Coercive Inequalities and U-Bounds

Esther Bou Dagher

October 11, 2021

Imperial College London

Introduction

In 1975, L. Gross obtained the following Logarithmic Sobolev inequality ([33]):

$$\int_{\mathbb{R}^n} f^2 log \left(\frac{f^2}{\int_{\mathbb{R}^n} f^2 d\mu} \right) d\mu \leq 2 \int_{\mathbb{R}^n} |\nabla f|^2 d\mu, \qquad (1)$$

where ∇ is the standard gradient on $\mathbb{R}^n$ and $d\mu = \frac{e^{-\frac{|x|^2}{2}}}{Z} d\lambda$ is the Gaussian measure. In a setup of a more general metric space, a natural question would be to try to find similar inequalities with different measures of the form $d\mu = \frac{e^{-U(d)}}{Z} d\lambda$, where U is a function of a metric d, and where the Euclidean gradient is replaced by a more general sub-gradient in $\mathbb{R}^n$.

Plan

Euclidean space: $\mathbb{R}^n$ → Carnot group: $\mathbb{G}$

Euclidean gradient: ∇_{euc} → Subgradient: $\nabla = (X_1, \ldots, X_n)$

Gaussian measure → More general probability measures

Plan

$$d\mu = \frac{e^{-U(d)}}{Z} d\lambda$$

d: Carnot-Carathéodory distance.

U-Bound

W. Hebisch and B. Zegarlinski (2009): $q - LSI$ for $U(d) = d^p$, $\frac{1}{p} + \frac{1}{q} = 1$.

EBD (2021): $q - LSI$ for $U(d)$, where $U'' \leq \beta U'$, $U \leq \gamma U'^q$, on $\{d(x) \geq 1\}$.

Applications:

EBD (2021):

LSI→Hypercontractivity

LSI→Talagrand inequality

$$d\mu = \frac{e^{-U(N)}}{Z} d\lambda.$$

N: a homogeneous norm

There is no LSI for such measures (W. Hebisch and B. Zegarlinski)

EBD, B. Zegarlinski (2021): Force a LSI by introducing a singularity through an additive or multiplicative term.

$$d\mu = \frac{e^{-U(|x|,N)}}{Z} d\lambda$$

Heisenberg: $N = \left(|x|^4 + 16|z|^2\right)^{\frac{1}{4}}$, solution to $\Delta N^{2-Q} = 0$ on $\mathbb{G}\backslash\{0\}$.

Step-2: $N = (|x|^4 + a|z|^2)^{\frac{1}{4}}$

Anisotropic Heisenberg: Compute $\widehat{N}$ s.t. $\Delta \widehat{N}^{2-Q} = 0$ on $\mathbb{G}\backslash\{0\}$.
(EBD, B. Zegarlinski (2021))

U-Bound

EBD, B. Zegarlisnki (2021):

Prove ϕ −LSI and q −Poincaré

U-Bound + Weak Poincaré

EBD, Y. Qiu, M. Zhang (2021):

Extend to Poincaré inequality to infinite dimensional Gibbs measures with unbounded interaction potentials.

EBD, Y. Wang (2021): Extending Poincaré and LSI to higher order.

Introduction

L.Gross also pointed out ([33]) the importance of the inequality (1) in the sense that it can be extended to infinite dimensions with additional useful results. (See also works: [34, 12, 60, 53, 13, 68, 59].) He proved that if $\mathcal{L}$ is the non-positive self-adjoint operator on $L^2(\mu)$ such that

$$(-\mathcal{L}f, f)_{L^2(\mu)} = \int_{\mathbb{R}^n} |\nabla f|^2 d\mu,$$

then (1) is equivalent to the fact that the semigroup $P_t = e^{t\mathcal{L}}$ generated by $\mathcal{L}$ is hypercontractive: i.e. for $q(t) \leq 1 + (q-1)e^{2t}$ with $q > 1$, we have $\| P_t f \|_{q(t)} \leq \| f \|_q$ for all $f \in L^q(\mu)$. ([33])

Introduction

In 1985, D. Bakry and M. Emery extended the Logarithmic Sobolev inequality for a larger class of probability measures defined on Riemaniann manifolds under an important Curvature-Dimension condition ([2]). More generally, if (Ω, F, μ) a probability space, and $\mathcal{L}$ is a non-positive self-adjoint operator acting on $L^2(\mu)$, we say that the measure μ satisfies a Logarithmic Sobolev inequality if there is a constant c such that, for $f \in D(\mathcal{L})$,

$$\int f^2 log \frac{f^2}{\int f^2 d\mu} d\mu \leq c \int f(-\mathcal{L}f)\, d\mu.$$

Introduction

Another generalisation, the so-called q-Logarithmic Sobolev inequality, in the setting of a metric measure space, was obtained by S. Bobkov and M. Ledoux in 2000 ([11]), in the form:

$$\int f^q log \frac{f^q}{\int f^q d\mu} d\mu \leq c \int |\nabla f|^q d\mu,$$

where $q \in (1, 2]$.In 2005 ([12]), S. Bobkov and B. Zegarliński showed that the q-Logarithmic Sobolev inequality is better than the classical $q = 2$ inequality in the sense that one gets a stronger decay of tail estimates i.e. if μ satisfies the Logarithmic Sobolev inequality for $q \in (1, 2]$, then for every bounded locally Lipschitz function f such that $|\nabla f| \leq M \ \mu - a.e.$ for $M \in (0, \infty)$, we have

$$\mu(e^{tf}) \leq exp\{\frac{cM^q}{q^q(q-1)} t^q + t\mu(f)\} \qquad \forall t > 0.$$

In addition, when the space is finite, and under weak conditions, they proved that the corresponding semigroup P_t is ultracontractive i.e.

$$\| P_t f \|_\infty \leq \| f \|_p$$

for all $t \geq 0$ and $p \in [1, \infty)$.

Introduction

The important q-Poincaré inequality

$$\int \left| f - \int f d\mu \right|^q d\mu \leq c \int |\nabla f|^q d\mu$$

can be obtained from the q-Logarithmic Sobolev inequality by simply replacing f by $1 + \varepsilon f$ in that inequality, and letting $\varepsilon \to 0$.

Introduction

Definition

We say that a Lie group on $\mathbb{R}^N$, $\mathbb{G} = (\mathbb{R}^N, \circ)$ is a (homogeneous) Carnot group if the following properties hold:
(C.1) $\mathbb{R}^N$ can be split as $\mathbb{R}^N = \mathbb{R}^{N_1} \times ... \times \mathbb{R}^{N_r}$, and the dilation $\delta_\lambda : \mathbb{R}^N \to \mathbb{R}^N$

$$\delta_\lambda(x) = \delta_\lambda(x^{(1)}, ..., x^{(r)}) = (\lambda x^{(1)}, \lambda^2 x^{(2)}, ..., \lambda^r x^{(r)}), \qquad x^{(i)} \in \mathbb{R}^{N_i},$$

is an automorphism of the group $\mathbb{G}$ for every $\lambda > 0$. Then $(\mathbb{R}^N, \circ, \delta_\lambda)$ is a homogeneous Lie group on $\mathbb{R}^N$. Moreover, the following condition holds:
(C.2) If N_1 is as above, let $X_1, ..., X_{N_1}$ be the left invariant vector fields on $\mathbb{G}$ such that $X_j(0) = \partial/\partial x_j|_0$ for $j = 1, ..., N_1$. Then

$$rank(Lie\{X_1, ..., X_{N_1}\}(x)) = N \qquad \forall x \in \mathbb{R}^N.$$

Definition

The vector valued operator $\nabla := (X_1, X_2, ..., X_{N_1})$ is called the sub-gradient on $\mathbb{G}$, and $\triangle = \sum_{i=1}^{N_1} X_i^2$ is called the sub-Laplacian on $\mathbb{G}$.

Introduction

In the setting of Carnot groups, D. Bakry and M. Emery's Curvature-Dimension condition in [2] will no longer hold true. In 2010 ([35]), a method of studying coercive inequalities on general metric spaces that does not require a bound on the curvature of space was developed. Working on a general metric space equipped with non-commuting vector fields $\{X_1, \ldots, X_n\}$, their method is based on U-bounds, which are inequalities of the form:

$$\int f^q \mathcal{U}(d)\, d\mu \leq C \int |\nabla f|^q d\mu + D \int f^q d\mu$$

where $d\mu = \frac{e^{-U(d)}}{Z} d\lambda$ is a probability measure, $U(d)$ and $\mathcal{U}(d)$ are functions having a suitable growth at infinity, λ is a natural measure like the Lebesgue measure for instance (which is the Haar measure for nilpotent Lie groups), d is a metric related to the gradient $\nabla = (X_1, \ldots, X_n)$, and $q \in (1, \infty)$.

$$\int f^q \mathcal{U}(d)\, d\mu \leq C \int |\nabla f|^q d\mu + D \int f^q d\mu; \; d\mu = \frac{e^{-U(d)}}{Z} d\lambda$$

$\mathcal{U}(d) \to \infty$

"in all directions"

↓

Hebsich-Zegarlinski Theorem 1 (2009)

↓

q —Poincaré

$\mathcal{U}(d) = U(d) + |\nabla U(d)|^q$

↓

Hebsich-Zegarlinski Theorem 2 (2009)

↓

q —Logarithmic Sobolev

$$d\mu = \frac{e^{-U(d)}}{Z} d\lambda$$

d: Carnot-Carathéodory distance.

U-Bound

W. Hebisch and B. Zegarlinski (2009): $q - LSI$ for $U(d) = d^p$, $\frac{1}{p} + \frac{1}{q} = 1.$

EBD (2021): $q - LSI$ for $U(d)$, where $U'' \leq \beta U'$, $U \leq \gamma U'^q$, on $\{d(x) \geq 1\}$.

Applications:

EBD (2021):

LSI→Hypercontractivity

LSI→Talagrand inequality

$$d\mu = \frac{e^{-U(N)}}{Z} d\lambda.$$

N: a homogeneous norm

There is no LSI for such measures (W. Hebisch and B. Zegarlinski)

EBD, B. Zegarlinski (2021): Force a LSI by introducing a singularity through an additive or multiplicative term.

$$d\mu = \frac{e^{-U(|x|,N)}}{Z} d\lambda$$

Heisenberg: $N = \left(|x|^4 + 16|z|^2\right)^{\frac{1}{4}}$, solution to $\Delta N^{2-Q} = 0$ on $\mathbb{G}\backslash\{0\}$.

Step-2: $N = (|x|^4 + a|z|^2)^{\frac{1}{4}}$

Anisotropic Heisenberg: Compute $\widehat{N}$ s.t. $\Delta \widehat{N}^{2-Q} = 0$ on $\mathbb{G}\backslash\{0\}$. (EBD, B. Zegarlinski (2021))

U-Bound

EBD, B. Zegarlisnki (2021):

Prove ϕ −LSI and q −Poincaré

U-Bound + Weak Poincaré

EBD, Y. Qiu, M. Zhang (2021):

Extend to Poincaré inequality to infinite dimensional Gibbs measures with unbounded interaction potentials.

EBD, Y. Wang (2021): Extending Poincaré and LSI to higher order.

Carnot-Carathéodory distance

Definition

We say that γ is horizontal if there exist measurable functions $a_1, \ldots, a_{N_1} : [0,1] \to \mathbb{R}$ such that $\gamma'(t) = \sum_{i=1}^{N_1} a_i(t) X_i(\gamma(t))$for almost all $t \in [0,1]$. For such a horizontal curve γ, we define the length of γ to be

$$|\gamma| = \int_0^1 \left(\sum_{i=1}^{N_1} a_i^2(t) \right)^{\frac{1}{2}} dt.$$

Definition

The Carnot-Carathéodory distance or the control distance between two points x and y is defined by

$$d(x,y) = inf \left\{ t | \gamma : [0,t] \to G, \gamma(0) = x, \gamma(t) = y \; |\gamma'(s)| \leq 1 \; \forall s \in [0,t] \right\},$$

where $\gamma : [0,1] \to G$ is an absolutely continuous horizontal path on $[0,1]$.

We are concerned with proving U-Bounds (to get Logarithmic Sobolev and Poincaré inequalities) for the measure $d\mu_U = \frac{e^{-U(d)}}{Z} d\lambda$.

U-Bound; $d\mu = \frac{e^{-U(d)}}{Z} d\lambda$

Theorem (EBD, 2021 [18])

Assume that outside the open unit ball $B = \{d(x) < 1\}$, the metric d satisfies the following: $|\nabla d|$ is bounded, say $|\nabla d| \leq 1$, and there exist finite positive constants K and c_0 such that

$$\Delta d \leq K + U'(d)\left(|\nabla d|^2 - c_0\right). \tag{2}$$

(i) If $U'' \leq \beta U'$ for some positive constant β, outside B, then for any $q \in (1, \infty)$, there exist constants C_q, D_q, independent of f, such that

$$\int |f|^q |U'(d)|^q d\mu_U \leq C_q \int |\nabla f|^q d\mu_U + D_q \int |f|^q d\mu_U.$$

(ii) If, in addition, $U \leq \gamma U'^q$ for some positive constant γ and some $q > 1$, outside B, then

$$\int |f|^q U(d) d\mu_U \leq C_q \int |\nabla f|^q d\mu_U + D_q \int |f|^q d\mu_U.$$

Take $\mathcal{U}(d) = |U'(d)|^q$, by Hebisch-Zegarlinski Theorem 1, (i)$\rightarrow q-$Poincaré. To apply Theorem 2, we need $\mathcal{U}(d) = U(d) + |\nabla U(d)|^q = U(d) + |U'(d)\nabla d|^q \leq U(d) + |U'(d)|^q$. Using (i) and (ii), we get $q-$LSI.

Examples of Logarithmic Sobolev inequality

Example (EBD, 2021 [18])

The q-Poincaré and a q-Logarithmic Sobolev inequality are satisfied for the measure

$$d\mu_U = \frac{e^{-(d+1)^p log(d+1)} d\lambda}{Z}$$

for $q \geq \beta$, where β is the finite index conjugate to p.

Example (EBD, 2021 [18])

For $U(d) = sinh(d)$, $U(d) = U^{''}(d) \leq cosh(d) = U^{'}(d)$.
So, by Corollary 9, the q-Poincaré and q-Logarithmic Sobolev inequalities hold true for the measure $d\mu_U = \dfrac{e^{-sinh(d)}}{Z} d\lambda$ for all $q \geq 1$.

Talagrand Inequality

In 2000, F. Otto and C. Villani showed [57] that in the setting of manifolds under D. Bakry and M. Emery's Curvature-Dimension condition, the Logarithmic Sobolev inequality implies the Talagrand transportation cost inequality. The Talagrand transportation cost inequality was first introduced in 1996 ([64]) by M. Talagrand:

$$T_w(\mu,\nu) \leq 2\int log(f)d\mu, \tag{3}$$

where μ is a measure on $\mathbb{R}^N$ absolutely continuous with respect to the Gaussian measure ν, $f = \frac{d\mu}{d\nu}$ is the relative density, $w(x,y) = \sum_{i=1}^{N}(x_i - y_i)^2$,and

$$T_w(\mu,\nu) = \inf_{\pi\in\Pi(\mu,\nu)} \int_{\mathbb{R}^N\times\mathbb{R}^N} w(x,y)d\pi(x,y),$$

where $\Pi(\mu,\nu)$ is the set of probability measures on $\mathbb{R}^N \times \mathbb{R}^N$ with μ the first marginal and ν the second marginal.

Talagrand Inequality

We would like to apply the $q-$Logarithmic Sobolev inequality to get hypercontractivity and to obtain the $p-$Talagrand inequality on (X, d, μ) with a constant K :

$$W_p(\mu, \nu)^p \leq \frac{1}{K} Ent_\mu \left(\frac{d\nu}{d\mu} \right), \tag{4}$$

with p finite index conjugate of q. The $p-$Wasserstein distance between two probability measures on X is defined as $W_p(\mu, \nu)^p = \inf_{\pi \in \Pi(\mu,\nu)} \int_{X \times X} d(x, y)^p d\pi(x, y)$, where $\Pi(\mu, \nu)$ is the set of probability measures on $X \times X$ with μ the first marginal and ν the second marginal. $Ent_\mu \left(\frac{d\nu}{d\mu} \right) = \int \frac{d\nu}{d\mu} log \left(\frac{d\nu}{d\mu} \right) d\mu$ is the entropy functional such that ν is a probability measure absolutely continuous with respect to μ. We note that for $p = 2$, (3) is a special case of (4).

Talagrand Inequality

For the quadratic case $p = q = 2$, in 2007, J. Lott and C. Villani [48] used the Hamilton-Jacobi infimum convolution operator under the assumption where the space (X, d, μ) supports local Poincaré inequality and the measure μ is a doubling measure i.e. the measure of any open ball is positive and finite and there exists a constant $c_d \geq 1$ such that for all $x \in X$ and $r > 0$,

$$\mu(B(x, 2r)) \leq c_d \mu(B(x, r)). \tag{5}$$

In our setting, we show hypercontractivity and the $p-$Talagrand inequality using the Hamilton-Jacobi equation in the setting of Carnot groups done by F. Dragoni in 2007 ([25]). The advantage of doing so is that the restriction (5) to have μ a doubling measure is no longer required!

Talagrand Inequality and Hypercontractivity

Theorem (EBD, 2021 [18] LSI→Talagrand)

Let $1 < q \leq 2$, and $p \geq 2$ be its finite index conjugate, so that $\frac{1}{p} + \frac{1}{q} = 1$. If (G, d, μ) satisfies the q-Logarithmic Sobolev inequality with constant $c = (q-1)\left(\frac{q}{K}\right)^{q-1}$ for some constant $K > 0$, then it also satisfies the p-Talagrand inequality with the same constant K.

Theorem (EBD, 2021 [18] LSI→Hypercontractivity)

Assume we have the following 2-Logarithmic Sobolev inequality with the measure $d\mu = \frac{e^{-U(d)}}{Z} d\lambda$, and in the setting of the Carnot group: then, for every bounded measurable function f on $\mathbb{G}$, every $t \geq 0$, and every $a \in \mathbb{R}$,

$$||e^{Q_t f}||_{a+\rho t} \leq ||e^f||_a.$$

$$d\mu = \frac{e^{-U(d)}}{Z} d\lambda$$

d: Carnot-Carathéodory distance.

U-Bound

W. Hebisch and B. Zegarlinski (2009): $q - LSI$ for $U(d) = d^p$, $\frac{1}{p} + \frac{1}{q} = 1.$

EBD (2021): $q - LSI$ for $U(d)$, where $U'' \leq \beta U',\ U \leq \gamma U'^q$, on $\{d(x) \geq 1\}$.

Applications:

EBD (2021):

LSI→Hypercontractivity

LSI→Talagrand inequality

$$d\mu = \frac{e^{-U(N)}}{Z} d\lambda.$$

N: a homogeneous norm

There is no LSI for such measures (W. Hebisch and B. Zegarlinski)

EBD, B. Zegarlinski (2021): Force a LSI by introducing a singularity through an additive or multiplicative term.

$$d\mu = \frac{e^{-U(|x|,N)}}{Z} d\lambda$$

Heisenberg: $N = (|x|^4 + 16|z|^2)^{\frac{1}{4}}$, solution to $\Delta N^{2-Q} = 0$ on $\mathbb{G}\backslash\{0\}$.

Step-2: $N = (|x|^4 + a|z|^2)^{\frac{1}{4}}$

Anisotropic Heisenberg: Compute $\widehat{N}$ s.t. $\Delta \widehat{N}^{2-Q} = 0$ on $\mathbb{G}\backslash\{0\}$. (EBD, B. Zegarlinski (2021))

U-Bound

EBD, B. Zegarlisnki (2021):

Prove ϕ −LSI and q −Poincaré

U-Bound + Weak Poincaré

EBD, Y. Qiu, M. Zhang (2021):

Extend to Poincaré inequality to infinite dimensional Gibbs measures with unbounded interaction potentials.

EBD, Y. Wang (2021): Extending Poincaré and LSI to higher order.

Setup

We define the step-two Carnot group $\mathbb{G}$, i.e. a group isomorphic to $\mathbb{R}^{n+m}$ with the group law

$$(x,z)\circ(x',z') = \left(x_i + x_i',\ z_j + z_j' + \frac{1}{2} < \Lambda^{(j)}x, x' >\right)_{i=1,..,n;j=1,..,m}$$

for $x, x' \in \mathbb{R}^n, z, z' \in \mathbb{R}^m$, where $< .,. >$ stands for the inner product on $\mathbb{R}^n$, and:

1) The matrices $\Lambda^{(j)}$ are $n \times n$ skew-symmetric

2) The matrices are linearly independent

We are in the setting of Heisenberg group, if in addition:

1) $\Lambda^{(j)}$ are orthogonal

2) $\Lambda^{(k)}\Lambda^{(j)} + \Lambda^{(j)}\Lambda^{(k)} = 0,\ \forall k \neq j.$

Setup; $d\mu = \frac{e^{-U(N)}}{Z} d\lambda$

Heisenberg:

$N \equiv \left(|x|^4 + 16|z|^2\right)^{\frac{1}{4}}$ is the Kaplan norm. In other words, N^{2-Q} is the unique fundamental solution of the sub-Laplacian $\triangle := \sum_{i=1}^{n} X_i^2$,where X_i is the Jacobian basis of $\mathfrak{g}$, the Lie algebra of $\mathbb{G} \cong \mathbb{R}^{n+m}$, and $Q = n + 2m$ is the homogeneous dimension.

Step-two:

We consider $N \equiv \left(|x|^4 + a|z|^2\right)^{\frac{1}{4}}$,where $(x, z) \in \mathbb{G}$ and $a \in (0, \infty)$.

Goal

J. Inglis (2010)

- Heisenberg-type group
- $d\mu = \frac{e^{-N^p}}{Z} d\lambda$
- $N = \left(|x|^4 + 16|z|^2\right)^{\frac{1}{4}}$
- No U-Bound

Our Setting

- Step-two Carnot groups
- $d\mu = \frac{e^{-U(N)}}{Z} d\lambda$
- $N = \left(|x|^4 + a|z|^2\right)^{\frac{1}{4}}$, where $a \in (0, \infty)$
- U-Bound

U-Bound

Theorem (EBD and B. Zegarliński, 2021 [15])

Let $N = \left(|x|^4 + a|z|^2\right)^{\frac{1}{4}}$ *with* $a \in (0, \infty)$ *be as above and* $g : [0, \infty) \to [0, \infty)$ *be a differentiable increasing function such that* $g''(N) \leq g'(N)^3 N^3$ *on* $\{N \geq 1\}$. *Let* $d\mu = \frac{e^{-g(N)}}{Z} d\lambda$ *be a probability measure, and* Z *the normalization constant. Then, for all locally Lipschitz functions* f,

$$\int \frac{g'(N)}{N^2} |f|^q d\mu \leq C \int |\nabla f|^q d\mu + D \int |f|^q d\mu \tag{6}$$

holds outside the unit ball $\{N < 1\}$ *with* C *and* D *positive constants and* $q \geq 2$.

By Hebisch-Zegarlinski Theorem 1, we choose $\mathcal{U}(N) = \frac{g'(N)}{N^2}$, to obtain $q-$Poincaré inequality.

Examples of Poincaré Inequality

Example (EBD and B. Zegarliński, 2021 [15])

The Poincaré inequality for $q \geq 2$ holds for the measure $d\mu = \frac{exp\left(-cosh\left(N^k\right)\right)}{Z} d\lambda$, where λ is the Lebesgue measure, and $k \geq 1$ in the setting of the step-two Carnot group.

Example (J. Inglis, 2010 [36])

The Poincaré inequality for $q \geq 2$ holds for the measure $d\mu = \frac{exp\left(-N^k\right)}{Z} d\lambda$, where λ is the Lebesgue measure, and $k \geq 4$ in the setting of the step-two Carnot group.

Example (EBD and B. Zegarliński, 2021 [15])

The Poincaré inequality for $q \geq 2$ holds for the measure $d\mu = \frac{exp\left(-N^k log\left(N+1\right)\right)}{Z} d\lambda$, where λ is the Lebesgue measure, and $k \geq 3$ in the setting of the step-two Carnot group.

$$d\mu = \frac{e^{-U(d)}}{Z} d\lambda$$

d: Carnot-Carathéodory distance.

U-Bound

W. Hebisch and B. Zegarlinski (2009): $q - LSI$ for $U(d) = d^p$, $\frac{1}{p} + \frac{1}{q} = 1$.

EBD (2021): $q - LSI$ for $U(d)$, where $U'' \leq \beta U'$, $U \leq \gamma U'^q$, on $\{d(x) \geq 1\}$.

Applications:

EBD (2021):

LSI→Hypercontractivity

LSI→Talagrand inequality

$$d\mu = \frac{e^{-U(N)}}{Z} d\lambda.$$

N: a homogeneous norm

There is no LSI for such measures (W. Hebisch and B. Zegarlinski)

EBD, B. Zegarlinski (2021): Force a LSI by introducing a singularity through an additive or multiplicative term.

$$d\mu = \frac{e^{-U(|x|, N)}}{Z} d\lambda$$

Heisenberg: $N = \left(|x|^4 + 16|z|^2\right)^{\frac{1}{4}}$, solution to $\Delta N^{2-Q} = 0$ on $\mathbb{G}\backslash\{0\}$.

Step-2: $N = (|x|^4 + a|z|^2)^{\frac{1}{4}}$

Anisotropic Heisenberg: Compute $\widehat{N}$ s.t. $\Delta \widehat{N}^{2-Q} = 0$ on $\mathbb{G}\backslash\{0\}$. (EBD, B. Zegarlinski (2021))

U-Bound

EBD, B. Zegarlisnki (2021):

Prove ϕ −LSI and q −Poincaré

U-Bound + Weak Poincaré

EBD, Y. Qiu, M. Zhang (2021):

Extend to Poincaré inequality to infinite dimensional Gibbs measures with unbounded interaction potentials.

EBD, Y. Wang (2021): Extending Poincaré and LSI to higher order.

$\mathcal{U}(d) \to \infty$ "in all directions"	$\mathcal{U}(d) = U(d) + \lvert\nabla U(d)\rvert^q$	$\mathcal{U}(d) = \phi(U(d)) + \lvert\nabla U(d)\rvert^q$
Hebsich-Zegarlinski Theorem 1 (2009)	Hebsich-Zegarlinski Theorem 2 (2009)	EBD-Zegarlinski (2021)
q —Poincaré	q —Logarithmic Sobolev	ϕ —Logarithmic Sobolev

$\phi-$Logarithmic Sobolev Inequality

Theorem (EBD and B. Zegarliński, 2021 [15])

Let U be a locally lipschitz function on $\mathbb{R}^N$ which is bounded below such that $Z = \int e^{-U} d\lambda < \infty$, and $d\mu = \frac{e^{-U}}{Z} d\lambda$. Let $\phi : [0,\infty) \to \mathbb{R}^+$ be a non-negative, non-decreasing, concave function such that $\phi(0) > 0$, and $\phi'(0) > 0$. Assume the following classical Sobolev inequality is satisfied:

$$\left(\int |f|^{q+\epsilon} d\lambda \right)^{\frac{q}{q+\epsilon}} \leq a \int |\nabla f|^q d\lambda + b \int |f|^q d\lambda$$

for some a, $b \in [0,\infty)$, and $\epsilon > 0$. Moreover, if for some A, $B \in [0,\infty)$, we have:

$$\mu \left(|f|^q (\phi(U) + |\nabla U|^q) \right) \leq A\mu|\nabla f|^q + B\mu|f|^q, \tag{7}$$

Then, there exists constants C, $D \in [0,\infty)$ such that:

$$\mu \left(|f|^q \phi \left(\left| log \frac{|f|^q}{\mu|f|^q} \right| \right) \right) \leq C\mu|\nabla f|^q + D\mu|f|^q,$$

for all locally Lipschitz functions f.

Higher order LSI

Choose $\phi(x) = (1+x)^{\beta}$, for $\beta \in (0,1)$. Then, ϕ satisfies the conditions of the theorem above and we have:

$$\mu\left(|f|^q \left|log\frac{|f|^q}{\mu|f|^q}\right|^{\beta}\right) \leq \mu\left(|f|^q\phi\left(\left|log\frac{|f|^q}{\mu|f|^q}\right|\right)\right) \leq C\mu|\nabla f|^q + D\mu|f|^q.$$

Theorem (EBD and Y. Wang, 2021)

Given the following Logarithmic-Sobolev inequality

$$\int |f|^2 \left|log\left(\frac{|f|^2}{\mu|f|^2}\right)\right|^{\beta} d\mu \leq C\mu|\nabla f|^2, \tag{8}$$

for $\beta \in (0,1]$. *Then, for all* $m \in \mathbb{N}$,

$$\int |f|^2 \left|log\left(\frac{|f|^2}{\mu|f|^2}\right)\right|^{\beta m} d\mu \leq D\sum_{|\alpha|=0}^{m}\int |\nabla^{\alpha} f|^2 d\mu, \tag{9}$$

where $\nabla^{\alpha} f = (X_1^{\alpha_1} X_2^{\alpha_2} ... X_n^{\alpha_n} f)$ *such that* $|\alpha| = \sum_{i=1}^{n}\alpha_i$, *and* $C, D \in (0,\infty)$.

$$d\mu = \frac{e^{-U(d)}}{Z} d\lambda$$

d: Carnot-Carathéodory distance.

U-Bound

W. Hebisch and B. Zegarlinski (2009): $q-LSI$ for $U(d) = d^p$, $\frac{1}{p} + \frac{1}{q} = 1.$

EBD (2021): $q-LSI$ for $U(d)$, where $U'' \leq \beta U'$, $U \leq \gamma U'^q$, on $\{d(x) \geq 1\}$.

Applications:

EBD (2021):
LSI→Hypercontractivity

LSI→Talagrand inequality

$$d\mu = \frac{e^{-U(N)}}{Z} d\lambda.$$

N: a homogeneous norm

There is no LSI for such measures (W. Hebisch and B. Zegarlinski)

EBD, B. Zegarlinski (2021): Force a LSI by introducing a singularity through an additive or multiplicative term.

$$d\mu = \frac{e^{-U(|x|,N)}}{Z} d\lambda$$

Heisenberg: $N = (|x|^4 + 16|z|^2)^{\frac{1}{4}}$, solution to $\Delta N^{2-Q} = 0$ on $\mathbb{G}\backslash\{0\}$.

Step-2: $N = (|x|^4 + a|z|^2)^{\frac{1}{4}}$

Anisotropic Heisenberg: Compute $\widehat{N}$ s.t. $\Delta \widehat{N}^{2-Q} = 0$ on $\mathbb{G}\backslash\{0\}$. (EBD, B. Zegarlinski (2021))

U-Bound

EBD, B. Zegarlinski (2021):

Prove ϕ –LSI and q –Poincaré

U-Bound + Weak Poincaré

EBD, Y. Qiu, M. Zhang (2021):

Extend to Poincaré inequality to infinite dimensional Gibbs measures with unbounded interaction potentials.

EBD, Y. Wang (2021): Extending Poincaré and LSI to higher order.

What is a polarizable Carnot group?

The Carnot group $\mathbb{G}$ is said to be polarizable if N, where N^{2-Q} is the fundamental solution to the sub-Laplacian, is $\infty-$harmonic in $\mathbb{G}\backslash\{0\}$, i.e. for $\nabla :=(X_i)_{1\leq i\leq n}$,

$$\triangle_\infty N := \frac{1}{2} < \nabla\left(|\nabla N|^2\right), \nabla N >= 0 \qquad in\ \ \mathbb{G}\backslash\{0\}. \tag{10}$$

The concept of polarizable Carnot groups was first introduced by Z. Balogh and J. Tyson in [4], where they used the $\infty-$harmonicity of N to create a procedure to construct polar coordinates. Moreover, they showed in [4] that the fundamental solution of the $p-$sub-Laplacian can be expressed as the fundamental solution N^{2-Q} of the sub-Laplacian, proved capacity formulas, and produced sharp constants for the Moser-Trudinger inequality.

Higher-Dimensional Anisotropic Heisenberg Group

For the time being, there is no classification of polarizable Carnot groups and the only examples till now are Euclidean spaces and Heisenberg-type groups. Z. Balogh and J. Tyson provided in [4] the anisotropic Heisenberg group in $\mathbb{R}^5$ as a counterexample with the following generators of the Lie algebra: $X = \frac{\partial}{\partial x} + 2ay \cdot \frac{\partial}{\partial t}$, $Y = \frac{\partial}{\partial y} - 2ax \cdot \frac{\partial}{\partial t}$, $Z = \frac{\partial}{\partial z} - 2w \cdot \frac{\partial}{\partial t}$, and $W = \frac{\partial}{\partial w} - 2z \cdot \frac{\partial}{\partial t}$, where $a = \frac{1}{2}$. (Note that if $a = 1$, we have the polarizable Heisenberg group.) We will start by extending Z. Balogh and J. Tyson's anisotropic Heisenberg group in $\mathbb{R}^5$ [4] to a higher-dimensional anisotropic Heisenberg group in $\mathbb{R}^{2n+1}$, and use R. Beals, B. Gaveau, and P. Greiner's [7] explicit intergal representation to compute the fundamental solution N^{2-Q}. We then compute bounds for $|\nabla N|$ and $x \cdot \nabla N$, which are essential to get the U-Bound of the form (6).

$$d\mu = \frac{e^{-U(d)}}{Z} d\lambda$$

d: Carnot-Carathéodory distance.

U-Bound

W. Hebisch and B. Zegarlinski (2009): $q - LSI$ for $U(d) = d^p$, $\frac{1}{p} + \frac{1}{q} = 1$.

EBD (2021): $q - LSI$ for $U(d)$, where $U'' \leq \beta U'$, $U \leq \gamma U'^q$, on $\{d(x) \geq 1\}$.

Applications:

EBD (2021):

LSI→Hypercontractivity

LSI→Talagrand inequality

$$d\mu = \frac{e^{-U(N)}}{Z} d\lambda.$$

N: a homogeneous norm

There is no LSI for such measures (W. Hebisch and B. Zegarlinski)

EBD, B. Zegarlinski (2021): Force a LSI by introducing a singularity through an additive or multiplicative term.

$$d\mu = \frac{e^{-U(|x|,N)}}{Z} d\lambda$$

Heisenberg: $N = (|x|^4 + 16|z|^2)^{\frac{1}{4}}$, solution to $\Delta N^{2-Q} = 0$ on $\mathbb{G}\backslash\{0\}$.

Step-2: $N = (|x|^4 + a|z|^2)^{\frac{1}{4}}$

Anisotropic Heisenberg: Compute $\widehat{N}$ s.t. $\Delta\widehat{N}^{2-Q} = 0$ on $\mathbb{G}\backslash\{0\}$. (EBD, B. Zegarlinski (2021))

U-Bound

EBD, B. Zegarlinski (2021):

Prove ϕ −LSI and q −Poincaré

U-Bound + Weak Poincaré

EBD, Y. Qiu, M. Zhang (2021):

Extend to Poincaré inequality to infinite dimensional Gibbs measures with unbounded interaction potentials.

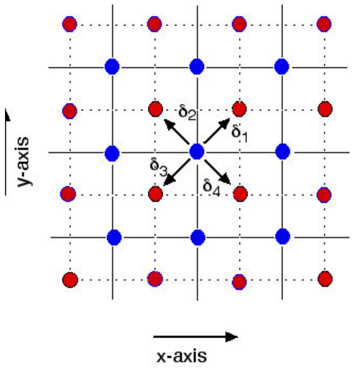
EBD, Y. Wang (2021): Extending Poincaré and LSI to higher order.

Gibbs measures

We aim to show that certain infinite dimensional Gibbs measures with unbounded interaction potentials as a function of homogeneous norms on an infinite product of Carnot groups satisfy the Poincaré inequality. So far, the passage to infinite dimensions in the setting of Nilpotent Lie groups required the condition $|\nabla d| \geq c$ outside the unit ball ([41, 38]), which is not true for homogeneous norms introduced. The methods known use the single site Poincaré inequality to pass to the global Poincaré inequality. To get results for measures as function of homogeneous norms, we use the U-Bound (6) proved in Theorem 1 to get a weak U-bound; coupled with a weak single-site Poincaré inequality, we are able to pass to the infinite dimensional setting.

For Carnot-Carathéodory distance: $|\nabla d| = 1$, so $|\nabla d| \geq c$ outside the unit ball $\{d(x) < 1\}$.
For Kaplan norm in Heisenberg group: $N = (|x|^2 + 16|z|^2)^{\frac{1}{4}}$, and $|\nabla N| = \dfrac{|x|}{N}$. For $|x| = 0$ and $|z|$ large, $|\nabla N|$ does not satisfy $|\nabla N| \geq c$ outside the unit ball $\{N(x,z) < 1\}$.

Gibbs measures: Setup



We have a Carnot group $\mathbb{G}$, and we give it a d-dimensional integer lattice structure: $\mathbb{G}^{\mathbb{Z}^d}$.For any compact $\Lambda \subset \mathbb{Z}$, denote the potential U_Λ^w by

$$U_\Lambda^w(x_\Lambda) := \sum_{i\in\Lambda} \phi(x_i) + \sum_{i,j\in\Lambda, i\sim j} \beta V(x_i, x_j) + \sum_{i\in\Lambda, j\notin\Lambda, i\sim j} \beta V(x_i, w_j),$$

where $\phi \in C^1(\mathbb{G}, \mathbb{R})$ is the phase and $V \in C^1(\mathbb{G}\times\mathbb{G}, \mathbb{R})$ is the interaction with strength $\beta \geq 0$. Let $\mathbb{E}_\Lambda^w := \dfrac{1}{Z_\Lambda^w} e^{-U_\Lambda^w} dx_\Lambda$ be the local Gibbs measure and ν be the associated global measure satisfying $\nu\mathbb{E}_\Lambda^w = \nu$ for all compact $\Lambda \subset \mathbb{Z}$. Denote $|\nabla_\Lambda f|^2 = \sum_{i\in\Lambda} |\nabla_i f|^2$ and $|\nabla f|^2 = |\nabla_\mathbb{Z} f|^2$.

Gibbs measures: Hypothesis

Consider the following two hypotheses:

(H1) For any $i \in \mathbb{Z}$, the (U-bound $\to$) weak U-Bound

$$\sum_{j:j\sim i} \nu(f^q|\nabla_j V(x_i,x_j)|^q) \leq A\left(\nu|\nabla_i f|^q + \nu|f|^q + \sum_{m=0}^{\infty} C_\beta^m \nu|\nabla_{\{i-1-m,i+1+m\}} f|^q\right)$$

holds for some constants $A > 0$ and $C_\beta \in [0,1)$ such that $A\beta$ and C_β vanish as $\beta \to 0$.

(H2) For any $i \in \mathbb{Z}$, the weak q-Poincaré inequality

$$\nu \mathbb{E}_i^w |f - \mathbb{E}_i^w f|^q \leq B_{SG}\left(\nu|\nabla_i f|^q + \sum_{m=0}^{\infty} C_\beta^m \nu|\nabla_{\{i-1-m,i+1+m\}} f|^q\right)$$

holds for some constants $B_{SG} > 0$ and the same $C_\beta \in [0,1)$ such that $A\beta$ and $B_{SG}\beta \to 0$ as $\beta \to 0$.

Gibbs measures: Theorem

Theorem (EBD, Y. Qiu, and M. Zhang, 2021)

*Suppose **(H1)** and **(H2)** are satisfied, then there exists $\tilde{\beta} > 0$ such that for all $\beta \in [0, \tilde{\beta})$ the global Poincaré inequality*

$$\nu|f - \nu f|^q \leq c_{SG}\nu|\nabla f|^q$$

holds for some constant $c_{SG} > 0$.

Thanks to the U-Bound in Theorem 1, we were able to get examples for ϕ and V as functions of the homogeneous norms introduced, as well as a mixture of those norms. Under an additional hypothesis, we were also able to prove a global Logarithmic-Sobolev inequality.

$$d\mu = \frac{e^{-U(d)}}{Z} d\lambda$$

d: Carnot-Carathéodory distance.

U-Bound

W. Hebisch and B. Zegarlinski (2009): $q - LSI$ for $U(d) = d^p$, $\frac{1}{p} + \frac{1}{q} = 1.$

EBD (2021): $q - LSI$ for $U(d)$, where $U'' \leq \beta U'$, $U \leq \gamma U'^q$, on $\{d(x) \geq 1\}$.

Applications:

EBD (2021):

LSI→Hypercontractivity

LSI→Talagrand inequality

$$d\mu = \frac{e^{-U(N)}}{Z} d\lambda.$$

N: a homogeneous norm

There is no LSI for such measures (W. Hebisch and B. Zegarlinski)

EBD, B. Zegarlinski (2021): Force a LSI by introducing a singularity through an additive or multiplicative term.

$$d\mu = \frac{e^{-U(|x|,N)}}{Z} d\lambda$$

Heisenberg: $N = (|x|^4 + 16|z|^2)^{\frac{1}{4}}$, solution to $\Delta N^{2-Q} = 0$ on $\mathbb{G}\backslash\{0\}$.

Step-2: $N = (|x|^4 + a|z|^2)^{\frac{1}{4}}$

Anisotropic Heisenberg: Compute $\widehat{N}$ s.t. $\Delta \widehat{N}^{2-Q} = 0$ on $\mathbb{G}\backslash\{0\}$. (EBD, B. Zegarlinski (2021))

U-Bound

EBD, B. Zegarlisnki (2021):

Prove ϕ −LSI and q −Poincaré

U-Bound + Weak Poincaré

EBD, Y. Qiu, M. Zhang (2021):

Extend to Poincaré inequality to infinite dimensional Gibbs measures with unbounded interaction potentials.

EBD, Y. Wang (2021): Extending Poincaré and LSI to higher order.

Proof idea of U-Bound Inequality

Theorem (EBD and B. Zegarliński, 2021 [15])

Let $N = \left(|x|^4 + a|z|^2\right)^{\frac{1}{4}}$ *with* $a \in (0, \infty)$ *be as above and* $g : [0, \infty) \to [0, \infty)$ *be a differentiable increasing function such that* $g''(N) \leq g'(N)^3 N^3$ *on* $\{N \geq 1\}$*. Let* $d\mu = \frac{e^{-g(N)}}{Z} d\lambda$ *be a probability measure, and* Z *the normalization constant. Then, for all locally Lipschitz functions* f*,*

$$\int \frac{g'(N)}{N^2} |f|^q d\mu \leq C \int |\nabla f|^q d\mu + D \int |f|^q d\mu \tag{11}$$

holds outside the unit ball $\{N < 1\}$ *with* C *and* D *positive constants and* $q \geq 2$.

Here, $\mathcal{U} = \frac{g'(N)}{N^2}$. First Question: How to choose $\mathcal{U}$?

We need a technical lemma first:

Heisenberg $\mathbb{R}^{n+m}$

- $N = \left(|x|^4 + 16|z|^2\right)^{\frac{1}{4}}$
- $|\nabla N|^2 = \frac{|x|^2}{N^2}$
- $|\Delta N| = (Q-1)|\nabla N|^2 = (n+2m-1)\frac{|x|^2}{N^2}$
- $\frac{x}{|x|} \cdot \nabla N = \frac{|x|^3}{N^3}$

Step-2 Carnot $\mathbb{R}^{n+m}$

- $N = (|x|^4 + a|z|^2)^{\frac{1}{4}}$
- 1) $A\frac{|x|^2}{N^2} \leq |\nabla N|^2 \leq C\,\frac{|x|^2}{N^2}$
- 2) $|\Delta N| \leq B\frac{|x|^2}{N^2}$
- 3) $\frac{x}{|x|} \cdot \nabla N = \frac{|x|^3}{N^3}$
- $a, A, B, C \in (0, \infty)$

Anisotropic Heisenberg $\mathbb{R}^{2n+1}$

- $N = \frac{(B^2+t^2)^{\frac{1}{4n}}\left(AB+t^2+A\sqrt{B^2+t^2}\right)^{\frac{1}{2}-\frac{1}{4n}}}{\left(B+\sqrt{B^2+t^2}\right)^{\frac{1}{2}}}$

  where
  $$A = \frac{x_1^2}{2} + \frac{x_{n+1}^2}{2} + \frac{1}{2}\sum_{j=2, j\neq n+1}^{2n} x_j^2$$
  and
  $$B = \frac{x_1^2}{4} + \frac{x_{n+1}^2}{4} + \frac{1}{2}\sum_{j=2, j\neq n+1}^{2n} x_j^2$$
- $\frac{|x|^2}{2^{5+\frac{2}{n}}N^2} \leq |\nabla N|^2 \leq \frac{(2n+1)^2|x|^2}{8n^2N^2}$
- $|\Delta N| = (Q-1)|\nabla N|^2 = (2n+1)\frac{|x|^2}{N^2}$
- $\frac{x}{|x|} \cdot \nabla N \geq -\frac{|x|^2}{4nN}$. Problem: this term could be negative, and so we need the dimension $n > 5$.

How to choose $\mathcal{U}$?

For $q=2$,using integration by parts:

$$\int (\nabla N)\cdot(\nabla f)\, e^{-g(N)} d\lambda = -\int \nabla\left(\nabla N e^{-g(N)}\right) f d\lambda$$

$$= -\int \Delta N f e^{-g(N)} d\lambda + \int |\nabla N|^2 f g^{'}(N)\, e^{-g(N)} d\lambda.$$

Netx, using 1) and 2),

$$A\int \frac{|x|^2}{N^2} f g^{'}(N) e^{-g(N)} d\lambda - B\int \frac{|x|^2}{N^3} f e^{-g(N)} d\lambda \leq \int (\nabla N)\cdot(\nabla f)\, e^{-g(N)} d\lambda.$$

First candidate for $\mathcal{U} = \frac{|x|^2}{N^2} g^{'}(N)$ .We need $\mathcal{U}\to\infty$ "in all directions" to apply Hebisch-Zegarlinski Theorem 1 (2009). Recall that $N = (|x|^2 + a|z|^2)^{\frac{1}{4}}$. For $|x|=0$, we can have $|z|^2\to\infty$, but $\mathcal{U}$ =0. So, the problem is around the $z-$axis.

Idea: Replace f by $\frac{f^2}{|x|^2}$:

Now we have the good candidate $\mathcal{U} = \frac{g'(N)}{N^2}$:

$$\int f^2 \left(\frac{Ag'(N)}{N^2} - \frac{B}{N^3} \right) e^{-g(N)} d\lambda \leq \int (\nabla N) \cdot \left(\nabla \left(\frac{f^2}{|x|^2} \right) \right) e^{-g(N)} d\lambda$$

$$= \int (\nabla N) \cdot \left[2f \frac{\nabla f}{|x|^2} - \frac{2f^2 \nabla |x|}{|x|^3} \right] e^{-g(N)} d\lambda$$

$$= \int \frac{2f}{|x|^2} \nabla N \cdot \nabla f e^{-g(N)} d\lambda - 2 \int f^2 \frac{\nabla N \cdot x}{|x|^4} e^{-g(N)} d\lambda$$

$$\leq 2 \int \frac{f}{|x|^2} |\nabla N| |\nabla f| e^{-g(N)} d\lambda$$

$$\leq 2\sqrt{C} \int \frac{|f|}{N|x|} |\nabla f| e^{-g(N)} d\lambda.$$

Where the last two inequalities use the calculation of $\nabla N \cdot x$, from 3) and the upper bound on $|\nabla N|$ from 1).

Trial 1: Use Hardy's Inequality

Applying Cauchy's inequality with ϵ : $ab \leq \frac{\epsilon a^2}{2} + \frac{b^2}{2\epsilon}$ with $a = \frac{|f|}{N|x|}e^{-\frac{g(N)}{2}}$ and $b = \sqrt{C}|\nabla f|e^{-\frac{g(N)}{2}}$,

$$\int f^2 \left(A\frac{g'(N)}{N^2} - \frac{B}{N^3} \right) e^{-g(N)} d\lambda \leq \epsilon \int \frac{|f|^2}{N^2|x|^2} e^{-g(N)} d\lambda + \frac{C}{\epsilon} \int |\nabla f|^2 e^{-g(N)} d\lambda.$$

For $f \in C_0^\infty(\mathbb{R}^{n+m})$, we want to use Hardy's inequality:

$$\int \frac{f^2}{|x|^2} d\lambda \leq \frac{4}{(n-2)^2} \int |\nabla f|^2 d\lambda.$$

The grey term becomes:

$$\epsilon \int \frac{(\frac{fe^{\frac{-g(N)}{2}}}{N})^2}{|x|^2} d\lambda \leq \frac{4\epsilon}{(n-2)^2} \int |\nabla \frac{fe^{\frac{-g(N)}{2}}}{N}|^2 d\lambda$$

$$= \frac{\epsilon}{(n-2)^2} \int \frac{f^2 g'(N)^2}{N^2} |\nabla N|^2 d\mu + other \ \ terms.$$

This last term cannot be absorbed in the left-hand side of our $U-$Bound inequality, and Trial 1 fails.

Trial 2: Use Hardy's Inequality with $f \in C_0^\infty(B_R \times B_1)$

Using Hardy's inequality on the grey term:

$$\epsilon \int \frac{|f|^2}{N^2|x|^2} e^{-g(N)} d\lambda = \epsilon \int_{B_R \times B_{\mathbf{1}}} \frac{|f|^2}{N^2|x|^2} e^{-g(N)} d\lambda + \epsilon \int_{(B_R \times B_{\mathbf{1}})^c} \frac{|f|^2}{N^2|x|^2} e^{-g(N)} d\lambda$$

$$\leq \epsilon \int_{B_R \times B_{\mathbf{1}}} \frac{|f|^2}{|x|^2} d\lambda + \epsilon \int_{(B_R \times B_{\mathbf{1}})^c} \frac{|f|^2}{N^2|x|^2} e^{-g(N)} d\lambda$$

$$\leq \frac{4\epsilon}{(n-2)^2} \int_{B_R \times B_{\mathbf{1}}} |\nabla f|^2 d\lambda + \epsilon \int_{(B_R \times B_{\mathbf{1}})^c} \frac{|f|^2}{N^2|x|^2} e^{-g(N)} d\lambda$$

$$\leq \frac{4\epsilon C}{(n-2)^2} \int_{B_R \times B_{\mathbf{1}}} |\nabla f|^2 d\mu + \epsilon \int_{(B_R \times B_{\mathbf{1}})^c} \frac{|f|^2}{N^2|x|^2} e^{-g(N)} d\lambda.$$

Where the last line is true since we can bound $e^{-g(N)}$ from below on $B_R \times B_1$.
Regarding the complement:

$$(B_R \times B_1)^c = B_R^c \times B_1^c \cup B_R^c \times B_1 \cup B_R \times B_1^c.$$

On $B_R^c \times B_1^c$ and $B_R^c \times B_1$, we have $\frac{1}{|x|^2} \leq \frac{1}{R^2}$,so we avoid the singularity. However, on $B_R \times B_1^c$,we face the same problem as Trial 1.

Trial 3: Introduce $F = \{(x,z) \in \mathbb{R}^{n+m} : |x|\sqrt{g'(N)} < 1\}$

$$\int f^2 \left(\frac{Ag'(N)}{N^2} - \frac{B}{N^3} \right) e^{-g(N)} d\lambda$$

$$\leq \epsilon \int \frac{|f|^2}{N^2|x|^2} e^{-g(N)} d\lambda + \frac{C}{\epsilon} \int |\nabla f|^2 e^{-g(N)} d\lambda$$

$$= \epsilon \int_F \frac{|f|^2}{N^2|x|^2} e^{-g(N)} d\lambda + \epsilon \int_{F^c} \frac{|f|^2}{N^2|x|^2} e^{-g(N)} d\lambda + \frac{C}{\epsilon} \int |\nabla f|^2 e^{-g(N)} d\lambda$$

$$\leq \epsilon \int_F \frac{|fe^{\frac{-g(N)}{2}}|^2}{N^2|x|^2} d\lambda + \epsilon \int \frac{g'(N)|f|^2}{N^2} e^{-g(N)} d\lambda + \frac{C}{\epsilon} \int |\nabla f|^2 e^{-g(N)} d\lambda.$$

If $f \in C_0^\infty(F)$, we apply Hardy's inequality, and we are done. However, this is not the case, and we must consider the boundary term.

Final trial: Hardy's inequality on $F_r = \{(x,z) \in \mathbb{R}^{n+m} : |x|\sqrt{g'(N)} < r\}$, where $1 \le r \le 2$.

$$\epsilon \int_{F_r} \frac{|fe^{\frac{-g(N)}{2}}|^2}{N^2|x|^2}\, d\lambda = \frac{\epsilon}{n-2} \int_{F_r} \frac{|fe^{\frac{-g(N)}{2}}|^2}{N^2} \nabla \cdot \left(\frac{x}{|x|^2}\right) d\lambda$$

$$= -\frac{2\epsilon}{n-2} \int_{F_r} \frac{fe^{\frac{-g(N)}{2}}}{N} \nabla \left(\frac{fe^{\frac{-g(N)}{2}}}{N}\right) \cdot \frac{x}{|x|^2}\, d\lambda + boundary\ term$$

$$\le \frac{\epsilon}{2} \int_{F_r} \frac{|fe^{\frac{-g(N)}{2}}|^2}{N^2|x|^2}\, d\lambda + \frac{2\epsilon}{(n-2)^2} \int_{F_r} \left|\nabla \left(\frac{fe^{\frac{-g(N)}{2}}}{N}\right)\right|^2 d\lambda + boundary\ term.$$

Where we have used Integration by parts in the first line, Cauchy's inequality in the last line, and $boundary\ term = \dfrac{\epsilon}{n-2} \displaystyle\int_{\partial F_r} \frac{f^2 e^{-g(N)}}{N^2|x|^2} \sum_{j=1}^{n} \frac{x_j < X_j I, \nabla_{euc}\left(|x|\sqrt{g'(N)}\right) >}{\left|\nabla_{euc}\left(|x|\sqrt{g'(N)}\right)\right|} dH^{n+m-1}$. So,

$$\epsilon \int_{F_r} \frac{|fe^{\frac{-g(N)}{2}}|^2}{N^2|x|^2}\, d\lambda \le \frac{4\epsilon}{(n-2)^2} \int_{F_r} \left|\nabla \left(\frac{fe^{\frac{-g(N)}{2}}}{N}\right)\right|^2 d\lambda + boundary\ term.$$

Using the fact that $F \subset F_r \subset F_2$,

$$\epsilon \int_{F} \frac{|fe^{\frac{-g(N)}{2}}|^2}{N^2|x|^2}\, d\lambda \le \frac{4\epsilon}{(n-2)^2} \int_{F_2} \left|\nabla \left(\frac{fe^{\frac{-g(N)}{2}}}{N}\right)\right|^2 d\lambda + boundary\ term.$$

Recover the full measure using the Coarea formula

$$\epsilon \int_1^2 \int_F \frac{|fe^{\frac{-g(N)}{2}}|^2}{N^2|x|^2} d\lambda dr \leq \frac{4\epsilon}{(n-2)^2} \int_1^2 \int_{F_2} \left| \nabla \left(\frac{fe^{\frac{-g(N)}{2}}}{N} \right) \right|^2 d\lambda dr$$

$$+ \frac{2\epsilon}{n-2} \int_1^2 \int_{\partial F_r} \frac{f^2 e^{-g(N)}}{N^2|x|^2} \sum_{j=1}^{n} \frac{x_j < X_j I, \nabla_{euc} \left(|x|\sqrt{g'(N)} \right) >}{\left| \nabla_{euc} \left(|x|\sqrt{g'(N)} \right) \right|} dH^{n+m-1} dr$$

Where we have Integrated both sides of the inequality from $r = 1$ to $r = 2$. To recover the full measure in the boundary term, we use the Coarea formula:

$$\epsilon \int_F \frac{|fe^{\frac{-g(N)}{2}}|^2}{N^2|x|^2} d\lambda \leq \frac{4\epsilon}{(n-2)^2} \int_{F_2} \left| \nabla \left(\frac{fe^{\frac{-g(N)}{2}}}{N} \right) \right|^2 d\lambda$$

$$+ \frac{2\epsilon}{n-2} \int_{\{1<|x|\sqrt{g'(N)}<2\}} \frac{f^2 e^{-g(N)}}{N^2|x|^2} \sum_{j=1}^{n} x_j < X_j I, \nabla_{euc} \left(|x|\sqrt{g'(N)} \right) > d\lambda.$$

The remainder of the proof is to use the condition of the theorem, the technical lemma, the domain of integrations, and the given fields X_j,to find a suitable ϵ, which turns out to be satisfying $\left(\frac{10\epsilon}{n-2} + \epsilon \right) < A$. (Recall: 1) $A\frac{|x|^2}{N^2} \leq |\nabla N|^2 \leq C\frac{|x|^2}{N^2}$.)

Thanks for your attention!