\|\varphi\|_{\mathrm{Lip}} \leq 1$ i.e. $|\varphi(x) - \varphi(y)| \leq |x - y|$, define $\phi(\mu) := \int_{\mathbb{R}^d} \varphi \, d\mu$. Then $\phi \in \mathcal{C}_1^{1,1}(\mathcal{P}_\beta(\mathbb{R}^d))$.

$\hookrightarrow$ **Rate of convergence w.r.t.** $\boldsymbol{W_1}$. More precisely for mean-field limit, if $\mu_0 \in \mathcal{P}_{2\delta}(\mathbb{R}^d)$, we have by the Kantorovich-Rubinstein theorem

$$
\begin{aligned}
W_1([X_T^{1,N}], \mu_T) &= \sup_{\varphi,\, \|\varphi\|_{\mathrm{Lip}} \leq 1} \left| \mathbb{E}\varphi(X_T^{1,N}) - \int_{\mathbb{R}^d} \varphi \, d\mu_T \right| \\
&= \sup_{\varphi,\, \|\varphi\|_{\mathrm{Lip}} \leq 1} \left| \mathbb{E}\left(\frac{1}{N} \sum_{k=1}^N \varphi(X_T^{k,N}) \right) - \int_{\mathbb{R}^d} \varphi \, d\mu_T \right| \\
&= \sup_{\varphi,\, \|\varphi\|_{\mathrm{Lip}} \leq 1} \left| \mathbb{E} \int_{\mathbb{R}^d} \varphi \, d\overline{\mu}_T^N - \mathbb{E} \int_{\mathbb{R}^d} \varphi \, d\mu_T \right| \\
&= \sup_{\varphi,\, \|\varphi\|_{\mathrm{Lip}} \leq 1} \left| \mathbb{E}\phi(\overline{\mu}_T^N) - \mathbb{E}\phi(\mu_T) \right| \\
&\leq \frac{C}{N^\gamma}.
\end{aligned}
$$

$\to$ Allows also **non-linear functions** of the empirical measure.

Idea of the proof

- Consider $U(t,\mu) = \phi([X_T^{t,\mu}])$ the solution to the Kolmogorov PDE.
- If $\mu_t := [X_t^{0,\mu_0}]$, the map $t \in [0,T] \mapsto U(t,\mu_t)$ is constant by the **flow property**.
- Compare $\phi(\overline{\mu}_T^N) - \phi(\mu_T)$ through dynamics of $t \mapsto U(t,\overline{\mu}_t^N) - U(t,\mu_t)$.
- Additional regularity on U: for $\gamma \in (0,1] \cap (0,(2\alpha-2)\wedge(\eta+\alpha-1))$

$$\left|\partial_v \frac{\delta}{\delta m} U(t,\mu_1)(v_1) - \partial_v \frac{\delta}{\delta m} U(t,\mu_2)(v_2)\right| \le C \underbrace{(T-t)^{\frac{\delta-1-\gamma}{\alpha}}}_{\textbf{integrable for } \gamma<\delta+\alpha-1} \left(|v_1-v_2|^\gamma + W_1^\gamma(\mu_1,\mu_2)\right).$$

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- Apply Itô's formula for the **empirical projection** $(t,\boldsymbol{x}) \in [0,T) \times (\mathbb{R}^d)^N \mapsto U(t,\overline{\mu}_{\boldsymbol{x}}^N)$, where $\overline{\mu}_{\boldsymbol{x}}^N = \frac{1}{N}\sum_{k=1}^N \delta_{x_k}$ and for the particle system using that

$$\partial_{x_i} U(t,\overline{\mu}_{\boldsymbol{x}}^N) = \frac{1}{N}\partial_v \frac{\delta}{\delta m} U(t,\overline{\mu}_{\boldsymbol{x}}^N)(x_i) \quad \rightarrow \textbf{choose } \boldsymbol{\gamma > \alpha - 1}.$$

Idea of the proof

$$
\begin{aligned}
&U(t,\overline{\mu}^N_t) - U(t,\mu_t) - \Big(U(0,\overline{\mu}^N_0) - U(0,\mu_0)\Big) \\
&= \int_0^t \partial_t U(s,\overline{\mu}^N_s)\,ds \\
&\quad + \frac{1}{N}\sum_{i=1}^N \int_0^t \partial_v \frac{\delta}{\delta m} U(s,\overline{\mu}^N_s)(X^{i,N}_s)\cdot b(s,X^{i,N}_s,\overline{\mu}^N_s)\,ds \\
&\quad + \int_0^t \int_{(\mathbb{R}^d)^N} \Big[U(s,\overline{\mu}^N_{\boldsymbol{X}^N_{s-}+\boldsymbol{z}}) - U(s,\overline{\mu}^N_{\boldsymbol{X}^N_{s-}}) - \partial_{\boldsymbol{x}} U(s,\overline{\mu}^N_{\boldsymbol{X}^N_{s-}})\cdot \boldsymbol{z}\Big]\, d\boldsymbol{\nu}^N(\boldsymbol{z})\,ds \\
&\quad + \int_0^t \int_{(D_1^c)^N} \Big[U(s,\overline{\mu}^N_{\boldsymbol{X}^N_{s-}+\boldsymbol{z}}) - U(s,\overline{\mu}^N_{\boldsymbol{X}^N_{s-}})\Big]\, \widetilde{\boldsymbol{\mathcal{N}}}^N(ds,d\boldsymbol{z}) \\
&\quad + \int_0^t \int_{(D_1)^N} \Big[U(s,\overline{\mu}^N_{\boldsymbol{X}^N_{s-}+\boldsymbol{z}}) - U(s,\overline{\mu}^N_{\boldsymbol{X}^N_{s-}})\Big]\, \widetilde{\boldsymbol{\mathcal{N}}}^N(ds,d\boldsymbol{z}) \\
&=: I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}
$$

$\rightarrow$ **Goal:** make appear the Kolmogorov PDE.

Idea of the proof

→ **Linearisation** with the linear derivative.

For I_3, setting $m^i_{s,z,w} := w\overline{\mu}^N_{\boldsymbol{X}^N_{s-}+\tilde{z}_i} + (1-w)\overline{\mu}^N_{\boldsymbol{X}^N_{s-}}$

$$I_3 = \frac{1}{N}\sum_{i=1}^N \int_0^t \int_{\mathbb{R}^d}\int_0^1 \Big[\frac{\delta}{\delta m}U(s, m^i_{s,z,w})(X^{i,N}_{s-}+z)$$

$$-\frac{\delta}{\delta m}U(s, m^i_{s,z,w})(X^{i,N}_{s-}) - \partial_v \frac{\delta}{\delta m}U(s, \overline{\mu}^N_{\boldsymbol{X}^N_{s-}})(X^{i,N}_{s-})\cdot z\Big]\, dw\, d\nu(z)\, ds,$$

$$= \int_0^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Big[\frac{\delta}{\delta m}U(s,\overline{\mu}^N_{s-})(x+z) - \frac{\delta}{\delta m}U(s,\overline{\mu}^N_{s-})(x)$$

$$-\partial_v \frac{\delta}{\delta m}U(s,\overline{\mu}^N_{s-})(x)\cdot z\Big]\, d\nu(z)\, d\overline{\mu}^N_{s-}(x)\, ds$$

$+$ Error term.

↪ Allows to use the PDE satisfied by U.

Idea of the proof

Using the PDE, it remains

$$\begin{aligned}
&U(t,\overline{\mu}^N_t) - U(t,\mu_t) - \Big(U(0,\overline{\mu}^N_0) - U(0,\mu_0)\Big) \\
&= \text{Error term} \\
&\quad + \int_0^t \int_{(D_1^c)^N} \Big[U(s,\overline{\mu}^N_{\mathbf{X}^N_{s-}+z}) - U(s,\overline{\mu}^N_{\mathbf{X}^N_{s-}}) \Big] \widetilde{\mathcal{N}}^N(ds,dz) \\
&\quad + \int_0^t \int_{(D_1)^N} \Big[U(s,\overline{\mu}^N_{\mathbf{X}^N_{s-}+z}) - U(s,\overline{\mu}^N_{\mathbf{X}^N_{s-}}) \Big] \widetilde{\mathcal{N}}^N(ds,dz) \\
&= \text{Error term} + I_4 + I_5.
\end{aligned}$$

- Use **martingale property/BDG's inequalities** for I_4 and I_5.
- Find upper-bounds for each term using **the bounds/Hölder controls** on the derivatives w.r.t. the measure of U $\to$ **time-integrability of the singularities.**
- Conclude using the continuity of U that $U(t,\overline{\mu}^N_t) - U(t,\mu_t) \underset{t\to T}{\longrightarrow} \phi(\overline{\mu}^N_T) - \phi(\mu_T)$ a.s.

The system of mean-field interacting Ornstein-Uhlenbeck processes

Let $\beta \in [1, \alpha)$, and $\xi \in L^\beta(\Omega, \mathcal{F}_0)$ with $[\xi] = \mu_0 \in \mathcal{P}_\beta(\mathbb{R}^d)$.

Drift: $b(x, \mu) = x + \int_{\mathbb{R}^d} y \, d\mu(y) \to$ Lipschitz but **unbounded w.r.t. space and measure variables.**

McKean-Vlasov SDE:

$$\begin{cases} dX_t &= (X_t + \mathbb{E}X_t)\, dt + dZ_t, \quad \forall t \in [0, T], \\ X_0 &= \xi, \quad [\xi] = \mu_0 \in \mathcal{P}_\beta(\mathbb{R}^d). \end{cases} \tag{McKean-Vlasov}$$

Particle system:

$$\begin{cases} dX_t^{i,N} = X_t^{i,N}\, dt + \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} X_t^{j,N}\, dt + dZ_t^i, \quad \forall t \in [0, T], \forall i \leq N, \\ X_0^{i,N} = X_0^i, \end{cases} \tag{Particle system}$$

where $(X_0^i)_i$ and $(Z^i)_i$ i.i.d. with the same distributions as ξ and Z.

Propagation of chaos and comparison

Recall: $\mathcal{C}_1^{1,1}(\mathcal{P}_\beta(\mathbb{R}^d))$ is the space of functions $\phi : \mathcal{P}_\beta(\mathbb{R}^d) \to \mathbb{R}$ admitting two linear derivatives s.t. $\frac{\delta}{\delta m}\phi(\mu)(\cdot)$ and $\frac{\delta^2}{\delta m^2}\phi(\mu)(\cdot)$ are Lipschitz with Lipschitz constant smaller than 1.

Theorem

There exists a positive constant C_T s.t. for all $\phi \in \mathcal{C}_1^{1,1}(\mathcal{P}_\beta(\mathbb{R}^d))$ and $N \geq 1$

$$\mathbb{E}\left|\phi(\overline{\mu}_T^N) - \phi(\mu_T)\right| \leq C_T\, \mathbb{E} W_1(\overline{\mu}_0^N, \mu_0) + \left. C_T \frac{\ln(N)^{\frac{1}{\alpha}}}{N^{1-\frac{1}{\alpha}}} \right\} < \underbrace{\frac{1}{N^{1-\frac{1}{\beta}}}, \quad \beta < \alpha,}_{\text{bounded-Lipschitz case } \eta = 1}$$

and

$$|\mathbb{E}(\phi(\overline{\mu}_T^N) - \phi(\mu_T))| \leq C_T\, \mathbb{E} W_1(\overline{\mu}_0^N, \mu_0) + \left. \frac{C_T}{N^{\alpha-1}} \right\} > \underbrace{\frac{C_T}{N^\gamma}, \gamma \in (0,1] \cap (0, 2\alpha - 2)}_{\text{bounded-Lipschitz case better}}.$$

If μ_0 belongs to $\mathcal{P}_2(\mathbb{R}^d)$, we have

$$|\mathbb{E}(\phi(\overline{\mu}_T^N) - \phi(\mu_T))| \leq \left. \frac{C_T}{N^{\alpha-1}} \right\} > \underbrace{\frac{C_T}{N^\gamma}, \gamma \in (0,1] \cap (0, 2\alpha - 2)}_{\text{bounded-Lipschitz case better}}.$$

Idea of proof

- **Remove the jumps of size bigger than the number of particles N** in the noises of (Particle system) and (McKean-Vlasov).
- Control the L^1 error resp. with (McKean-Vlasov) and (Particle system)
- Follow the same reasoning as in the bounded-Hölder case.
  $\hookrightarrow$ transition density of (McKean-Vlasov) explicit $\to$ no need of parametrix expansion !
- Need to truncate the noises because $\displaystyle\int_{D_1^c} |z|^2 \, d\nu(z) = +\infty$ appears... but we control

$$\int_{1 \leq |z| \leq N} |z|^2 \, d\nu(z) \underset{N \to +\infty}{\sim} N^{2-\alpha}.$$

- Truncation also explains the factor $\ln(N)$ in the first estimate by

$$\int_{1 \leq |z| \leq N} |z|^\alpha \, d\nu(z) \underset{N \to +\infty}{\sim} \ln(N).$$

THANK YOU FOR YOUR ATTENTION !

References: 2 papers soon on arXiv.