Time averages for kinetic Fokker-Planck equations

EFI project workshop - Université Paris-Dauphine

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13/10/2021

About myself and the project

- I was born in Pavia, Italy on 11/12/1995.
- PhD student at CEREMADE – Université Paris-Dauphine. Director: Jean Dolbeault. Co-director: Giuseppe Savaré (Università Bocconi, Milano).
- This research project is funded by the European Union's Horizon 2020 research and innovation program under the Marie Sklodowska-Curie grant agreement No 754362.
- Partial support has been obtained from the EFI ANR-17-CE40-0030 Project of the French National Research Agency.
- Preprint version available at `https://arxiv.org/pdf/2106.12801.pdf`.

Outline

1. Statement of the problem
2. Scientific background and quick bibliography
3. Results

Notation and setting

- Let $x \in \mathbb{T}^d = [0, L]^d$, $v \in \mathbb{R}^d$, and $t \in (0, \infty)$ be the space, velocity, and time variables, respectively.
- Call

$$\gamma = (2\pi)^{-\frac{d}{2}} \, \mathrm{e}^{-\frac{|v|^2}{2}}.$$

- Functional spaces:
  1. Velocity regularity: weighted L^2-space L^2_γ. Corresponding Sobolev space H^1_γ with dual H^{-1}_γ.
  2. Space-time regularity: $\mathrm{L}^2(dt\,dx)$ space, with usual Sobolev space H^1_0 and dual H^{-1}.
- Ornstein-Uhlenbeck operator

$$\Delta_\gamma = \Delta_v - v \cdot \nabla_v.$$

The Ornstein-Uhlenbeck operator

$$\Delta_\gamma h = \Delta_v h - v \cdot \nabla_v h.$$

Properties:

1. $-\Delta_\gamma$ is positive semi-definite (kernel = constants), self adjoint with compact resolvent over L^2_γ.
2. Intregration by parts:

$$-\int \Delta_\gamma h \, k \, d\gamma = \int \nabla_v h \cdot \nabla_v k \, d\gamma.$$

3. Spectrum = $\mathbb{N}$. In particular

$$\int |\nabla_v h|^2 \, d\gamma = -\int h \Delta_\gamma h \, d\gamma \geq \|h\|^2_{\mathrm{L}^2_\gamma},$$

for all null-avg functions (Nash 1965).

Vlasov-Ornstein-Uhlenbeck equation

Vlasov-Ornstein-Uhlenbeck equation:

$$\begin{cases} \partial_t h_t + v \cdot \nabla_x h_t = \Delta_v h_t - v \cdot \nabla_v h_t =: \Delta_\gamma h_t; \\ h(t=0,x,v) = h_0(x,v) \in \mathrm{L}^2(dx\, d\gamma). \end{cases} \tag{VOU}$$

Structural properties:

- Linear, mass-preserving equation (henceforth null-avg data considered!).
- Existence and uniqueness of solutions (Green's function: Kolmogorov 1934 or BDMMS 2019).
- Elliptic regularity in velocity.
- Hypo-elliptic (Hörmander 1967) structure overall

$$[v \cdot \nabla_x, \nabla_v] = \nabla_x.$$

Main question

- The (VOU) equation generates a linear **flow** on $\mathrm{L}^2(dx\, d\gamma)$.
- Only equilibrium point for null-avg solution is

$$h_\infty = 0.$$

Question: convergence to equilibrium? Explicit decay rates?

Degeneracy and hypocoercivity

- The equation is degenerate diffusive, so the standard **entropy** method fails:

$$\frac{1}{2}\frac{d}{dt}\int h_t^2\,dxd\gamma = \int h\Delta_\gamma h\,dxd\gamma = -\int |\nabla_v h|^2\,dxd\gamma \le 0.$$

  Control/coercivity in velocity directions, miss the control in x.
- Hypo-ellipticity:
  commutators $\Longrightarrow$ interaction between v and x.
  Can we do it for coercivity?
- **Hypocoercivity** = coercivity in a twisted norm.

Strategy of hypocoercivity

1. Fix a reference norm, as L^2 or H^1.
2. Twist/perturb the norm and find an equivalent norm H such that
$$\frac{d}{dt}H(h_t) = -\mathcal{D}^2(h_t) \leq -\lambda H(h_t),$$
along the **flow** of (VOU).
3. Exponential decay in H, Gronwall's **inequality**.
4. Equivalence in reference norm:
$$\|h_t\|^2 \leq C\,\mathrm{e}^{-\lambda t}\,\|h_0\|^2.$$
Hypocoercivity constant $C > 1$.

Quick review of the literature

- → Decay of H^1 norm. Villani 2006, Dolbeault-Li 2018, Baudoin 2014-2019, ...
- → Decay of L^2 norm. Dolbeault-Mouhot-Schmeiser 2015, Bouin-Dolbeault-Mouhot-Mischler-Schmesier 2019, Arnold-Erb 2014, Achleitner-Arnold-Carlen 2016-2018, Arnold-Dolbeault-Schmeiser-Wöhrer 2021, Hérau 2005, Bernard-Fathi-Levitt-Stoltz 2020,...
- → Usage of weak norms. Armstrong-Mourrat 2019, Cao-lu-Wang 2021, ...

Strategy of time averages

Evolution not coercive, but on (time) average it is!

Construction:

1. Lions' lemma (Amrouche-alii 2015).
2. Averaging Lemma (Armstrong-Mourrat 2019-B. 2021)
3. Modified Poincaré **inequality**
4. Explicit and constructive decay rates

Proof 1

Fix $\tau > 0$ small enough.
Define

$$H_\tau(h_t) = \tau^{-1} \int_t^{t+\tau} \|h_s\|^2_{\mathrm{L}^2(dx\,d\gamma)}\,ds.$$

Energy estimate:

$$\frac{d}{dt} \int_t^{t+\tau} \|h(s,\cdot,\cdot)\|^2_{\mathrm{L}^2(dx\,d\gamma)}\,ds = -2 \int_t^{t+\tau} \|\nabla_v h(s,\cdot,\cdot)\|^2_{\mathrm{L}^2(dx\,d\gamma)}\,ds.$$

Exponential decay if:

$$\int_t^{t+\tau} \|\nabla_v h(s,\cdot,\cdot)\|^2_{\mathrm{L}^2(dx\,d\gamma)}\,ds \geq \lambda \int_t^{t+\tau} \|h(s,\cdot,\cdot)\|^2_{\mathrm{L}^2(dx\,d\gamma)}\,ds.$$

First passage

Let $\rho(t,x) = \int h(t,x,v)\, d\gamma$. Then,

$$\int_t^{t+\tau} \|h(s,\cdot,\cdot)\|^2_{\mathrm{L}^2(dx\, d\gamma)}\, ds = \|\rho\|^2_{\mathrm{L}^2(dt\, dx)} + \|h - \rho\|^2_{\mathrm{L}^2(dt\, dx\, d\gamma)}.$$

Which we control with Gauss-Poincaré **inequality** with

$$\|\rho\|^2_{\mathrm{L}^2(dt\, dx)} + \|\nabla_v h\|^2_{\mathrm{L}^2(dt\, dx\, d\gamma)}.$$

The second term is ok. What about the first one?

Lions' Lemma

Lemma

Let $\mathcal{O}$ be a bounded, open and Lipschitz-regular subset in $\mathbb{R}^{d+1}$. Then, for all $u \in \mathcal{D}^*(\mathcal{O})$, we have that $u \in \mathrm{L}^2(\mathcal{O})$ if and only if the weak gradient ∇u belongs to $\mathrm{H}^{-1}(\mathcal{O})$. Moreover, there exists a constant $C_L(\mathcal{O})$ such that

$$\left\| u - \int_{\mathcal{O}} u \, dx \, dt \right\|^2_{\mathrm{L}^2(\mathcal{O})} \leq C_L \|\nabla u\|^2_{H^{-1}(\mathcal{O})},$$

for any $u \in \mathrm{L}^2(\mathcal{O})$.

Second passage

Let $\mathcal{O} = (t, t+\tau) \times \mathbb{T}^d$. Use Lions' Lemma to get

$$\|\rho\|^2_{\mathrm{L}^2(dt\,dx)} \leq C_L \|\nabla_{t,x}\rho\|^2_{\mathrm{H}^{-1}}.$$

Very explicit structure of Lions' constant:

$$C_L = 4\,|S^d|\,\frac{\sqrt{d\,L^2 + \tau^2}}{\tau}. \tag{1}$$

Averaging Lemma

Lemma

We have

$$\|\nabla_{t,x}\rho\|^2_{\mathrm{H}^{-1}(\Omega)} \leq$$
$$d_2\left(\|h-\rho\|^2_{\mathrm{L}^2(dt\,dx\,d\gamma_\alpha)} + \|\partial_t h + v\cdot\nabla_x h\|^2_{\mathrm{L}^2(\Omega;H^{-1}_\alpha)}\right)$$

with

$$d_2 = 2\left(\|v_1|v|^2\|^2_{\mathrm{L}^2_\gamma} + \left(1+\frac{L^2}{4\pi^2}\right)\||v|^2\|^2_{\mathrm{L}^2_\gamma} + \frac{d^2L^2}{4\pi^2}\|v\|^2_{\mathrm{L}^2_\gamma}\right). \quad (2)$$

Modified Poincaré inequality

Using the (VOU) equation ...

Lemma

$$\|(\partial_t + v \cdot \nabla_x)\, h\|_{\mathrm{L}^2(\Omega;\mathrm{H}_\alpha^{-1})} \leq \|\nabla_v h\|_{\mathrm{L}^2(\Omega;\mathrm{L}_\alpha^2)}. \tag{3}$$

... we recover Poincaré inequality *on average*!

Proposition

Zero average solutions to (VOU) satisfy

$$\|h\|^2_{\mathrm{L}^2(dt\,dx\,d\gamma)} \leq \kappa_2\, \|\nabla_v h\|^2_{\mathrm{L}^2(dt\,dx\,d\gamma)}, \tag{4}$$

with $\kappa_2 = 2(1 + C_L\, d_2)$.

The result

Theorem

Null-average solutions to (VOU) satisfy

$$\tau^{-1}\int_t^{t+\tau}\|h(s,\cdot,\cdot)\|^2_{\mathrm{L}^2(dx\,d\gamma)}\,ds \leq \|h_0\|^2_{\mathrm{L}^2(dx\,d\gamma)}\,e^{-\lambda\,t},\quad \forall\, t\geq 0, \tag{5}$$

with

$$\frac{1}{\lambda}=2\,\frac{1}{\tau}\left(\tau+\sqrt{d\,L^2+\tau^2}\right)\left(2\,d_2\,|\mathbb{S}^{d-1}|\right).$$

Generalizations

Let $\alpha > 0$. Consider

$$\partial_t h + v \cdot \nabla_x h = \Delta_\alpha h, \qquad h(0,\cdot,\cdot) = h_0, \tag{6}$$

with

$$\Delta_\alpha h := \Delta_v h - \alpha\, v\, \langle v \rangle^{\alpha-2} \cdot \nabla_v h.$$

We use the notation

$$\langle v \rangle = \sqrt{1 + |v|^2}, \quad \forall\, v \in \mathbb{R}^d.$$

Local equilibria in velocity:

$$\gamma_\alpha(v) = \frac{1}{Z_\alpha}\, \mathrm{e}^{-\langle v \rangle^\alpha}, \quad \forall\, v \in \mathbb{R}^d.$$

Exponential and super-exponential case

Theorem (B. 2021)

Let $\alpha \geq 1$. Then, for all $L > 0$ and $\tau > 0$, there exists an explicit and constructive constant $\lambda > 0$ such that, for all $h_0 \in \mathrm{L}^2(dx\, d\gamma_\alpha)$ with zero-average, the solution to (6) satisfies

$$\tau^{-1} \int_t^{t+\tau} \|h(s,\cdot,\cdot)\|^2_{\mathrm{L}^2(dx\, d\gamma_\alpha)}\, ds \leq \|h_0\|^2_{\mathrm{L}^2(dx\, d\gamma_\alpha)}\, e^{-\lambda\, t}, \quad \forall\, t \geq 0. \tag{7}$$

Fat tails

Theorem (B. 2021)

Let $\alpha \in (0,1)$ *Then, for all* $L > 0$ *and* $\tau > 0$*, for all* $\sigma > 0$*, there exists an explicit and constructive constant* $K > 0$ *such that all solutions to* (6) *decay according to*

$$\tau^{-1} \int_t^{t+\tau} \|h(s,\cdot,\cdot)\|^2_{\mathrm{L}^2(dx\, d\gamma_\alpha)}\, ds \leq$$

$$\leq\ K\,(1+t)^{-\frac{\sigma}{2\,(1-\alpha)}} \iint_{Q\times\mathbb{R}^d} \langle v\rangle^\sigma\, h_0^2\, dx\, d\gamma_\alpha, \quad \forall\, t \geq 0.$$

Thank you for your attention.