Short- and long-time behavior in (hypo)coercive ODE-systems and kinetic equations

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Part I

Motivation

motivation

Evolution equations

$$\frac{\mathrm{d}}{\mathrm{d}t} f = -\mathbf{L} f\,, \qquad t \geq 0\,,$$

operator $\mathbf{L}$ is independent of time t

and has a unique steady state f_∞: $\mathbf{L} f_\infty = 0$

- ▷ Goal: find an estimate on $\|f(t) - f_\infty\|$
- ▷ Strategy: construct a (strict) Lyapunov functional $\mathcal{E}(f, f_\infty) \sim \|f - f_\infty\|^2$

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  - possibly with exponential decay: $\|f(t) - f_\infty\| \leq c\mathrm{e}^{-\mu t} \|f(0) - f_\infty\|$
  - possibly with sharp (=maximal) rate $\mu > 0$ and minimal $c \geq 1$ [uniform for all $f(0)$]
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- Strategy: construct a (strict) Lyapunov functional $\mathcal{E}(f, f_\infty) \sim \|f - f_\infty\|^2$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(f(t), f_\infty) \leq -2\mu\, \mathcal{E}(f(t), f_\infty) \,, \qquad t \geq 0$$

$$\Downarrow \qquad \text{Gronwall lemma}$$

$$\mathcal{E}(f(t), f_\infty) \leq \mathrm{e}^{-2\mu t}\mathcal{E}(f(0), f_\infty) \,, \qquad t \geq 0$$

$$\Downarrow \qquad \mathcal{E}(f, f_\infty) \sim \|f - f_\infty\|^2$$

$$\|f(t) - f_\infty\|^2 \leq c\mathrm{e}^{-2\mu t}\|f(0) - f_\infty\|^2 \,, \qquad t \geq 0$$

Part II

BGK-type kinetic equations

nonlinear BGK-type model with constant collision frequency σ[1]

$$\partial_t f + v \cdot \nabla_x f = \sigma\big(M_f(x,v,t) - f(x,v,t)\big)\,, \quad t \geq 0\,,\ x \in \mathbb{T}^d\,,\ v \in \mathbb{R}^d\,.$$

- relaxation towards local Maxwellian $M_f(x,v) = \frac{\rho(x)}{(2\pi T(x))^{d/2}}\, \mathrm{e}^{-\frac{|v-u(x)|^2}{2T(x)}}$
  with density $\rho(x) := \int f \, \mathrm{d}v$, mean velocity $u(x) := \frac{1}{\rho(x)} \int v \ f \, \mathrm{d}v$, temperature $T(x) := \frac{1}{d\rho(x)} \int |v - u(x)|^2 f \, \mathrm{d}v$.
- Consider normalized initial data $f^I(x,v)$ with unit mass $\iint f^I \, \mathrm{d}x \, \mathrm{d}v = 1$, zero mean momentum $\iint v \ f^I \, \mathrm{d}x \, \mathrm{d}v = 0$ and unit mean pressure $\iint |v|^2 f^I \, \mathrm{d}x \, \mathrm{d}v = d$.
- normalization is conserved under the flow of (BGK)
- $f^\infty(v) := M_1(v) = (2\pi)^{-d/2}\, \mathrm{e}^{-\frac{|v|^2}{2}}$ is the unique, normalized, space-homogeneous steady state of (BGK) via standard argument using Boltzmann entropy, but no information about rate of convergence.

Our result: For normalized initial data f^I "sufficiently close" to f^∞, the solutions of (BGK) converge to f^∞ exp. fast: Construction of a strict Lyapunov functional and derivation of explicit exponential decay rate.

[1] BGK: Bhatnagar-Gross-Krook (1954), Welander (1954), Kogan (1958)

1D nonlinear BGK-type model - local exponential stability

$$\partial_t f + v\partial_x f = M_f(x,v,t) - f\,, \quad x \in \mathbb{T}\,,\ v \in \mathbb{R}\,,$$

with local Maxwellian $M_f = \frac{\rho(x)}{\sqrt{2\pi T(x)}} \mathrm{e}^{-v^2/2T(x)}$ here mean velocity=0.

Theorem 1 (FA-Arnold-Carlen 2016)

Let $\frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{R}}(1,v^2)\, f^I \,\mathrm{d}v\,\mathrm{d}x = (1,1)$ *and* $M_1 = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-v^2/2}$.
If $\gamma > \frac{1}{2}$, $\|f^I - M_1\|_{\mathcal{H}^\gamma} < \delta_\gamma$ *(=explicit constant) then*

$$\mathcal{E}_\gamma(f(t), M_1) \le \mathrm{e}^{-t/25}\mathcal{E}_\gamma(f^I, M_1)\,, \quad t \ge 0\,.$$

Strict Lyapunov functional (with $h = f - M_1$ and $\mathcal{H}^\gamma := H^\gamma(0,2\pi)\otimes L^2(\mathbb{R}; M_1^{-1})$):

$$\mathcal{E}_\gamma(f, M_1) := \sum_{k\in\mathbb{Z}}(1+k^2)^\gamma \langle h_k(v), \widetilde{\mathbf{P}}_k h_k(v)\rangle_{L^2(M_1^{-1})} \sim \|f - M_1\|^2_{\mathcal{H}^\gamma}$$

$\widetilde{\mathbf{P}}_k$ is a bounded operator on $L^2(M_1^{-1})$, represented by "infinite" matrix $\mathbf{P}_k$

Idea of proof:

- For $h := f - M_1$, rewrite nonlinear model as

$$\partial_t h + v\partial_x h = Q_2 h + R_f$$

- Analyze the **linearized model** $\partial_t h + v\partial_x h = Q_2 h$ with

$$Q_2 h := \left(\tfrac{3-v^2}{2}\right) M_1(v) \int h \,\mathrm{d}v + \left(\tfrac{v^2-1}{2}\right) M_1(v) \int v^2 h \,\mathrm{d}v - h\,.$$

Theorem 2 (FA-Arnold-Carlen 2016)

Let $\frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{R}} \binom{1}{v^2} f^I \,\mathrm{d}v\,\mathrm{d}x = \binom{1}{1}$. *Then, for* $\gamma \geq 0$,

$$\mathcal{E}_\gamma(f(t), M_1) \leq \mathrm{e}^{-t/25}\mathcal{E}_\gamma(f^I, M_1)\,, \qquad t \geq 0\,.$$

- remainder: $\|R_f\|_{\mathcal{H}^\gamma} \leq c\|f - M_1\|^2_{\mathcal{H}^\gamma}$ if $\gamma > \frac{1}{2}$, $\|f^I - M_1\|_{\mathcal{H}^\gamma} < \delta_\gamma$

$$\Longrightarrow \qquad \frac{d}{dt}\mathcal{E}_\gamma(f) \leq -\underbrace{0.0412}_{>1/25}\,\underbrace{\mathcal{E}_\gamma(f)}_{=\mathcal{O}(h^2)} + c\|h\|^3_{\mathcal{H}^\gamma}$$

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- For $h := f - M_1$, rewrite nonlinear model as

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$$\mathcal{E}_\gamma(f(t), M_1) \leq \mathrm{e}^{-t/25}\mathcal{E}_\gamma(f^I, M_1)\,, \qquad t \geq 0\,.$$

Proof of Theorem 2:

- expansion of h: Fourier in x / Hermite functions in v:

$$\partial_t \hat{\mathbf{h}}_k(t) + \mathrm{i}k\,\mathbf{L}_1\hat{\mathbf{h}}_k(t) = -\mathbf{L}_3\hat{\mathbf{h}}_k(t)\,, \quad k \in \mathbb{Z}\,,$$

for some Hermitian matrices $\mathbf{L}_1$, $\mathbf{L}_3$ with $\mathbf{L}_3 \geq 0$

1D linearized BGK model - simplified Lyapunov functional

- For $k \in \mathbb{Z}$, $\partial_t \hat{\mathbf{h}}_k(t) = -\mathbf{C}_k \hat{\mathbf{h}}_k(t)$ where $\mathbf{C}_k := \mathrm{i}k\,\mathbf{L}_1 + \mathbf{L}_3$

$$\mathbf{L}_1 = \begin{bmatrix} 0 & \sqrt{1} & 0 & \cdots \\ \sqrt{1} & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} \\ \vdots & 0 & \sqrt{3} & \ddots \end{bmatrix},$$

$\mathbf{L}_3 = \mathrm{diag}(0, 1, 0, 1, 1, \dots)$

2 conserved quantities:
mass & energy

- simplified ansatz

$$\mathbf{P}_k = \mathbf{I} + \left(\begin{array}{cccc|c} 0 & -\mathrm{i}\alpha/k & 0 & 0 & \\ \mathrm{i}\alpha/k & 0 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 0 & -\mathrm{i}\beta/2k & \\ 0 & 0 & \mathrm{i}\beta/2k & 0 & \\ \hline & & \mathbf{0} & & \mathbf{0} \end{array}\right)$$

- For $\mathbf{C}_k := \mathrm{i}k\mathbf{L}_1 + \mathbf{L}_3$ and $\mathbf{P}_k$ with $\alpha = \beta = \frac{1}{3}$, $\exists \mu_0 > 0$ such that

$$\mathbf{P}_k\mathbf{C}_k + \mathbf{C}_k^{\mathrm{H}}\mathbf{P}_k \geq 2\mu_0\mathbf{P}_k \quad \text{uniform-in-}k$$

some references in kinetic theory/hypocoercivity

general theory on hypocoercivity:

MOUHOT-NEUMANN (2006) weighted Sobolev spaces
general class of linear inhomogeneous kinetic equations on the torus

HÉRAU (200x) Fokker–Planck equation with confining potential, linear inhomogeneous relaxation Boltzmann equation (= BGK-type equation)

VILLANI (2009) abstract operators: $\mathbf{L} := \mathbf{A}^*\mathbf{A} + \mathbf{B}$ where $\mathbf{B} = -\mathbf{B}^*$

Quote: *"Construct a [strict] Lyapunov functional by adding carefully chosen lower-order terms to the 'natural' [non-strict] Lyapunov functional."*

linear kinetic equations: $\partial_t f + \mathbf{T} f = \widetilde{\mathbf{L}} f$:

DOLBEAULT-MOUHOT-SCHMEISER (2009, 2015) weighted L^2 spaces
linear kinetic equations with **only one** conservation law

DUAN (2011) macro-micro decomposition combined with Kawashima's argument on dissipation of the hyperbolic-parabolic system +Korn ineq.

CARRAPATOSO-DOLBEAULT-HÉRAU-MISCHLER-MOUHOT-SCHMEISER (2021)
Special modes and hypocoercivity for linear kinetic equations with **several** conservation laws and a confining potential

Part III

ODEs: Hypocoercive matrices

ODEs $\frac{\mathrm{d}}{\mathrm{d}t} u = -\mathbf{C} u$ with (hypo)coercive matrices $\mathbf{C}$

Definition 3 ((Hypo)coercive matrices)

Let $\mathbf{C} \in \mathbb{C}^{n \times n}$ (with trivial $\mathcal{K} = \ker \mathbf{C} = \{0\}$) and $\mathcal{H} = \mathbb{C}^n$ be endowed with Euclidean scalar product and norm.

- The operator $\mathbf{C}$ is called coercive on $(\mathbb{C}^n, \|\cdot\|)$ if

$$\exists\, \kappa > 0 : \qquad \forall\, u \in \mathbb{C}^n, \qquad \Re\langle u, \mathbf{C} u\rangle \geq \kappa \|u\|^2 \ .$$

- The operator $\mathbf{C}$ is called hypocoercive on $(\mathbb{C}^n, \|\cdot\|)$ if

$$\exists\, \kappa > 0,\ c \geq 1 : \qquad \forall\, u \in \mathbb{C}^n,\ t \geq 0, \qquad \|\mathrm{e}^{-\mathbf{C} t} u\| \leq c\, \mathrm{e}^{-\kappa t} \|u\| \ .$$

- $\mathbf{C}$ is *coercive* with $\kappa > 0 \iff \mathbf{C}^{\mathrm{H}} + \mathbf{C} \geq 2\kappa \mathbf{I}$

  Energy method: $\frac{\mathrm{d}}{\mathrm{d}t} \|u(t)\|^2 = -\langle u(t), (\mathbf{C}^{\mathrm{H}} + \mathbf{C}) u(t)\rangle \leq -2\kappa \|u(t)\|^2$

$$\Longrightarrow \quad \|u(t)\|^2 \leq \|u(0)\|^2\, \mathrm{e}^{-2\kappa t} \quad \text{for } t \geq 0 \ .$$

- hypocoercive operator: coercive $\iff$ $c = 1$

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$$\exists\ \kappa > 0,\ c \geq 1: \qquad \forall\ u \in \mathbb{C}^n,\ t \geq 0, \qquad \|\mathrm{e}^{-\mathbf{C}t} u\| \leq c\,\mathrm{e}^{-\kappa t}\|u\|\ .$$

- u_∞ is *asymptotically stable* $:\iff$ $\|u(t) - u_\infty\| \to 0$ as $t \to \infty$

$\iff$ All eigenvalues λ_j of $\mathbf{C}$ satisfy $\Re\lambda_j > 0$.

$\iff$ $\exists\ \mathbf{P} \in \mathbb{H}_n^>$: $\mathbf{C}^{\mathrm{H}}\mathbf{P} + \mathbf{P}\mathbf{C} > 0$ with $\mathbb{H}_n^> := \{\mathbf{P} \in \mathbb{C}^{n\times n} | \mathbf{P} = \mathbf{P}^{\mathrm{H}}, \mathbf{P} > 0\}$

$\Rightarrow \|u(t)\|_{\mathbf{P}}^2 := \langle u(t), \mathbf{P}u(t)\rangle$ is a strict Lyapunov functional

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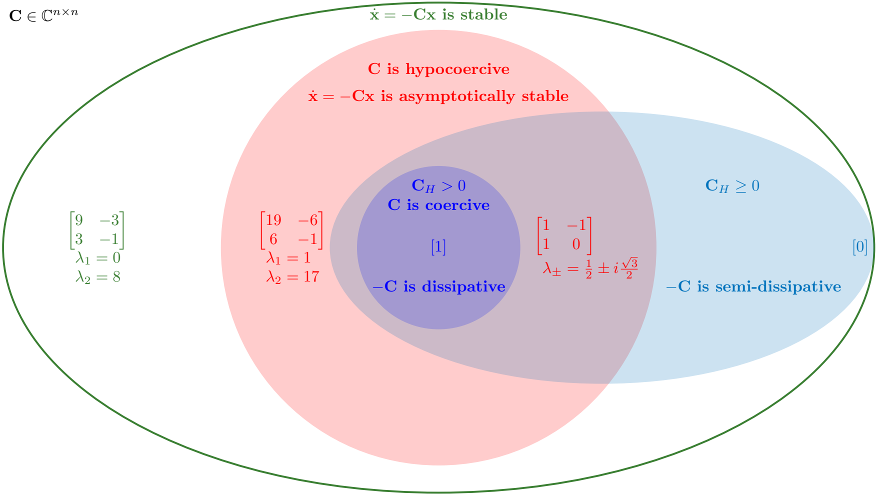
$$\exists\, \kappa > 0,\ c \geq 1: \qquad \forall\, u \in \mathbb{C}^n,\ t \geq 0, \qquad \|\mathrm{e}^{-\mathbf{C}t} u\| \leq c\, \mathrm{e}^{-\kappa t} \|u\| \,.$$

- If $\mathbf{C}^{\mathsf{H}}\mathbf{P} + \mathbf{P}\mathbf{C} \geq 2\kappa \mathbf{P}$ for some $\kappa > 0$ and $\mathbf{P} \in \mathbb{H}_n^{>}$ then

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{\mathbf{P}}^2 = -\langle u(t), (\mathbf{C}^{\mathsf{H}}\mathbf{P} + \mathbf{P}\mathbf{C}) u(t)\rangle &\leq -2\kappa \|u(t)\|_{\mathbf{P}}^2 \\
\Longrightarrow \quad \|u(t)\|_{\mathbf{P}}^2 \leq \|u(0)\|_{\mathbf{P}}^2\, \mathrm{e}^{-2\kappa t} &\quad \text{for } t \geq 0 \\
\Longrightarrow \quad \|u(t)\|^2 \leq \operatorname{cond}(\mathbf{P})\, \|u(0)\|^2\, \mathrm{e}^{-2\kappa t} &\,.
\end{aligned}$$

Venn diagram of matrices $\mathbf{C} \in \mathbb{C}^{n\times n}$

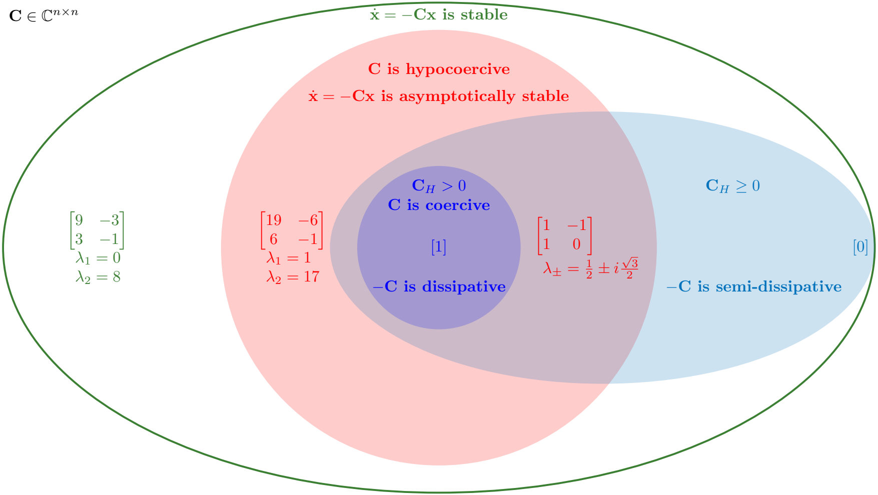


<u>characterization</u>

coercive $\mathbf{C}$: $\|\cdot\|_2^2$ is a strict Lyapunov functional

hypocoercive $\mathbf{C}$: $\exists \mathbf{P} \in \mathbb{H}_n^>$, $\|\cdot\|_{\mathbf{P}}^2$ is a strict Lyapunov functional

Venn diagram of matrices $\mathbf{C} \in \mathbb{C}^{n \times n}$



▷If $\mathbf{C} = \mathbf{A}^*\mathbf{A} + \mathbf{B}$ with $\mathbf{B} = -\mathbf{B}^*$ then $\mathbf{C} = \mathbf{C}_H + \mathbf{C}_S$ yields

$$\mathbf{C}_H = \tfrac{1}{2}(\mathbf{C} + \mathbf{C}^{\mathsf{H}}) = \mathbf{A}^*\mathbf{A} \geq 0 \quad \text{and} \quad \mathbf{C}_S = \tfrac{1}{2}(\mathbf{C} - \mathbf{C}^{\mathsf{H}}) = \mathbf{B}$$

ODEs $\frac{\mathrm{d}}{\mathrm{d}t}u = -\mathbf{C}u$ with $\mathbf{C}_H \geq 0$

solutions $u(t)$ of ODE satisfy $\|u(t)\|^2 \leq \|u(0)\|^2$ for $t \geq 0$

Lemma Let $\mathbf{C} \in \mathbb{C}^{n\times n}$ satisfy $\mathbf{C}_H \geq 0$. Then, $\mathbf{C}$ has a purely imaginary eigenvalue if and only if $\mathbf{C}_H w = 0$ for some eigenvector w of $\mathbf{C}_S$.

Lemma Let $\mathbf{C} \in \mathbb{C}^{n\times n}$ satisfy $\mathbf{C}_H \geq 0$. Then the following conditions are equivalent:

- $\mathbf{C}$ is hypocoercive
- Shizuta-Kawashima: No eigenvector of $\mathbf{C}_S$ lies in the kernel of $\mathbf{C}_H$.
- Kalman: $\operatorname{rank}[\mathbf{C}_H, \mathbf{C}_S\mathbf{C}_H, \dots, \mathbf{C}_S^{n-1}\mathbf{C}_H] = n$
- $\sum_{j=0}^{n-1} (\mathbf{C}_S)^j \mathbf{C}_H (\mathbf{C}_S^{\mathrm{H}})^j > 0$
- Popov-Belevitch-Hautus: $\operatorname{rank}[\lambda\mathbf{I} - \mathbf{C}_S, \mathbf{C}_H] = n$ for every $\lambda \in \mathbb{C}$, in particular for every eigenvalue λ of $\mathbf{C}_S$.

Construction of strict Lyapunov functional/solution $\mathbf{P}$ of $\mathbf{C}^{\mathrm{H}}\mathbf{P} + \mathbf{P}\mathbf{C} > 0$.

Hypocoercivity index for $\mathbf{C} \in \mathbb{C}^{n\times n}$ with $\mathbf{C}_H \geq 0$

Let $\mathbf{C} \in \mathbb{C}^{n\times n}$ be a positive conservative-dissipative matrix.

Definition 4

The *hypocoercivity index of* $\mathbf{C} = \mathbf{C}_S + \mathbf{C}_H$ with $\mathbf{C}_H \geq 0$ is defined as the smallest integer $m_{HC} \in \mathbb{N}_0$ (if it exists) such that $\sum_{j=0}^{m_{HC}} \mathbf{C}_S^j \mathbf{C}_H (\mathbf{C}_S^{\mathrm{H}})^j > 0$. If $\mathbf{C}$ is not hypocoercive we set $m_{HC} = \infty$.

- $\mathbf{C}$ is coercive $\iff$ $\mathbf{C}_H > 0$ $\iff$ $m_{HC} = 0$
- $\mathbf{C}$ is hypocoercive $\iff$ $m_{HC} < \infty$
- If $\mathbf{C}$ is hypocoercive then $\frac{n - \operatorname{rank} \mathbf{C}_H}{\operatorname{rank} \mathbf{C}_H} \leq m_{HC} \leq n - \operatorname{rank} \mathbf{C}_H$
- m_{HC} describes the structural complexity of $\mathbf{C}$

Examples:

$$\mathbf{C}_S = \mathrm{i} \begin{bmatrix} 0 & \sqrt{1} & 0 & 0 \\ \sqrt{1} & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}$$

1. $\mathbf{C}_H = \operatorname{diag}(0, 1, 0, 1)$
   $\implies$ HC-index $m_{HC} = 1$
2. $\mathbf{C}_H = \operatorname{diag}(0, 0, 1, 1)$
   $\implies$ HC-index $m_{HC} = 2$

Hypocoercivity index for $\mathbf{C} \in \mathbb{C}^{n\times n}$ with $\mathbf{C}_H \geq 0$

Let $\mathbf{C} \in \mathbb{C}^{n\times n}$ be a hypocoercive, positive conservative-dissipative matrix.

Lemma 1 (Equivalent conditions)

- no (non-trivial) subspace of $\ker \mathbf{C}_H$ is invariant under $\mathbf{C}_S$.
- $\exists \tau \in \mathbb{N}_0$: $\sum_{j=0}^{\tau} \mathbf{C}_S^j \mathbf{C}_H (\mathbf{C}_S^{\mathrm{H}})^j > 0$.
- $\exists \tau \in \mathbb{N}_0$: $\operatorname{rank}\{\sqrt{\mathbf{C}_H}, \mathbf{C}_S\sqrt{\mathbf{C}_H}, \ldots, \mathbf{C}_S^\tau\sqrt{\mathbf{C}_H}\} = n$
- $\exists \tau \in \mathbb{N}_0$: $\bigcap_{j=0}^{\tau} \ker(\sqrt{\mathbf{C}_H}(\mathbf{C}_S^{\mathrm{H}})^j) = \{0\}$
- $\exists \tau \in \mathbb{N}_0$: $\sum_{j=0}^{\tau} \mathbf{C}_j^{\mathrm{H}} \mathbf{C}_j > 0$ with $\mathbf{C}_0 := \sqrt{\mathbf{C}_H}$; $\mathbf{C}_{j+1} := [\mathbf{C}_j, \mathbf{C}_S]$, $j \in \mathbb{N}_0$.

Examples:

$$\mathbf{C}_S = \mathrm{i}\begin{bmatrix} 0 & \sqrt{1} & 0 & 0 \\ \sqrt{1} & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}$$

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Short-time behavior for $\frac{\mathrm{d}}{\mathrm{d}t}u = -\mathbf{C}u$ with $\mathbf{C}_H \geq 0$

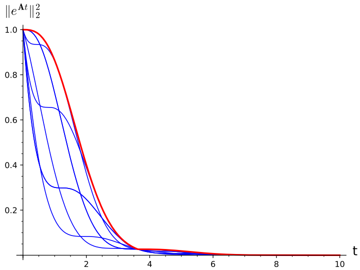
Lemma 2 (A-Arnold-Carlen (2020))

Let $\mathbf{C} \in \mathbb{C}^{n\times n}$ satisfy $\mathbf{C}_H \geq 0$. Its HC-index is $m_{HC} \in \mathbb{N}_0$ if and only if

$$\|\mathrm{e}^{-\mathbf{C}t}\|_2 = 1 - c\; t^{2m_{HC}+1} + \mathcal{O}\big(t^{2m_{HC}+2}\big)\;, \quad t \to 0^+\;,$$

for some $c > 0$.

Example (continued)



ODE $\frac{\mathrm{d}}{\mathrm{d}t}u(t) = -\mathbf{C}u$ with $\mathbf{C} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$

The squared propagator norm $\|e^{-\mathbf{C}t}\|_2^2$ satisfies $\|e^{-\mathbf{C}t}\|_2^2 \sim 1 - t^3/6 + \mathcal{O}(t^4)$ for $t \to 0+$.

Moreover, it is the envelope of $\|u(t)\|_2^2$ for all solutions with $\|u(0)\|_2^2 = 1$.

Propagator norm of (normalized) Fokker-Planck equations

$$\partial_t f = \operatorname{div}_\xi \left(\mathbf{D}\nabla_\xi f + \mathbf{C}\xi f \right) =: \mathbf{L} f \ , \qquad \mathbf{D} = \mathbf{C}_H \geq 0 \ . \qquad \text{(nFP)}$$

Condition A (for hypocoercivity)

- No (nontrivial) subspace of $\ker \mathbf{D}$ is invariant under $\mathbf{C}^\top$
  (Hörmander: $\mathbf{L}$ is hypoelliptic, i.e. (nFP) has smooth solutions.)
- Let $\mathbf{C}_H := (\mathbf{C} + \mathbf{C}^\top)/2 \in \mathbb{R}^{d\times d}$ and $\mathbf{C}_H \geq 0$.

Condition A $\Longrightarrow$ $\mathbf{C}$ is positively stable (i.e. $\Re\lambda_{\mathbf{C}} > 0$) $\Longrightarrow$ $\exists f_\infty$: $\mathbf{L} f_\infty = 0$.

Theorem 5 (Arnold-Schmeiser-Signorello (2021))

Let $\mathbf{L}$ satisfy Condition A (i.e. $\mathbf{L}$ is hypocoercive). Then

$$\|\mathrm{e}^{-\mathbf{L}t} - \Pi_0\|_{\mathcal{B}(\mathcal{H})} = \|\mathrm{e}^{-\mathbf{C}t}\|_2 \ , \qquad t \geq 0 \ ,$$

where $\mathcal{H} = L^2(f_\infty^{-1}\,\mathrm{d}\xi)$ and Π_0 is projection onto $\operatorname{span}\{f_\infty\}$.

Conclusion

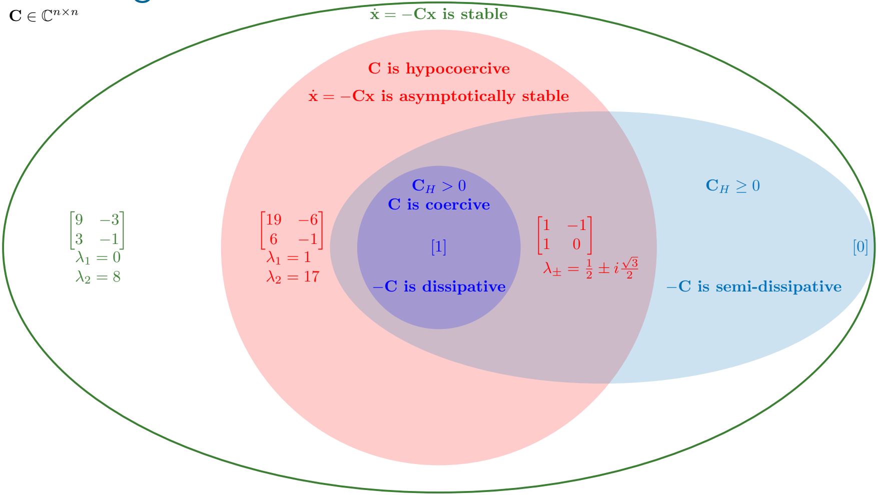
- Optimal decay estimates of (drift) ODEs carry over to **Fokker-Planck equations**
- HC-index characterizes the short-time behavior of ODEs and (normalized) Fokker-Planck equations. It also characterizes the regularization rate in Fokker-Planck equations:

$$\left\| \nabla_\xi \frac{f(t)}{f_\infty} \right\|_{L^2(f_\infty \, \mathrm{d}\xi)} \leq c t^{-(2m_{HC}+1)} \left\| \frac{f_0}{f_\infty} \right\|_{L^2(f_\infty \, \mathrm{d}\xi)} , \qquad 0 < t \leq \delta \, ,$$

  see Villani using Hörmander rank, Arnold-Erb using HC-index.

- analysis of kinetic **BGK-type models**: exponential decay for discrete / continuous velocities, linearized / nonlinear (similar to Kawashima). modal decomposition yields ODE with "infinite" matrices: extension of HC-index m_{HC} to "infinite" matrices and algorithm to construct strict Lyapunov functional in m_{HC} number of steps.

Venn diagram of matrices $\mathbf{C} \in \mathbb{C}^{n\times n}$



- Extension to $\frac{\mathrm{d}}{\mathrm{d}t}u = -\mathbf{C}u$ with $\mathbf{C} \in \mathbb{C}^{n\times n}$
- Extension to differential-algebraic equations (DAEs) $\mathbf{E}\frac{\mathrm{d}}{\mathrm{d}t}u = -\mathbf{C}u$ with $\mathbf{E} \in \mathbb{H}_n^{\geq}$ and $\mathbf{C} \in \mathbb{C}^{n\times n}$ such that $\mathbf{C}_H \geq 0$

Thank you
for your attention!

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