Uniform in time propagation of chaos for the 2D vortex model and other singular stochastic systems.

Pierre Le Bris
Joint work with : Arnaud Guillin (LMBP), Pierre Monmarché (LJLL)

LJLL, Sorbonne Université - Paris

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I. Introduction

Idea

In a system of N interacting particles, as N increases, two particles become more and more statistically independent.

Formal limit of SDE

N-particle system on the torus $\mathbb{T}^d$

$$dX_t^i = \sqrt{2}dB_t^i + \frac{1}{N}\sum_{j=1}^{N} K(X_t^i - X_t^j)dt.$$

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$$dX_t^i = \sqrt{2}dB_t^i + K * \mu_t^N(X_t^i)dt,$$

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$$\begin{cases} d\bar{X}_t = \sqrt{2} dB_t + K * \bar{\rho}_t(\bar{X}_t) dt, \\ \bar{\rho}_t = \text{Law}(\bar{X}_t). \end{cases}$$

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N-particle system on the torus $\mathbb{T}^d$

$$dX_t^i = \sqrt{2}dB_t^i + \frac{1}{N}\sum_{j=1}^{N} K(X_t^i - X_t^j)dt. \quad \text{(PS)}$$

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Liouville equations

For the particle system

$$dX_t^i = \sqrt{2}dB_t^i + \frac{1}{N}\sum_{j=1}^N K(X_t^i - X_t^j)dt$$

$$\longleftrightarrow$$

$$\partial_t \rho_t^N = -\sum_{i=1}^N \nabla_{x_i} \cdot \left(\left(\frac{1}{N}\sum_{j=1}^N K(x_i - x_j) \right) \rho_t^N \right) + \sum_{i=1}^N \Delta_{x_i} \rho_t^N.$$

For the non linear equation

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Main example : 2D vortex model

The Biot-Savart kernel, defined in $\mathbb{R}^2$ by

$$K(x) = \frac{1}{2\pi}\frac{x^\perp}{|x|^2} = \frac{1}{2\pi}\left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}\right).$$

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Consider the 2D incompressible Navier-Stokes system on $u \in \mathbb{R}^2$

$$\begin{aligned} \partial_t u =& - u \cdot \nabla u - \nabla p + \Delta u \\ \nabla \cdot u =& 0, \end{aligned}$$

where p is the local pressure.

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Goal : Obtain a limit "$\rho_t^N \to \bar\rho_t$" as N tends to infinity for this Biot-Savart kernel.

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To quantify this "more and more", we compare the law of any subset of k particles within the N particles system to the law of k independent non-linear particles.
We denote, for any $k \leq N$

$$\rho_t^{k,N}(x_1, .., x_k) = \int_{\mathbb{T}^{(N-k)d}} \rho_t^N(x_1, .., x_N) dx_{k+1}...dx_N$$
$$\bar{\rho}_t^k = \bar{\rho}_t^{\otimes k}$$

(Rescaled) relative entropy

Definition

Let μ and ν be two probability measures on $\mathbb{T}^{dN}$. We consider the rescaled relative entropy

$$\mathcal{H}_N(\nu,\mu) = \begin{cases} \frac{1}{N}\mathbb{E}_\mu\left(\frac{d\nu}{d\mu}\log\frac{d\nu}{d\mu}\right) & \text{if } \nu \ll \mu, \\ +\infty & \text{otherwise.} \end{cases}$$

Results

Theorem (adapted from Jabin-Wang ('18))

Under some assumptions (satisfied by the Biot-Savart kernel) there are constants C_1 and C_2 such that for all $N \in \mathbb{N}$, all exchangeable probability density ρ_0^N and all $t \geq 0$

$$\mathcal{H}_N(\rho_t^N, \bar{\rho}_t^N) \leq e^{C_1 t} \left(\mathcal{H}_N(\rho_0^N, \bar{\rho}_0^N) + \frac{C_2}{N} \right)$$

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Theorem (Guillin-LB-Monmarché ('21))

Under some assumptions (satisfied by the Biot-Savart kernel) there are constants C_1, C_2 and C_3 such that for all $N \in \mathbb{N}$, all exchangeable probability density ρ_0^N and all $t \geq 0$

$$\mathcal{H}_N(\rho_t^N, \bar{\rho}_t^N) \leq C_1 e^{-C_2 t}\mathcal{H}_N(\rho_0^N, \bar{\rho}_0^N) + \frac{C_3}{N}$$

Various distances

For $\mathbf{x} = (x_i)_{i \in [\![1,N]\!]} \in \mathbb{T}^{dN}$, we write $\pi(\mathbf{x}) = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$ the associated empirical measure.

Corollary

Under some assumptions (satisfied by the Biot-Savart kernel), assuming moreover that $\rho_0^N = \bar{\rho}_0^N$, there is a constant C such that for all $k \leq N \in \mathbb{N}$ and all $t \geq 0$,

$$\|\rho_t^{k,N} - \bar{\rho}_t^k\|_{L^1} + \mathcal{W}_2\left(\rho_t^{k,N}, \bar{\rho}_t^k\right) \leq C\left(\left\lfloor \frac{N}{k} \right\rfloor\right)^{-\frac{1}{2}}$$

and

$$\mathbb{E}_{\rho_t^N}\left(\mathcal{W}_2(\pi(\boldsymbol{X}), \bar{\rho}_t)\right) \leqslant C\alpha(N)$$

where $\alpha(N) = N^{-1/2}\ln(1+N)$ if $d = 2$ and $\alpha(N) = N^{-1/d}$ if $d > 2$.

II. Proof

Step one : Time evolution of the relative entropy

We write

$$\mathcal{H}_N(t) = \mathcal{H}_N(\rho_t^N, \bar{\rho}_t^N), \;\; \mathcal{I}_N(t) = \frac{1}{N}\sum_i \int_{\mathbb{T}^{dN}} \rho_t^N \left| \nabla_{x_i} \log \frac{\rho_t^N}{\bar{\rho}_t^N} \right|^2 d\mathbf{X}^N.$$

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It has been shown, by Jabin-Wang, that

$$\begin{aligned}\frac{d}{dt}\mathcal{H}_N(t) \leq & -\mathcal{I}_N(t) \\ & - \frac{1}{N^2}\sum_{i,j} \int_{\mathbb{T}^{dN}} \rho_t^N \left(K(x_i - x_j) - K * \rho(x_i)\right) \cdot \nabla_{x_i} \log \bar{\rho}_t^N d\mathbf{X}^N \\ & - \frac{1}{N^2}\sum_{i,j} \int_{\mathbb{T}^{dN}} \rho_t^N \left(\text{div } K(x_i - x_j) - \text{div } K * \bar{\rho}_t(x_i)\right) d\mathbf{X}^N.\end{aligned}$$

Assumptions ?

$$\textbf{Goal :} \; K(x) = \frac{1}{2\pi}\frac{x^{\perp}}{|x|^2} = \frac{1}{2\pi}\left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}\right)$$

Justifying the calculations

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- $\bar\rho \in \mathcal{C}^\infty(\mathbb{R}^+ \times \mathbb{T}^d)$ and there is $\lambda > 1$, s.t $\frac{1}{\lambda} \leq \bar\rho \leq \lambda$

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Dealing with the terms

- In the sense of distributions, $\nabla \cdot K = 0$.

Step one : Time evolution of the relative entropy

We write

$$\mathcal{H}_N(t) = \mathcal{H}_N(\rho_t^N, \bar{\rho}_t^N), \;\; \mathcal{I}_N(t) = \frac{1}{N}\sum_i \int_{\mathbb{T}^{dN}} \rho_t^N \left| \nabla_{x_i} \log \frac{\rho_t^N}{\bar{\rho}_t^N} \right|^2 d\mathbf{X}^N.$$

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Step two : Integration by part

We are left with

$$\frac{d}{dt}\mathcal{H}_N(t) \leq -\mathcal{I}_N(t) \\ - \frac{1}{N^2}\sum_{i,j}\int_{\mathbb{T}^{dN}} \rho_t^N \left(K(x_i - x_j) - K * \rho(x_i)\right) \cdot \nabla_{x_i} \log \bar{\rho}_t^N d\mathbf{X}^N.$$

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Idea : Use the regularity of $\bar{\rho}$ to deal with the singularity of K
Remark : Notice that, for the Biot-Savart kernel on the whole space $\mathbb{R}^2$

$$\tilde{K}(x) = \frac{1}{2\pi}\frac{x^\perp}{|x|^2},$$

we have $\tilde{K} = \nabla \cdot \tilde{V}$ with

$$\tilde{V}(x) = \frac{1}{2\pi}\begin{pmatrix} -\arctan\left(\frac{x_1}{x_2}\right) & 0 \\ 0 & \arctan\left(\frac{x_2}{x_1}\right) \end{pmatrix}.$$

Assumptions ?

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Justifying the calculations

- There is $\lambda > 1$ such that $\bar\rho_0 \in \mathcal{C}^\infty_\lambda(\mathbb{T}^d)$
  $\implies \bar\rho \in \mathcal{C}^\infty_\lambda(\mathbb{R}^+ \times \mathbb{T}^d)$ (Ben-Artzi '94)
- $\rho^N \in \mathcal{C}^\infty_\lambda(\mathbb{R}^+ \times \mathbb{T}^{Nd})$ (???)

Dealing with the terms

- In the sense of distributions, $\nabla \cdot K = 0$.
- There is a matrix field $V \in L^\infty$ such that $K = \nabla \cdot V$, i.e for $1 \leq \alpha \leq d$, $K_\alpha = \sum_{\beta=1}^d \partial_\beta V_{\alpha,\beta}$ (Phuc-Torres '08).

Step two : Integration by part

For all $t \geqslant 0$,

$$\frac{d}{dt}\mathcal{H}_N(t) \leq A_N(t) + \frac{1}{2}B_N(t) - \frac{1}{2}\mathcal{I}_N(t),$$

with

$$A_N(t) := \frac{1}{N^2}\sum_{i,j}\int_{\mathbb{T}^{dN}} \rho_t^N \left(V(x_i - x_j) - V * \bar{\rho}(x_i)\right) : \frac{\nabla^2_{x_i}\bar{\rho}_t^N}{\bar{\rho}_t^N} d\mathbf{X}^N$$

$$B_N(t) := \frac{1}{N}\sum_{i}\int_{\mathbb{T}^{dN}} \rho_t^N \frac{\left|\nabla_{x_i}\bar{\rho}_t^N\right|^2}{|\bar{\rho}_t^N|^2} \left|\frac{1}{N}\sum_j V(x_i - x_j) - V * \bar{\rho}(x_i)\right|^2 d\mathbf{X}^N.$$

Step three : Change of reference measure and large deviation estimates

Lemma

For two probability densities μ and ν on a set Ω, and any $\Phi \in L^\infty(\Omega)$, $\eta > 0$ and $N \in \mathbb{N}$,

$$\mathbb{E}^\mu \Phi \leq \eta \mathcal{H}_N(\mu, \nu) + \frac{\eta}{N} \log \mathbb{E}^\nu e^{N\Phi/\eta}.$$

Large deviation estimates -1

Theorem (Jabin-Wang '18)

Consider any probability measure μ on $\mathbb{T}^d$, $\epsilon > 0$ and a scalar function $\psi \in L^\infty(\mathbb{T}^d \times \mathbb{T}^d)$ with $\|\psi\|_{L^\infty} < \frac{1}{2\epsilon}$ and such that for all $z \in \mathbb{T}^d$, $\int_{\mathbb{T}^d} \psi(z,x)\mu(dx) = 0$. Then there exists a constant C such that

$$\int_{\mathbb{T}^{dN}} \exp\left(\frac{1}{N}\sum_{j_1,j_2=1}^{N} \psi(x_1,x_{j_1})\psi(x_1,x_{j_2})\right)\mu^{\otimes N} d\boldsymbol{X}^N \leq C,$$

where C depends on

$$\alpha = (\epsilon\|\psi\|_{L^\infty})^4 < 1\ ,\ \ \beta = \left(\sqrt{2\epsilon}\|\psi\|_{L^\infty}\right)^4 < 1.$$

Large deviation estimates -2

Theorem (Jabin-Wang '18)

Consider any probability measure μ on $\mathbb{T}^d$ and $\phi \in L^\infty(\mathbb{T}^d \times \mathbb{T}^d)$ with

$$\gamma := \left(1600^2 + 36e^4\right)\left(\sup_{p\geq 1} \frac{\| \sup_z |\phi(\cdot, z)| \|_{L^p(\mu))}}{p}\right)^2 < 1.$$

Assume that ϕ satisfies the following cancellations

$$\forall z \in \mathbb{T}^d, \quad \int_{\mathbb{T}^d} \phi(x,z)\mu(dx) = 0 = \int_{\mathbb{T}^d} \phi(z,x)\mu(dx)\,.$$

Then, for all $N \in \mathbb{N}$,

$$\int_{\mathbb{T}^{dN}} \exp\left(\frac{1}{N}\sum_{i,j=1}^N \phi(x_i,x_j)\right)\mu^{\otimes N} d\boldsymbol{X}^N \leq \frac{2}{1-\gamma} < \infty.$$

Conclusion

For all $t \geqslant 0$,

$$\frac{d}{dt}\mathcal{H}_N(t) \leq C\left(\mathcal{H}_N(t) + \frac{1}{N}\right) - \frac{1}{2}\mathcal{I}_N(t),$$

with

$$C = \hat{C}_1 \|\nabla^2 \bar{\rho}_t\|_{L^\infty} \|V\|_{L^\infty} \lambda + \hat{C}_2 \|V\|_{L^\infty}^2 \lambda^2 d^2 \|\nabla \bar{\rho}_t\|_{L^\infty}^2$$

where $\hat{C}_1, \hat{C}_2$ are universal constants.

Step four : Uniform bounds and logarithmic Sobolev inequality

Two goals :

- A logarithmic Sobolev inequality for $\bar{\rho}^N$: $\mathcal{H}_N(t) \leq C\mathcal{I}_N(t)$

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- A logarithmic Sobolev inequality for $\bar{\rho}^N$: $\mathcal{H}_N(t) \leq C\mathcal{I}_N(t)$
- Uniform in time bounds on $\|\nabla \bar{\rho}_t\|_{L^\infty}$ and $\|\nabla^2 \bar{\rho}_t\|_{L^\infty}$

A logarithmic Sobolev inequality

Lemma (Tensorization)

If ν is a probability measure on $\mathbb{T}^d$ satisfying a LSI with constant C_ν^{LS}, then for all N ≥ 0, $\nu^{\otimes N}$ satisfies a LSI with constant C_ν^{LS}

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If ν is a probability measure on $\mathbb{T}^d$ satisfying a LSI with constant C_ν^{LS}, then for all $N \geq 0$, $\nu^{\otimes N}$ satisfies a LSI with constant C_ν^{LS}

Lemma (Perturbation)

If ν is a probability measure on $\mathbb{T}^d$ satisfying a LSI with constant C_ν^{LS}, and μ is a probability measure with density h with respect to ν such that, for some constant $\lambda > 0$, $\frac{1}{\lambda} \leq h \leq \lambda$, then μ satisfies a LSI with constant $C_\mu^{LS} = \lambda^2 C_\nu^{LS}$.

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Lemma (LSI for the uniform distribution)

The uniform distribution u on $\mathbb{T}^d$ satisfies a LSI with constant $\frac{1}{8\pi^2}$.

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Lemma (LSI for the uniform distribution)

The uniform distribution u on $\mathbb{T}^d$ satisfies a LSI with constant $\frac{1}{8\pi^2}$.

For all $N \in \mathbb{N}$, $t \geq 0$ and all probability density $\mu_N \in \mathcal{C}^\infty_{>0}(\mathbb{T}^{dN})$,

$$\mathcal{H}_N\left(\mu_N, \bar{\rho}_t^N\right) \leq \frac{\lambda^2}{8\pi^2}\frac{1}{N}\sum_{i=1}^N \int_{\mathbb{T}^d} \mu_N \left|\nabla_{x_i} \log \frac{\mu_N}{\bar{\rho}_t^N}\right|^2 d\mathbf{X}^N$$

Uniform in time bounds on the derivatives

Lemma

For all $n \geqslant 1$ *and* $\alpha_1, ..., \alpha_n \in [\![1, d]\!]$, *there exist* $C_n^u, C_n^\infty > 0$ *such that for all* $t \geqslant 0$,

$$\|\partial_{\alpha_1,\ldots,\alpha_n}\bar{\rho}_t\|_{L^\infty} \leq C_n^u \quad \textit{and} \quad \int_0^t \|\partial_{\alpha_1,\ldots,\alpha_n}\bar{\rho}_s\|_{L^\infty}^2 ds \leq C_n^\infty$$

Uniform in time bounds on the derivatives

Lemma

For all $n \geqslant 1$ and $\alpha_1, ..., \alpha_n \in [\![1, d]\!]$, there exist $C_n^u, C_n^\infty > 0$ such that for all $t \geqslant 0$,

$$\|\partial_{\alpha_1,\dots,\alpha_n}\bar{\rho}_t\|_{L^\infty} \leq C_n^u \quad \text{and} \quad \int_0^t \|\partial_{\alpha_1,\dots,\alpha_n}\bar{\rho}_s\|_{L^\infty}^2 ds \leq C_n^\infty$$

Thanks to Morrey's inequality and Sobolev embeddings, it is sufficient to prove such bounds in the Sobolev space H^m for all m, i.e in L^2

Uniform in time bounds on the derivatives-2

By induction on the order of the derivative

$$\frac{1}{2}\frac{d}{dt}\|\bar{\rho}_t\|_{L^2}^2 + \|\nabla\bar{\rho}_t\|_{L^2}^2 = 0,$$

Uniform in time bounds on the derivatives-2

By induction on the order of the derivative

$$\frac{1}{2}\frac{d}{dt}\|\bar{\rho}_t\|_{L^2}^2 + \|\nabla\bar{\rho}_t\|_{L^2}^2 = 0,$$

$$\frac{1}{2}\frac{d}{dt}\|\partial_{\alpha_1}\bar{\rho}_t\|_{L^2}^2 + \frac{1}{2}\sum_{\alpha_2}\|\partial_{\alpha_1,\alpha_2}\bar{\rho}_t\|_{L^2}^2 \leq \frac{1}{2}\|K\|_{L^1}^2\|\bar{\rho}_t\|_{L^\infty}^2\|\nabla\bar{\rho}_t\|_{L^2}^2,$$

Uniform in time bounds on the derivatives-2

By induction on the order of the derivative

$$\frac{1}{2}\frac{d}{dt}\|\bar{\rho}_t\|_{L^2}^2 + \|\nabla\bar{\rho}_t\|_{L^2}^2 = 0,$$

$$\frac{1}{2}\frac{d}{dt}\|\partial_{\alpha_1}\bar{\rho}_t\|_{L^2}^2 + \frac{1}{2}\sum_{\alpha_2}\|\partial_{\alpha_1,\alpha_2}\bar{\rho}_t\|_{L^2}^2 \leq \frac{1}{2}\|K\|_{L^1}^2\|\bar{\rho}_t\|_{L^\infty}^2\|\nabla\bar{\rho}_t\|_{L^2}^2,$$

$$\frac{1}{2}\frac{d}{dt}\|\partial_{\alpha_1,\alpha_2}\bar{\rho}_t\|_{L^2}^2 + \frac{1}{2}\sum_{\alpha_3}\|\partial_{\alpha_1,\alpha_2,\alpha_3}\bar{\rho}_t\|_{L^2}^2 \leq \|V\|_{L^\infty}^2\|\partial_{\alpha_1}\nabla\bar{\rho}_t\|_{L^2}^2\|\nabla\bar{\rho}_t\|_{L^2}^2$$

$$+\|K\|_{L^1}^2\|\bar{\rho}_t\|_{L^\infty}^2\|\partial_{\alpha_1}\nabla\bar{\rho}_t\|_{L^2}^2,$$

Uniform in time bounds on the derivatives-2

By induction on the order of the derivative

$$\frac{1}{2}\frac{d}{dt}\|\bar\rho_t\|_{L^2}^2 + \|\nabla\bar\rho_t\|_{L^2}^2 = 0,$$

$$\frac{1}{2}\frac{d}{dt}\|\partial_{\alpha_1}\bar\rho_t\|_{L^2}^2 + \frac{1}{2}\sum_{\alpha_2}\|\partial_{\alpha_1,\alpha_2}\bar\rho_t\|_{L^2}^2 \leq \frac{1}{2}\|K\|_{L^1}^2\|\bar\rho_t\|_{L^\infty}^2\|\nabla\bar\rho_t\|_{L^2}^2,$$

$$\frac{1}{2}\frac{d}{dt}\|\partial_{\alpha_1,\alpha_2}\bar\rho_t\|_{L^2}^2 + \frac{1}{2}\sum_{\alpha_3}\|\partial_{\alpha_1,\alpha_2,\alpha_3}\bar\rho_t\|_{L^2}^2 \leq \|V\|_{L^\infty}^2\|\partial_{\alpha_1}\nabla\bar\rho_t\|_{L^2}^2\|\nabla\bar\rho_t\|_{L^2}^2 + \|K\|_{L^1}^2\|\bar\rho_t\|_{L^\infty}^2\|\partial_{\alpha_1}\nabla\bar\rho_t\|_{L^2}^2,$$

etc

Assumptions ?

$$\textbf{Goal :} K(x) = \frac{1}{2\pi}\frac{x^\perp}{|x|^2} = \frac{1}{2\pi}\left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}\right)$$

Justifying the calculations

- There is $\lambda > 1$ such that $\bar\rho_0 \in \mathcal{C}^\infty_\lambda(\mathbb{T}^d)$ $\implies \bar\rho \in \mathcal{C}^\infty_\lambda(\mathbb{R}^+ \times \mathbb{T}^d)$ (Ben-Artzi '94)
- $\rho^N \in \mathcal{C}^\infty_\lambda(\mathbb{R}^+ \times \mathbb{T}^{Nd})$ (? ? ?)

Dealing with the terms

- In the sense of distributions, $\nabla \cdot K = 0$.
- There is a matrix field $V \in L^\infty$ such that $K = \nabla \cdot V$, i.e for $1 \leq \alpha \leq d$, $K_\alpha = \sum_{\beta=1}^d \partial_\beta V_{\alpha,\beta}$ (Phuc-Torres '08).

Uniformity in time

- For all $n \geq 1$, $C^0_n := \|\nabla^n \bar\rho_0\|_{L^\infty} < \infty$
- $\|K\|_{L^1} < \infty$ (also used to show regularity).

Step five : Conclusion

There are constants $C_1, C_2^\infty, C_3 > 0$ and a function $t \mapsto C_2(t) > 0$ with $\int_0^t C_2(s)ds \leq C_2^\infty$ for all $t \geq 0$ such that for all $t \geq 0$

$$\frac{d}{dt}\mathcal{H}_N(t) \leq -(C_1 - C_2(t))\mathcal{H}_N(t) + \frac{C_3}{N}.$$

Multiplying by $\exp(C_1 t - \int_0^t C_2(s)ds)$ and integrating in time we get

$$\mathcal{H}_N(t) \leq e^{-C_1 t + \int_0^t C_2(s)ds}\mathcal{H}_N(0) + \frac{C_3}{N}\int_0^t e^{C_1(s-t)+\int_s^t C_2(u)du}ds$$
$$\leq e^{C_2^\infty - C_1 t}\mathcal{H}_N(t) + \frac{C_3}{C_1 N}e^{C_2^\infty},$$

which concludes.

On $\rho^N \in \mathcal{C}^\infty_\lambda(\mathbb{R}^+ \times \mathbb{T}^{Nd})$

Everything works for regularized kernels K^ϵ, and the final result is independent of ϵ.

Assumptions

On the initial condition

- There is $\lambda > 1$ such that $\bar{\rho}_0 \in \mathcal{C}^\infty_\lambda(\mathbb{T}^d)$
- For all $n \geq 1$, $C^0_n := \|\nabla^n \bar{\rho}_0\|_{L^\infty} < \infty$

On the potential K

- $\|K\|_{L^1} < \infty$.
- In the sense of distributions, $\nabla \cdot K = 0$,
- There is a matrix field $V \in L^\infty$ such that $K = \nabla \cdot V$, i.e for $1 \leq \alpha \leq d$, $K_\alpha = \sum_{\beta=1}^d \partial_\beta V_{\alpha,\beta}$.

Thank you