Some recent results in propagation of chaos.

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Joint works with :

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- Arnaud Guillin *(LMBP, Clermont-Ferrand)*
- Pierre Monmarché *(LJLL, Paris)*
- Christophe Poquet *(ICJ, Lyon)*

I. Introduction

General setting

Consider the N-particle system in mean-field interaction

$$dX_t^i = \sqrt{2\sigma}dB_t^i + \frac{1}{N}\sum_{j=1}^{N} K(X_t^i - X_t^j)dt, \quad i \in \{1, ..., N\}.$$

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Question : What happens when $N \to \infty$?

In a system of N interacting particles, as N increases, two particles become more and more statistically independent.

Formal limit of SDE

N-particle system in a space $\mathcal{X}$

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Limit as *N* goes to infinity ?

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$$dX_t^i = \sqrt{2\sigma}\, dB_t^i + K * \mu_t^N(X_t^i)\, dt,$$

$$\mu_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^i}.$$

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$$\begin{aligned} dX_t^i &= \sqrt{2\sigma} dB_t^i + K * \mu_t^N(X_t^i) dt, \\ \mu_t^N &:= \frac{1}{N} \sum_{i=1}^N \delta_{X_t^i}. \end{aligned}$$

Limit as N goes to infinity ? Formally

$$\begin{cases} d\bar{X}_t = \sqrt{2\sigma} dB_t + K * \bar{\rho}_t(\bar{X}_t) dt, \\ \bar{\rho}_t = \mathrm{Law}(\bar{X}_t). \end{cases}$$

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Fokker-Planck equations

For the particle system

$$dX_t^i = \sqrt{2\sigma}dB_t^i + \frac{1}{N}\sum_{j=1}^N K(X_t^i - X_t^j)dt$$

$$\longleftrightarrow$$

$$\partial_t \rho_t^N = -\sum_{i=1}^N \nabla_{x_i} \cdot \left(\left(\frac{1}{N}\sum_{j=1}^N K(x_i - x_j) \right) \rho_t^N \right) + \sigma \sum_{i=1}^N \Delta_{x_i} \rho_t^N .$$

For the non linear equation

$$\begin{cases} d\bar{X}_t = \sqrt{2\sigma}dB_t + K * \bar{\rho}_t(\bar{X}_t)dt, \\ \bar{\rho}_t = \text{Law}(\bar{X}_t). \end{cases} \longleftrightarrow \partial_t \bar{\rho}_t = -\nabla \cdot (\bar{\rho}_t (K * \bar{\rho}_t)) + \sigma \Delta \bar{\rho}_t .$$

Several variations

Kinetic setting/Degenerate noise in a non-convex confining potential

$$\begin{cases} dX_t^i = V_t^i dt \\ dV_t^i = \sqrt{2\sigma} dB_t^i - V_t^i dt - \nabla U(X_t^i) dt - \frac{1}{N}\sum_{j=1}^N \nabla W(X_t^i - X_t^j) dt. \end{cases}$$

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Singular interactions

$$\text{(dim=2)} \quad dX_t^i = \sqrt{2\sigma} dB_t^i + \frac{1}{N}\sum_{j\neq i} K(X_t^i - X_t^j) dt, \quad K(x) = \frac{1}{2\pi}\left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}\right)$$

$$\text{(dim=1)} \quad dX_t^i = \sqrt{\frac{2\sigma}{N}} dB_t^i - \lambda X_t^i dt + \frac{1}{N}\sum_{j\neq i} \frac{X_t^i - X_t^j}{\left|X_t^i - X_t^j\right|^{\alpha+1}} dt.$$

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"Incomplete" interactions

$$dX_t^i = F\left(X_t^i, \omega_i\right) dt + \frac{\alpha_N}{N}\sum_{j=1}^N \xi_{i,j}^{(N)} \Gamma\left(X_t^i, \omega_i, X_t^j, \omega_j\right) dt + \sqrt{2\sigma} dB_t^i.$$

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or, equivalently, "$\mu_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^i} \to \bar{\rho}_t$".

Usual (or current) methods

Goal : Show $\mu_t^N \to \bar{\rho}_t$ or $\rho_t^{k,N} \to \bar{\rho}_t^{\otimes k}$ as $N \to \infty$, if possible uniformly in t.

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- Coupling methods *(McKean, Sznitman, Eberle...) :*
  $\mathcal{W}_p(\mu,\nu)^p = \inf_{X\sim\mu, Y\sim\nu} \mathbb{E}\left(|X-Y|^p\right)$. Show $\mathcal{W}_p\left(\rho_t^{k,N}, \bar{\rho}_t^{\otimes k}\right) \to 0$.

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- Weak norm and Lions derivative calculus *(Delarue-Tse, Chassagneux, Szpruch...)*

II. Coupling methods : kinetic setting or incomplete interactions

Synchronous coupling

$$\begin{cases} dX_t^i = \frac{1}{N}\sum_{j=1}^N K(X_t^i - X_t^j)dt + \sqrt{2\sigma}dB_t^i, \\ d\bar{X}_t^i = K * \bar{\rho}_t(\bar{X}_t^i)dt + \sqrt{2\sigma}dB_t^i, \\ \bar{\rho}_t = \mathrm{Law}(\bar{X}_t). \end{cases}$$

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Thus, for all $i \in \{1, ..., N\}$

$$d|X_t^i - \bar{X}_t^i| = A_t dt,$$

with

$$\begin{aligned} A_t &\leq \left| \frac{1}{N}\sum_{j=1}^N K(X_t^i - X_t^j) - K * \bar{\rho}_t(\bar{X}_t^i) \right| \\ &\leq \left| \frac{1}{N}\sum_{j=1}^N K(X_t^i - X_t^j) - \frac{1}{N}\sum_{j=1}^N K(\bar{X}_t^i - \bar{X}_t^j) \right| + \left| \frac{1}{N}\sum_{j=1}^N K(\bar{X}_t^i - \bar{X}_t^j) - K * \bar{\rho}_t(\bar{X}_t^i) \right|. \end{aligned}$$

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$$\leq \underbrace{\left| \frac{1}{N}\sum_{j=1}^N K(X_t^i - X_t^j) - \frac{1}{N}\sum_{j=1}^N K(\bar{X}_t^i - \bar{X}_t^j) \right|}_{\text{Lipschitz continuity}} + \underbrace{\left| \frac{1}{N}\sum_{j=1}^N K(\bar{X}_t^i - \bar{X}_t^j) - K * \bar{\rho}_t(\bar{X}_t^i) \right|}_{\text{Law of large numbers}}.$$

Synchronous coupling-2

By Gronwall's lemma and exchangeability

$$\mathbb{E}\left(|X_t^i - \bar{X}_t^i|\right) \leq e^{2Lt}\left(\mathbb{E}\left(|X_0^i - \bar{X}_0^i|\right) + \frac{C}{\sqrt{N}}\right).$$

and thus

$$\mathcal{W}_1\left(\rho_t^{k,N}, \bar{\rho}_t^{\otimes k}\right) \leq \mathbb{E}\left(\sum_{i=1}^k |X_t^i - \bar{X}_t^i|\right) \leq e^{2Lt}\left(\mathbb{E}\left(\sum_{i=1}^k |X_0^i - \bar{X}_0^i|\right) + \frac{Ck}{\sqrt{N}}\right).$$

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Because this holds for any initial coupling of $\rho_0^{k,N}$ and $\bar{\rho}_0^{\otimes k}$,

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Problem : Not uniform in time !

Kinetic setting

Particle system $((X_t^{i,N}, V_t^{i,N}))_{i=1,\dots,N}$, with $X_t^{i,N}, V_t^{i,N} \in \mathbb{R}^d$

$$\begin{cases} dX_t^{i,N} = V_t^{i,N} dt \\ dV_t^{i,N} = \sqrt{2\sigma} dB_t^i - V_t^{i,N} dt - \nabla U(X_t^{i,N}) dt - \frac{1}{N} \sum_{j=1}^N \nabla W(X_t^{i,N} - X_t^{j,N}) dt \end{cases}$$

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Underdamped Langevin diffusion (Non linear particle)

$$\begin{cases} d\bar{X}_t = \bar{V}_t dt \\ d\bar{V}_t = \sqrt{2\sigma} dB_t - \bar{V}_t dt - \nabla U(\bar{X}_t) dt - \nabla W * \bar{\mu}_t(\bar{X}_t) dt \\ \bar{\mu}_t = Law(\bar{X}_t) \end{cases}$$

with

$$\nabla W * \bar{\mu}_t(x) = \int_{\mathbb{R}^d} \nabla W(x-y) \bar{\mu}_t(dy)$$

Assumptions on the confinement potential

Assumption

The potential U is non-negative and there exist $\lambda > 0$ and $A \geq 0$ such that

$$\forall x \in \mathbb{R}^d, \qquad \frac{1}{2}\nabla U(x) \cdot x \geq \lambda \left(U(x) + \frac{|x|^2}{4} \right) - A.$$

Furthermore, there is a constant $L_U > 0$ such that

$$\forall x, y \in \mathbb{R}^d \times \mathbb{R}^d, \qquad |\nabla U(x) - \nabla U(y)| \leq L_U |x - y|.$$

Assumptions on the confinement potential

The double-well potential given by

$$U(x) = \begin{cases} (x^2-1)^2 & \text{if } |x| \leq 1, \\ (|x|-1)^2 & \text{otherwise.} \end{cases}$$

satisfies the previous assumptions.

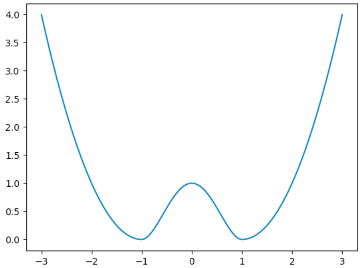

FIGURE – Double well potential

Assumptions on the interaction potential

Assumption

$\nabla W(0) = 0$ *and there exists* $L_W \leq \lambda/8$ *such that*

$$\forall x, y \in \mathbb{R}^d \times \mathbb{R}^d, \qquad |\nabla W(x) - \nabla W(y)| \leq L_W |x - y|.$$

In particular $|\nabla W(x)| \leq L_W |x|$ *for all* $x \in \mathbb{R}^d$*.*

Constructing solutions

We need to construct simultaneously two solutions

$$\begin{cases} d\bar{X}_t^i = \bar{V}_t^i dt \\ d\bar{V}_t^i = -\bar{V}_t^i dt - \nabla U\left(\bar{X}_t^i\right) dt - \nabla W * \bar{\mu}_t \left(\bar{X}_t^i\right) dt + \sqrt{2\sigma} dB_t^{i,1} \\ \bar{\mu}_t = \mathcal{L}\left(\bar{X}_t^i\right) \\ dX_t^{i,N} = V_t^{i,N} dt \\ dV_t^{i,N} = -V_t^{i,N} dt - \nabla U(X_t^{i,N}) dt - \frac{1}{N}\sum_{j=1}^N \nabla W(X_t^{i,N} - X_t^{j,N}) dt \\ \qquad\qquad + \sqrt{2\sigma} dB_t^{i,2}, \end{cases}$$

Coupling

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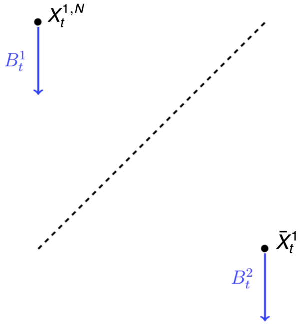

FIGURE – Synchronous coupling

Choosing $B^{i,1} = B^{i,2}$:

• the Brownian noise is canceled out in the infinitesimal evolution of the difference $(Z_t^i, W_t^i) = \left(X_t^{1,N} - \bar{X}_t^1, V_t^{1,N} - \bar{V}_t^1\right)$,

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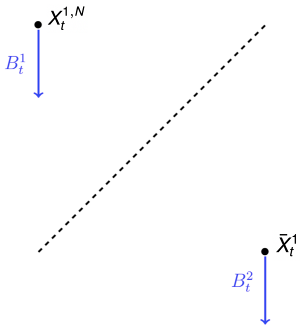

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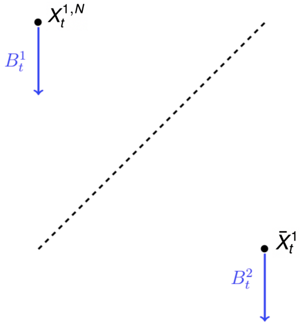

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- the contraction of a distance between the processes can only be induced by the deterministic drift.
- Here : contraction when $Z_t^i + W_t^i = 0$

Coupling

Outside of $\{(z, v) \in \mathbb{R}^{2d}, z + w = 0\}$, it is necessary to make use of the noise to get the processes closer to one another.

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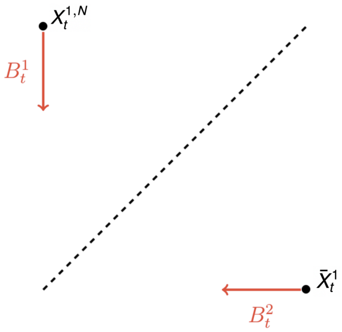


FIGURE – Reflection coupling

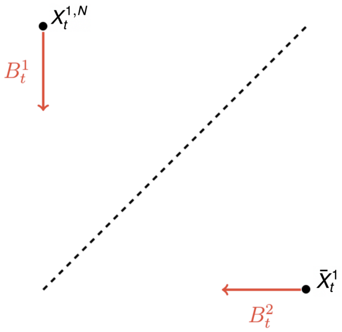
Writing

$$e_t^i = \begin{cases} \frac{Z_t^i+W_t^i}{|Z_t^i+W_t^i|} & \text{if } Z_t^i+W_t^i\neq 0 \\ 0 & \text{otherwise} \end{cases}$$

we consider
$dB_t^{i,2} = \left(Id - 2e_t^i e_t^{i,T}\right) dB_t^{i,1}$:

Coupling

Outside of $\{(z, v) \in \mathbb{R}^{2d}, z + w = 0\}$, it is necessary to make use of the noise to get the processes closer to one another.



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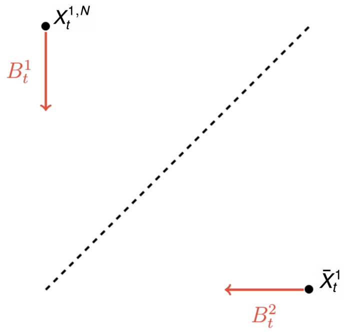

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- this maximizes the variance of the noise in the desired direction,
- requires a modification of the distance by some concave function $\implies$ only within a compact set.

Three behaviors

• When any of the particle ventures at infinity (i.e $|X_t|$ or $|V_t|$ becomes sufficiently big), the friction and confinement potential will tend to bring the particle back,
$\implies$ use a Lyapunov function (i.e H such that $\frac{d}{dt}\mathbb{E}H \leq B - \gamma\mathbb{E}H$).

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• When the particles are near the space

$$\left\{\left(X_t^{i,N}, \bar{X}_t^i, V_t^{i,N}, \bar{V}_t^i\right) \in \mathbb{R}^{4d}, X_t^{i,N} - \bar{X}_t^i + V_t^{i,N} - \bar{V}_t^i = 0\right\},$$

the L^1 distance will naturally contract,
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the L^1 distance will naturally contract,
$\implies$ use a synchronous coupling.

• Otherwise, the particles are in a compact set,
$\implies$ use a reflection coupling.

Coupling

We consider the following coupling

$$\begin{cases} d\bar{X}_t^i = \bar{V}_t^i dt \\ d\bar{V}_t^i = -\bar{V}_t^i dt - \nabla U\left(\bar{X}_t^i\right) dt - \nabla W * \bar{\mu}_t\left(\bar{X}_t^i\right) dt + \sqrt{2} rc\left(Z_t^i, W_t^i\right) dB_t^{rc,i} \\ \qquad + \sqrt{2} sc\left(Z_t^i, W_t^i\right) dB_t^{sc,i} \\ \bar{\mu}_t = \mathcal{L}\left(\bar{X}_t^i\right) \\ dX_t^{i,N} = V_t^{i,N} dt \\ dV_t^{i,N} = -V_t^{i,N} dt - \nabla U(X_t^{i,N}) dt - \frac{1}{N}\sum_{j=1}^N \nabla W(X_t^{i,N} - X_t^{j,N}) dt \\ \qquad + \sqrt{2}\left(rc\left(Z_t^i, W_t^i\right)\left(Id - 2e_t^i e_t^{i,T}\right) dB_t^{rc,i} + sc\left(Z_t^i, W_t^i\right) dB_t^{sc,i}\right), \end{cases}$$

with

$$rc^2 + sc^2 = 1,$$

$$rc(z,w) = 0 \text{ if } |z+w| \leq \frac{\xi}{2} \text{ or } \alpha|z| + |z+w| \geq R_1 + \xi,$$

$$rc(z,w) = 1 \text{ if } |z+w| \geq \xi \text{ and } \alpha|z| + |z+w| \leq R_1.$$

Semimetrics

Define, for f a well chosen concave function and H a Lyapunov function

$$
\begin{aligned}
r_t^i =& \alpha|X_t^{i,N} - \bar{X}_t^i| + |X_t^{i,N} - \bar{X}_t^i + V_t^{i,N} - \bar{V}_t^i|, \\
\rho_t =& \frac{1}{N}\sum_{i=1}^N f\left(r_t^i\right)\left(1 + \epsilon H\left(\bar{X}_t^i, \bar{V}_t^i\right) + \epsilon H(X_t^{i,N}, V_t^{i,N})\right. \\
& \left. + \frac{\epsilon}{N}\sum_{j=1}^N H\left(\bar{X}_t^j, \bar{V}_t^j\right) + \frac{\epsilon}{N}\sum_{j=1}^N H(X_t^{j,N}, V_t^{j,N})\right) \\
:=& \frac{1}{N}\sum_{i=1}^N f\left(r_t^i\right) G_t^i.
\end{aligned}
$$

Main result

Theorem (Guillin-LB-Monmarché ('22))

Let $\mathcal{C}^0 > 0$ and $a > 0$. Let $U \in \mathcal{C}^1(\mathbb{R}^d)$ satisfy the previous assumption. There is an explicit $c^W > 0$ such that, for all $W \in \mathcal{C}^1(\mathbb{R}^d)$ satisfying $L_W < c^W$, there exist explicit $B_1, B_2 > 0$, such that for all probability measures ν_0 on $\mathbb{R}^{2d}$ (under some initial moment assumption depending on $\mathcal{C}^0$ and a) and for all $t \geq 0$,

$$\mathcal{W}_1\left(\nu_t^{k,N}, \bar{\nu}_t^{\otimes k}\right) \leq \frac{kB_1}{\sqrt{N}}, \qquad \mathcal{W}_2^2\left(\nu_t^{k,N}, \bar{\nu}_t^{\otimes k}\right) \leq \frac{kB_2}{\sqrt{N}},$$

for all $k \in \mathbb{N}$, where $\nu_t^{k,N}$ is the marginal distribution at time t of the first k particles $((X_t^1, V_t^1), ..., (X_t^k, V_t^k))$ of an N particle system (PS) with initial distribution $(\nu_0)^{\otimes N}$, while $\bar{\nu}_t$ is the probability densities of (NL) with initial distribution ν_0.

Somes references on reflection coupling

Contraction :

• Andreas Eberle. *Reflection couplings and contraction rates for diffusions*. Probab. Theory Relat. Fields (2016)

• Andreas Eberle, Arnaud Guillin, and Raphael Zimmer. *Couplings and quantitative contraction rates for Langevin dynamics*. Ann. Probab. (2019)

• Andreas Eberle, Arnaud Guillin, and Raphael Zimmer. *Quantitative Harris-type theorems for diffusions and McKean-Vlasov processes*. Trans. Am. Math. Soc.(2019)

Propagation of chaos :

• Alain Durmus, Andreas Eberle, Arnaud Guillin, and Raphael Zimmer. *An elementary approach to uniform in time propagation of chaos*. Proc. Amer. Math. Soc. (2020)

• Arnaud Guillin, Pierre Le Bris, and Pierre Monmarché. *Convergence rates for the Vlasov-Fokker-Planck equation and uniform in time propagation of chaos in non convex cases.* Electron. J. Probab. (2022)

• Katharina Schuh. *Global contractivity for Langevin dynamics with distribution-dependent forces and uniform in time propagation of chaos.* arXiv preprint arXiv :2206.03082 (2022)

FitzHugh-Nagumo model

Show uniform in time propagation of chaos of

$$\begin{cases} dX_t^{i,N} = (X_t^{i,N} - (X_t^{i,N})^3 - C_t^{i,N} - \alpha)dt + \frac{1}{N}\sum_{j=1}^N K_X(Z_t^{i,N} - Z_t^{j,N}) \\ \qquad\qquad + \sigma_X dB_t^{i,X} \\ dC_t^{i,N} = (\gamma X_t^{i,N} - C_t^{i,N} + \beta)dt + \frac{1}{N}\sum_{j=1}^N K_C(Z_t^{i,N} - Z_t^{j,N}) + \sigma_C dB_t^{i,C}, \end{cases}$$

towards

$$\begin{cases} d\bar{X}_t = (\bar{X}_t - (\bar{X}_t)^3 - \bar{C}_t - \alpha)dt + K_X * \bar{\mu}_t(\bar{Z}_t)dt + \sigma_X d\bar{B}_t^X \\ d\bar{C}_t = (\gamma\bar{X}_t - \bar{C}_t + \beta)dt + K_C * \bar{\mu}_t(\bar{Z}_t)dt + \sigma_C d\bar{B}_t^C, \end{cases}$$

where we allow either σ_X or σ_C to be equal to 0.

Reference :

• Laetitia Colombani and Pierre Le Bris. *Chaos propagation in mean field networks of FitzHugh-Nagumo neurons*. arXiv preprint arXiv :2206.13291 (2022)

In a graph

$$dX_t^i = F\left(X_t^i, \omega_i\right) dt + \frac{\alpha_N}{N} \sum_{j=1}^{N} \xi_{i,j}^{(N)} \Gamma\left(X_t^i, \omega_i, X_t^j, \omega_j\right) dt + \sqrt{2\sigma} dB_t^i,$$

where

- $\xi^{(N)} = \left(\xi_{i,j}^{(N)}\right)_{i,j \in \{1,\dots,N\}}, \ \xi_{i,j}^{(N)} \in \{0,1\}$: graph,
- $\{\omega_i\}_{i \in \{1,\dots,N\}}$: environmental disorder,
- $(\alpha_N)_{N \geq 1}$: scaling,
- $F : \mathbb{R}^d \times \mathcal{X} \mapsto \mathbb{R}^d$: outside force,
- $\Gamma : \left(\mathbb{R}^d \times \mathcal{X}\right)^2 \mapsto \mathbb{R}^d$: interaction

In a graph-2

Assuming there is $p \in [0,1]$

$$\sup_{i\in\{1,\dots,N\}} \left| \alpha_N \frac{d_i^{(N)}}{N} - p \right| \xrightarrow[N\to\infty]{a.s} 0,$$

uniform in time propagation of chaos towards

$$\begin{cases} d\bar{X}_t^\omega = F\left(\bar{X}_t^\omega, \omega\right) dt + p \int_{\mathbb{R}^d \times \mathcal{X}} \Gamma\left(\bar{X}_t^\omega, \omega, y, \tilde{\omega}\right) \bar{\rho}_t(dy, d\tilde{\omega}) dt + \sqrt{2\sigma} dB_t, \\ \bar{\rho}_t = \mathrm{Law}(\bar{X}_t^\omega, \omega) \end{cases},$$

Reference :

• Pierre Le Bris and Christophe Poquet. *A note on uniform in time mean-field limit in graphs*, arXiv preprint arXiv :2211.11519 (2022).

• Sylvain Delattre, Giambattista Giacomin, and Eric Luçon. *A note on dynamical models on randomgraphs and Fokker-Planck equations*, J. Stat. Phys. (2016).

Some remarks on the method

Pros :

- Quantitative,
- Yields/Uses a probabilistic understanding of the result,
- "Quite" robust...

Cons :

- Not sharp in N *(cf. Lacker)*,
- So far, restricted to "nice" interactions...

III. Singular kernels : entropy method or Cauchy-type method

The 2D vortex model

The Biot-Savart kernel, defined in $\mathbb{R}^2$ by

$$K(x) = \frac{1}{2\pi}\frac{x^\perp}{|x|^2} = \frac{1}{2\pi}\left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}\right).$$

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Consider the 2D incompressible Navier-Stokes system on $u \in \mathbb{R}^2$

$$\begin{aligned}\partial_t u &= -u\cdot\nabla u - \nabla p + \Delta u\\ \nabla\cdot u &= 0,\end{aligned}$$

where p is the local pressure.

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N-particle system on the torus $\mathbb{T}^d$

$$dX_t^i = \sqrt{2}dB_t^i + \frac{1}{N}\sum_{j=1}^N K(X_t^i - X_t^j)dt.$$

(Rescaled) relative entropy

Definition

Let μ and ν be two probability measures on $\mathbb{T}^{dN}$. We consider the rescaled relative entropy

$$\mathcal{H}_N(\nu,\mu) = \begin{cases} \frac{1}{N}\mathbb{E}_\mu\left(\frac{d\nu}{d\mu}\log\frac{d\nu}{d\mu}\right) & \text{if } \nu \ll \mu, \\ +\infty \text{ otherwise.} \end{cases}$$

Results

Theorem (adapted from Jabin-Wang ('18))

Under some assumptions (satisfied by the Biot-Savart kernel) there are constants C_1 and C_2 such that for all $N \in \mathbb{N}$, all exchangeable probability density ρ_0^N and all $t \geq 0$

$$\mathcal{H}_N(\rho_t^N, \bar{\rho}_t^N) \leq e^{C_1 t}\left(\mathcal{H}_N(\rho_0^N, \bar{\rho}_0^N) + \frac{C_2}{N}\right)$$

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Theorem (Guillin-LB-Monmarché ('21))

Under some assumptions (satisfied by the Biot-Savart kernel) there are constants C_1, C_2 and C_3 such that for all $N \in \mathbb{N}$, all exchangeable probability density ρ_0^N and all $t \geq 0$

$$\mathcal{H}_N(\rho_t^N, \bar{\rho}_t^N) \leq C_1 e^{-C_2 t} \mathcal{H}_N(\rho_0^N, \bar{\rho}_0^N) + \frac{C_3}{N}$$

Various distances

For $\mathbf{x} = (x_i)_{i\in[\![1,N]\!]} \in \mathbb{T}^{dN}$, we write $\pi(\mathbf{x}) = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$ the associated empirical measure.

Corollary

Under some assumptions (satisfied by the Biot-Savart kernel), assuming moreover that $\rho_0^N = \bar\rho_0^N$, there is a constant C such that for all $k \leq N \in \mathbb{N}$ and all $t \geq 0$,

$$\|\rho_t^{k,N} - \bar\rho_t^k\|_{L^1} + \mathcal{W}_2\left(\rho_t^{k,N}, \bar\rho_t^k\right) \leq C\left(\left\lfloor \frac{N}{k} \right\rfloor\right)^{-\frac{1}{2}}$$

and

$$\mathbb{E}_{\rho_t^N}\left(\mathcal{W}_2(\pi(\boldsymbol{X}), \bar\rho_t)\right) \leqslant C\alpha(N)$$

where $\alpha(N) = N^{-1/2}\ln(1+N)$ if $d = 2$ and $\alpha(N) = N^{-1/d}$ if $d > 2$.

Some references on 2D vortex model

• Pierre-Emmanuel Jabin and Zhenfu Wang. *Quantitative estimates of propagation of chaos for stochastic systems with* $w^{-1,\infty}$ *kernels*. Invent. Math., (2018)

• Arnaud Guillin, Pierre Le Bris, and Pierre Monmarché. *Uniform in time propagation of chaos for the 2D vortex model and other singular stochastic systems.* arXiv preprint arXiv :2108.08675 (2021)

• Nicolas Fournier, Maxime Hauray, and Stéphane Mischler. *Propagation of chaos for the 2D viscous vortex model*. J. Eur. Math. Soc. (JEMS), (2014)

• Matthew Rosenzweig and Sylvia Serfaty. *Global-in-time mean-field convergence for singular Riesz-type diffusive flows.* arXiv preprint arXiv :2108.09878 (2021)

Step one : Time evolution of the relative entropy

We write

$$\mathcal{H}_N(t) = \mathcal{H}_N(\rho_t^N, \bar{\rho}_t^N), \quad \mathcal{I}_N(t) = \frac{1}{N}\sum_i \int_{\mathbb{T}^{dN}} \rho_t^N \left| \nabla_{x_i} \log \frac{\rho_t^N}{\bar{\rho}_t^N} \right|^2 d\mathbf{X}^N.$$

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It can be shown that

$$\begin{aligned}\frac{d}{dt}\mathcal{H}_N(t) = &- \mathcal{I}_N(t) \\ &- \frac{1}{N^2}\sum_{i,j}\int_{\mathbb{T}^{dN}} \rho_t^N \left(K(x_i - x_j) - K * \bar{\rho}_t(x_i)\right) \cdot \nabla_{x_i} \log \bar{\rho}_t^N d\mathbf{X}^N \\ &- \frac{1}{N^2}\sum_{i,j}\int_{\mathbb{T}^{dN}} \rho_t^N \left(\mathrm{div}\, K(x_i - x_j) - \mathrm{div}\, K * \bar{\rho}_t(x_i)\right) d\mathbf{X}^N.\end{aligned}$$

Assumptions ?

$$\textbf{Goal :}\ K(x) = \frac{1}{2\pi}\frac{x^\perp}{|x|^2} = \frac{1}{2\pi}\left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}\right)$$

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- $\bar{\rho} \in \mathcal{C}^\infty(\mathbb{R}^+ \times \mathbb{T}^d)$

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Justifying the calculations

- $\bar\rho \in \mathcal{C}^\infty(\mathbb{R}^+ \times \mathbb{T}^d)$ and there is $\lambda > 1$, s.t $\frac{1}{\lambda} \leq \bar\rho \leq \lambda$

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- $\rho^N \in \mathcal{C}^\infty_\lambda(\mathbb{R}^+ \times \mathbb{T}^{Nd})$ (? ? ?)

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- $\rho^N \in \mathcal{C}^{\infty}_{\lambda}(\mathbb{R}^+ \times \mathbb{T}^{Nd})$ (? ? ?)

Dealing with the terms

- In the sense of distributions, $\nabla \cdot K = 0$.

Step one : Time evolution of the relative entropy

We write

$$\mathcal{H}_N(t) = \mathcal{H}_N(\rho_t^N, \bar{\rho}_t^N), \quad \mathcal{I}_N(t) = \frac{1}{N}\sum_i \int_{\mathbb{T}^{dN}} \rho_t^N \left|\nabla_{x_i} \log \frac{\rho_t^N}{\bar{\rho}_t^N}\right|^2 d\mathbf{X}^N.$$

It can be shown that

$$\begin{aligned}\frac{d}{dt}\mathcal{H}_N(t) = &-\mathcal{I}_N(t)\\ &-\frac{1}{N^2}\sum_{i,j}\int_{\mathbb{T}^{dN}} \rho_t^N \left(K(x_i - x_j) - K * \bar{\rho}_t(x_i)\right) \cdot \nabla_{x_i} \log \bar{\rho}_t^N d\mathbf{X}^N\\ &-\frac{1}{N^2}\sum_{i,j}\int_{\mathbb{T}^{dN}} \rho_t^N \left(\operatorname{div} K(x_i - x_j) - \operatorname{div} K * \bar{\rho}_t(x_i)\right) d\mathbf{X}^N.\end{aligned}$$

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It can be shown that

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Step two : Integration by part

We are left with

$$\frac{d}{dt}\mathcal{H}_N(t) = -\mathcal{I}_N(t) \\ -\frac{1}{N^2}\sum_{i,j}\int_{\mathbb{T}^{dN}} \rho_t^N \left(K(x_i - x_j) - K * \rho(x_i)\right) \cdot \nabla_{x_i} \log \bar{\rho}_t^N d\mathbf{X}^N.$$

Idea : Use the regularity of $\bar{\rho}$ to deal with the singularity of K

Step two : Integration by part

We are left with

$$\frac{d}{dt}\mathcal{H}_N(t) = -\mathcal{I}_N(t) - \frac{1}{N^2}\sum_{i,j}\int_{\mathbb{T}^{dN}} \rho_t^N \left(K(x_i - x_j) - K * \rho(x_i)\right) \cdot \nabla_{x_i} \log \bar{\rho}_t^N d\mathbf{X}^N.$$

Idea : Use the regularity of $\bar{\rho}$ to deal with the singularity of K

Remark : Notice that, for the Biot-Savart kernel on the whole space $\mathbb{R}^2$

$$\tilde{K}(x) = \frac{1}{2\pi}\frac{x^\perp}{|x|^2},$$

we have $\tilde{K} = \nabla \cdot \tilde{V}$ with

$$\tilde{V}(x) = \frac{1}{2\pi}\begin{pmatrix} -\arctan\left(\frac{x_1}{x_2}\right) & 0 \\ 0 & \arctan\left(\frac{x_2}{x_1}\right) \end{pmatrix}.$$

Assumptions ?

$$\textbf{Goal :}\ K(x) = \frac{1}{2\pi}\frac{x^\perp}{|x|^2} = \frac{1}{2\pi}\left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}\right)$$

Justifying the calculations

- There is $\lambda > 1$ such that $\bar\rho_0 \in \mathcal{C}^\infty_\lambda(\mathbb{T}^d)$ $\implies \bar\rho \in \mathcal{C}^\infty_\lambda(\mathbb{R}^+ \times \mathbb{T}^d)$ (Ben-Artzi '94)
- $\rho^N \in \mathcal{C}^\infty_\lambda(\mathbb{R}^+ \times \mathbb{T}^{Nd})$ (???)

Dealing with the terms

- In the sense of distributions, $\nabla \cdot K = 0$.
- There is a matrix field $V \in L^\infty$ such that $K = \nabla \cdot V$, i.e for $1 \leq \alpha \leq d$, $K_\alpha = \sum_{\beta=1}^d \partial_\beta V_{\alpha,\beta}$ (Phuc-Torres '08).

Step two : Integration by part

For all $t \geqslant 0$,

$$\frac{d}{dt}\mathcal{H}_N(t) \leq A_N(t) + \frac{1}{2}B_N(t) - \frac{1}{2}\mathcal{I}_N(t),$$

with

$$A_N(t) := \frac{1}{N^2}\sum_{i,j}\int_{\mathbb{T}^{dN}} \rho_t^N \left(V(x_i - x_j) - V * \bar{\rho}(x_i)\right) : \frac{\nabla^2_{x_i}\bar{\rho}_t^N}{\bar{\rho}_t^N} d\mathbf{X}^N$$

$$B_N(t) := \frac{1}{N}\sum_{i}\int_{\mathbb{T}^{dN}} \rho_t^N \frac{\left|\nabla_{x_i}\bar{\rho}_t^N\right|^2}{|\bar{\rho}_t^N|^2}\left|\frac{1}{N}\sum_j V(x_i - x_j) - V * \bar{\rho}(x_i)\right|^2 d\mathbf{X}^N.$$

Step three : Change of reference measure and large deviation estimates

Lemma

For two probability densities μ and ν on a set Ω, and any $\Phi \in L^\infty(\Omega)$, $\eta > 0$ and $N \in \mathbb{N}$,

$$\mathbb{E}^\mu \Phi \leq \eta \mathcal{H}_N(\mu, \nu) + \frac{\eta}{N} \log \mathbb{E}^\nu e^{N\Phi/\eta}.$$

Large deviation estimates -1

Theorem (Jabin-Wang '18)

Consider any probability measure μ on $\mathbb{T}^d$, $\epsilon > 0$ and a scalar function $\psi \in L^\infty(\mathbb{T}^d \times \mathbb{T}^d)$ with $\|\psi\|_{L^\infty} < \frac{1}{2\epsilon}$ and such that for all $z \in \mathbb{T}^d$, $\int_{\mathbb{T}^d} \psi(z,x)\mu(dx) = 0$. Then there exists a constant C such that

$$\int_{\mathbb{T}^{dN}} \exp\left(\frac{1}{N}\sum_{j_1,j_2=1}^{N} \psi(x_1, x_{j_1})\psi(x_1, x_{j_2})\right)\mu^{\otimes N} d\boldsymbol{X}^N \leq C,$$

where C depends on

$$\alpha = (\epsilon\|\psi\|_{L^\infty})^4 < 1\ ,\ \ \beta = \left(\sqrt{2\epsilon}\|\psi\|_{L^\infty}\right)^4 < 1.$$

Large deviation estimates -2

Theorem (Jabin-Wang '18)

Consider any probability measure μ on $\mathbb{T}^d$ and $\phi \in L^\infty(\mathbb{T}^d \times \mathbb{T}^d)$ with

$$\gamma := \left(1600^2 + 36e^4\right)\left(\sup_{p\geq 1} \frac{\| \sup_z |\phi(\cdot, z)| \|_{L^p(\mu)})}{p}\right)^2 < 1.$$

Assume that ϕ satisfies the following cancellations

$$\forall z \in \mathbb{T}^d, \quad \int_{\mathbb{T}^d} \phi(x,z)\mu(dx) = 0 = \int_{\mathbb{T}^d} \phi(z,x)\mu(dx)\,.$$

Then, for all $N \in \mathbb{N}$,

$$\int_{\mathbb{T}^{dN}} \exp\left(\frac{1}{N}\sum_{i,j=1}^{N} \phi(x_i, x_j)\right)\mu^{\otimes N} d\boldsymbol{X}^N \leq \frac{2}{1-\gamma} < \infty.$$

Partial conclusion

For all $t \geqslant 0$,

$$\frac{d}{dt}\mathcal{H}_N(t) \leq C\left(\mathcal{H}_N(t) + \frac{1}{N}\right) - \frac{1}{2}\mathcal{I}_N(t),$$

with

$$C = \hat{C}_1 \|\nabla^2 \bar{\rho}_t\|_{L^\infty} \|V\|_{L^\infty} \lambda + \hat{C}_2 \|V\|_{L^\infty}^2 \lambda^2 d^2 \|\nabla \bar{\rho}_t\|_{L^\infty}^2$$

where $\hat{C}_1, \hat{C}_2$ are universal constants.

Step four : Uniform bounds and logarithmic Sobolev inequality

Two goals :

- A logarithmic Sobolev inequality for $\bar{\rho}^N$: $\mathcal{H}_N(t) \leq C\mathcal{I}_N(t)$

Step four : Uniform bounds and logarithmic Sobolev inequality

Two goals :

- A logarithmic Sobolev inequality for $\bar{\rho}^N$: $\mathcal{H}_N(t) \leq C\mathcal{I}_N(t)$
- Uniform in time bounds on $\|\nabla \bar{\rho}_t\|_{L^\infty}$ and $\|\nabla^2 \bar{\rho}_t\|_{L^\infty}$

A logarithmic Sobolev inequality

Lemma (Tensorization)

If ν is a probability measure on $\mathbb{T}^d$ satisfying a LSI with constant C_ν^{LS}, then for all $N \geq 0$, $\nu^{\otimes N}$ satisfies a LSI with constant C_ν^{LS}

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If ν is a probability measure on $\mathbb{T}^d$ satisfying a LSI with constant C_ν^{LS}, and μ is a probability measure with density h with respect to ν such that, for some constant $\lambda > 0$, $\frac{1}{\lambda} \leq h \leq \lambda$, then μ satisfies a LSI with constant $C_\mu^{LS} = \lambda^2 C_\nu^{LS}$.

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Lemma (LSI for the uniform distribution)

The uniform distribution u on $\mathbb{T}^d$ satisfies a LSI with constant $\frac{1}{8\pi^2}$.

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Lemma (LSI for the uniform distribution)

The uniform distribution u on $\mathbb{T}^d$ satisfies a LSI with constant $\frac{1}{8\pi^2}$.

For all $N \in \mathbb{N}$, $t \geq 0$ and all probability density $\mu_N \in \mathcal{C}^\infty_{>0}(\mathbb{T}^{dN})$,

$$\mathcal{H}_N\left(\mu_N, \bar{\rho}_t^N\right) \leq \frac{\lambda^2}{8\pi^2}\frac{1}{N}\sum_{i=1}^N \int_{\mathbb{T}^d} \mu_N \left|\nabla_{x_i} \log \frac{\mu_N}{\bar{\rho}_t^N}\right|^2 d\mathbf{X}^N$$

Uniform in time bounds on the derivatives

Lemma

For all $n \geqslant 1$ and $\alpha_1, ..., \alpha_n \in [\![1, d]\!]$, there exist $C_n^u, C_n^\infty > 0$ such that for all $t \geqslant 0$,

$$\|\partial_{\alpha_1,\ldots,\alpha_n}\bar{\rho}_t\|_{L^\infty} \leq C_n^u \quad \text{and} \quad \int_0^t \|\partial_{\alpha_1,\ldots,\alpha_n}\bar{\rho}_s\|_{L^\infty}^2 ds \leq C_n^\infty$$

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Thanks to Morrey's inequality and Sobolev embeddings, it is sufficient to prove such bounds in the Sobolev space H^m for all m, i.e in L^2

Uniform in time bounds on the derivatives-2

By induction on the order of the derivative

$$\frac{1}{2}\frac{d}{dt}\|\bar{\rho}_t\|_{L^2}^2 + \|\nabla\bar{\rho}_t\|_{L^2}^2 = 0,$$

Uniform in time bounds on the derivatives-2

By induction on the order of the derivative

$$\frac{1}{2}\frac{d}{dt}\|\bar{\rho}_t\|^2_{L^2}+\|\nabla\bar{\rho}_t\|^2_{L^2}=0,$$

$$\frac{1}{2}\frac{d}{dt}\|\partial_{\alpha_1}\bar{\rho}_t\|^2_{L^2}+\frac{1}{2}\sum_{\alpha_2}\|\partial_{\alpha_1,\alpha_2}\bar{\rho}_t\|^2_{L^2}\leq\frac{1}{2}\|K\|^2_{L^1}\|\bar{\rho}_t\|^2_{L^\infty}\|\nabla\bar{\rho}_t\|^2_{L^2},$$

Uniform in time bounds on the derivatives-2

By induction on the order of the derivative

$$\frac{1}{2}\frac{d}{dt}\|\bar\rho_t\|^2_{L^2} + \|\nabla\bar\rho_t\|^2_{L^2} = 0,$$

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$$\frac{1}{2}\frac{d}{dt}\|\partial_{\alpha_1,\alpha_2}\bar\rho_t\|^2_{L^2} + \frac{1}{2}\sum_{\alpha_3}\|\partial_{\alpha_1,\alpha_2,\alpha_3}\bar\rho_t\|^2_{L^2} \leq \|V\|^2_{L^\infty}\|\partial_{\alpha_1}\nabla\bar\rho_t\|^2_{L^2}\|\nabla\bar\rho_t\|^2_{L^2}$$
$$+\|K\|^2_{L^1}\|\bar\rho_t\|^2_{L^\infty}\|\partial_{\alpha_1}\nabla\bar\rho_t\|^2_{L^2},$$

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$$+\|K\|^2_{L^1}\|\bar\rho_t\|^2_{L^\infty}\|\partial_{\alpha_1}\nabla\bar\rho_t\|^2_{L^2},$$

etc

Assumptions ?

$$\textbf{Goal :}\ K(x) = \frac{1}{2\pi}\frac{x^\perp}{|x|^2} = \frac{1}{2\pi}\left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}\right)$$

Justifying the calculations

- There is $\lambda > 1$ such that $\bar\rho_0 \in \mathcal{C}^\infty_\lambda(\mathbb{T}^d)$
  $\implies \bar\rho \in \mathcal{C}^\infty_\lambda(\mathbb{R}^+ \times \mathbb{T}^d)$ (Ben-Artzi '94)
- $\rho^N \in \mathcal{C}^\infty_\lambda(\mathbb{R}^+ \times \mathbb{T}^{Nd})$ (???)

Dealing with the terms

- In the sense of distributions, $\nabla \cdot K = 0$.
- There is a matrix field $V \in L^\infty$ such that $K = \nabla \cdot V$, i.e for $1 \leq \alpha \leq d$, $K_\alpha = \sum_{\beta=1}^d \partial_\beta V_{\alpha,\beta}$ (Phuc-Torres '08).

Uniformity in time

- For all $n \geq 1$, $C_n^0 := \|\nabla^n \bar\rho_0\|_{L^\infty} < \infty$
- $\|K\|_{L^1} < \infty$ (also used to show regularity).

Step five : Conclusion

There are constants $C_1, C_2^\infty, C_3 > 0$ and a function $t \mapsto C_2(t) > 0$ with $\int_0^t C_2(s)ds \leq C_2^\infty$ for all $t \geq 0$ such that for all $t \geq 0$

$$\frac{d}{dt}\mathcal{H}_N(t) \leq -(C_1 - C_2(t))\mathcal{H}_N(t) + \frac{C_3}{N}.$$

Multiplying by $\exp(C_1 t - \int_0^t C_2(s)ds)$ and integrating in time we get

$$\begin{aligned}\mathcal{H}_N(t) &\leq e^{-C_1 t + \int_0^t C_2(s)ds}\mathcal{H}_N(0) + \frac{C_3}{N}\int_0^t e^{C_1(s-t) + \int_s^t C_2(u)du}ds \\ &\leq e^{C_2^\infty - C_1 t}\mathcal{H}_N(t) + \frac{C_3}{C_1 N}e^{C_2^\infty},\end{aligned}$$

which concludes.

On $\rho^N \in \mathcal{C}^\infty_\lambda(\mathbb{R}^+ \times \mathbb{T}^{Nd})$

Everything works for regularized kernels K^ϵ, and the final result is independent of ϵ.

Assumptions

On the initial condition

- There is $\lambda > 1$ such that $\bar{\rho}_0 \in \mathcal{C}^\infty_\lambda(\mathbb{T}^d)$
- For all $n \geq 1$, $C^0_n := \|\nabla^n \bar{\rho}_0\|_{L^\infty} < \infty$

On the potential K

- $\|K\|_{L^1} < \infty$.
- In the sense of distributions, $\nabla \cdot K = 0$,
- There is a matrix field $V \in L^\infty$ such that $K = \nabla \cdot V$, i.e for $1 \leq \alpha \leq d$, $K_\alpha = \sum_{\beta=1}^d \partial_\beta V_{\alpha,\beta}$.

Log and Riesz gases in dimension 1

1D N-particle system in mean field interaction

$$dX_t^i = \sqrt{2\sigma_N} dB_t^i - U'(X_t^i)dt - \frac{1}{N}\sum_{j\neq i} V'(X_t^i - X_t^j)dt,$$

where

- σ_N diffusion coefficient,
- $(B^i)_i$ independent Brownian motions,
- U confining potential such that either U' is Lipschitz continuous or $U'(x) = \lambda x$,
- $\exists \alpha \geq 0,\ \forall x \in \mathbb{R}^*,\ V'(x) = -\frac{x}{|x|^{\alpha+1}}$.

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Motivation : The (generalized) Dyson Brownian motion

$$dX_t^i = \sqrt{\frac{2\sigma}{N}} dB_t^i - \lambda X_t^i dt + \frac{1}{N}\sum_{j\neq i} \frac{1}{X_t^i - X_t^j} dt.$$

Some references

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• Songzi Li, Xiang-Dong Li, and Yong-Xiao Xie. *On the law of large numbers for the empirical measure process of generalized Dyson brownian motion*. J. Stat. Phys. (2020)

• Arnaud Guillin, Pierre Le Bris, and Pierre Monmarché. *On systems of particles in singular repulsive interaction in dimension one : log and Riesz gas.* arXiv preprint arXiv :2204.10653 (2022).

Existence, uniqueness, no collisions

Theorem

Consider $N \geq 2$, and $-\infty < x_1 < ... < x_N < \infty$.

- *If $\alpha > 1$, for any $\sigma_N \geq 0$, there exists a unique strong solution $X = (X^1, ..., X^N)$ to the particle system with initial condition $X_0^1 = x_1$, ..., $X_0^N = x_N$, which furthermore satisfies $X_t^1 < ... < X_t^N$ for all $t \geq 0$, $\mathbb{P}$-a.s.*
- *The same result holds for $\alpha = 1$ and $\sigma_N \leq \frac{1}{N}$.*

"Cauchy sequence"

Lemma

Let $(\mu^N)_{N\in\mathbb{N}}$ be any sequence of independent empirical measures, such that μ_t^N is the empirical measure of the N particle system at time t. Then (for $\lambda > 0$, $\alpha = 1$, $U'(x) = \lambda x$ and $\sigma_N = \frac{1}{N}$), we have for all $t \geq 0$ and all $N, M \geq 1$

$$\mathbb{E}\left(\mathcal{W}_2\left(\mu_t^N, \mu_t^M\right)^2\right) \leq e^{-2\lambda t}\mathbb{E}\left(\mathcal{W}_2\left(\mu_0^N, \mu_0^M\right)^2\right) + \frac{C}{N \wedge M}.$$

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Remark : The same result holds...

"Cauchy sequence"

Lemma

Let $(\mu^N)_{N\in\mathbb{N}}$ be any sequence of independent empirical measures, such that μ^N_t is the empirical measure of the N particle system at time t. Then (for $\lambda > 0$, $\alpha = 1$, $U'(x) = \lambda x$ and $\sigma_N = \frac{1}{N}$), we have for all $t \geq 0$ and all $N, M \geq 1$

$$\mathbb{E}\left(\mathcal{W}_2\left(\mu_t^N, \mu_t^M\right)^2\right) \leq e^{-2\lambda t}\mathbb{E}\left(\mathcal{W}_2\left(\mu_0^N, \mu_0^M\right)^2\right) + \frac{C}{N \wedge M}.$$

Remark : The same result holds...

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Remark : The same result holds...

- for $U = 0$, but no longer uniform in time,
- for $\alpha \in [1, 2[$, with rate $N^{-\frac{2-\alpha}{\alpha}}$,

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Lemma

Let $(\mu^N)_{N\in\mathbb{N}}$ be any sequence of independent empirical measures, such that μ_t^N is the empirical measure of the N particle system at time t. Then (for $\lambda > 0$, $\alpha = 1$, $U'(x) = \lambda x$ and $\sigma_N = \frac{1}{N}$), we have for all $t \geq 0$ and all $N, M \geq 1$

$$\mathbb{E}\left(\mathcal{W}_2\left(\mu_t^N, \mu_t^M\right)^2\right) \leq e^{-2\lambda t}\mathbb{E}\left(\mathcal{W}_2\left(\mu_0^N, \mu_0^M\right)^2\right) + \frac{C}{N \wedge M}.$$

Remark : The same result holds...

- for $U = 0$, but no longer uniform in time,
- for $\alpha \in [1, 2[$, with rate $N^{-\frac{2-\alpha}{\alpha}}$,
- for U' only Lipschitz continuous, but no longer uniform in time,

"Cauchy sequence"

Lemma

Let $(\mu^N)_{N\in\mathbb{N}}$ be any sequence of independent empirical measures, such that μ_t^N is the empirical measure of the N particle system at time t. Then (for $\lambda > 0$, $\alpha = 1$, $U'(x) = \lambda x$ and $\sigma_N = \frac{1}{N}$), we have for all $t \geq 0$ and all $N, M \geq 1$

$$\mathbb{E}\left(\mathcal{W}_2\left(\mu_t^N, \mu_t^M\right)^2\right) \leq e^{-2\lambda t}\mathbb{E}\left(\mathcal{W}_2\left(\mu_0^N, \mu_0^M\right)^2\right) + \frac{C}{N \wedge M}.$$

Remark : The same result holds...

- for $U = 0$, but no longer uniform in time,
- for $\alpha \in [1, 2[$, with rate $N^{-\frac{2-\alpha}{\alpha}}$,
- for U' only Lipschitz continuous, but no longer uniform in time,
- for the supremum, but no longer uniform in time.

Conclusion

Using independence, this implies that there exists a (deterministic) $\bar{\rho}_t$ such that

$$\mathbb{E}\left(\mathcal{W}_2(\mu_t^N,\bar{\rho}_t)^2\right)\to 0,$$

which satisfies, for all functions f "sufficiently nice" and $\forall t\geqslant 0$,

$$\int_{\mathbb{R}} f(x)\bar{\rho}_t(dx)=\int_{\mathbb{R}} f(x)\bar{\rho}_0(dx)-\int_0^t\int_{\mathbb{R}} f'(x)U'(x)\bar{\rho}_s(dx)ds + \frac{1}{2}\int_0^t\int\int_{\{x\neq y\}}\frac{(f'(x)-f'(y))(x-y)}{|x-y|^{\alpha+1}}\bar{\rho}_s(dx)\bar{\rho}_s(dy)ds.$$

Proof of the estimate

For two sets of points $(x_i)_{i\in\{1,\dots,N\}}$ and $(y_j)_{j\in\{1,\dots,N\}}$, with $x_1 \leq \dots \leq x_N$ and $y_1 \leq \dots \leq y_N$, and two measures $\mu = \frac{1}{N}\sum_i \delta_{x_i}$ and $\nu = \frac{1}{N}\sum_j \delta_{y_j}$:

$$\mathcal{W}_2(\mu,\nu)^2 = \frac{1}{N}\sum_i |x_i - y_i|^2.$$

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For two sets of points $(x_i)_{i\in\{1,\dots,N\}}$ and $(y_j)_{j\in\{1,\dots,N\}}$, with $x_1 \leq \dots \leq x_N$ and $y_1 \leq \dots \leq y_N$, and two measures $\mu = \frac{1}{N}\sum_i \delta_{x_i}$ and $\nu = \frac{1}{N}\sum_j \delta_{y_j}$:

$$\mathcal{W}_2(\mu,\nu)^2 = \frac{1}{N}\sum_i |x_i - y_i|^2.$$

Let

$$-\infty < X_t^1 = \dots = X_t^N < \dots < X_t^{N(M-1)+1} = \dots = X_t^{NM} < \infty$$
$$-\infty < Y_t^1 = \dots = Y_t^M < \dots < Y_t^{M(N-1)+1} = \dots = Y_t^{NM} < \infty.$$

Thus

$$\mu_t^M = \frac{1}{M}\sum_{i=1}^{M} \delta_{\tilde{X}_t^{i,M}} = \frac{1}{NM}\sum_{i=1}^{NM} \delta_{X_t^i} \quad \text{and} \quad \mu_t^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{\tilde{Y}_t^{i,N}} = \frac{1}{NM}\sum_{i=1}^{NM} \delta_{Y_t^i},$$

and

$$\mathcal{W}_2\left(\mu_t^N, \mu_t^M\right)^2 = \frac{1}{NM}\sum_{i=1}^{NM} \left|X_t^i - Y_t^i\right|^2.$$

Proof of the estimate-2

Direct calculations yield :

$$d\left(\mathcal{W}_2\left(\mu_t^N,\mu_t^M\right)^2\right)$$
$$= -2\lambda\mathcal{W}_2\left(\mu_t^N,\mu_t^M\right)^2 dt + 2\sigma\left(\frac{1}{N}+\frac{1}{M}\right)dt + dM_t$$
$$- \frac{2}{(NM)^2}\sum_i\left(X_t^i - Y_t^i\right)\sum_j\left(V'(X_t^i - X_t^j) - V'(Y_t^i - Y_t^j)\right)dt.$$

Proof of the estimate-3

$$
\begin{aligned}
\sum_i \left(X_t^i - Y_t^i\right) \sum_j \left(V'(X_t^i - X_t^j) - V'(Y_t^i - Y_t^j)\right) \\
&= \sum_{i>j} \left(V'(X_t^i - X_t^j) - V'(Y_t^i - Y_t^j)\right) \left(\left(X_t^i - Y_t^i\right) - \left(X_t^j - Y_t^j\right)\right) \\
&= \sum_{i>j} \left(V'(X_t^i - X_t^j) - V'(Y_t^i - Y_t^j)\right) \left(\left(X_t^i - X_t^j\right) - \left(Y_t^i - Y_t^j\right)\right) .
\end{aligned}
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\sum_i \left(X_t^i - Y_t^i\right) \sum_j \left(V'(X_t^i - X_t^j) - V'(Y_t^i - Y_t^j)\right) \\
&= \sum_{i>j} \left(V'(X_t^i - X_t^j) - V'(Y_t^i - Y_t^j)\right) \left(\left(X_t^i - Y_t^i\right) - \left(X_t^j - Y_t^j\right)\right) \\
&= \sum_{i>j} \left(V'(X_t^i - X_t^j) - V'(Y_t^i - Y_t^j)\right) \left(\left(X_t^i - X_t^j\right) - \left(Y_t^i - Y_t^j\right)\right). \\
&\geq \sum_{i>j \text{ s.t } Y_t^i = Y_t^j} V'(X_t^i - X_t^j)\left(X_t^i - X_t^j\right) + \sum_{i>j \text{ s.t } X_t^i = X_t^j} V'(Y_t^i - Y_t^j)\left(Y_t^i - Y_t^j\right) \\
&\geq \sum_{i>j \text{ s.t } Y_t^i = Y_t^j} -1 + \sum_{i>j \text{ s.t } X_t^i = X_t^j} -1 \\
&= -\frac{M(M-1)}{2} N - \frac{N(N-1)}{2} M.
\end{aligned}
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\sum_i \left(X_t^i - Y_t^i\right) \sum_j \left(V'(X_t^i - X_t^j) - V'(Y_t^i - Y_t^j)\right) \\
&= \sum_{i>j} \left(V'(X_t^i - X_t^j) - V'(Y_t^i - Y_t^j)\right) \left(\left(X_t^i - Y_t^i\right) - \left(X_t^j - Y_t^j\right)\right) \\
&= \sum_{i>j} \left(V'(X_t^i - X_t^j) - V'(Y_t^i - Y_t^j)\right) \left(\left(X_t^i - X_t^j\right) - \left(Y_t^i - Y_t^j\right)\right). \\
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&= -\frac{M(M-1)}{2} N - \frac{N(N-1)}{2} M.
\end{aligned}
$$

Hence

$$
\mathbb{E}\left(\mathcal{W}_2\left(\mu_t^N, \mu_t^M\right)^2\right) \leq e^{-2\lambda t} \mathbb{E}\left(\mathcal{W}_2\left(\mu_0^N, \mu_0^M\right)^2\right) + \frac{C}{N \wedge M}.
$$

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- **Ongoing work :** Minibatching, Propagation of chaos and Metastability.

Thank you