The Vlasov-Poisson-Boltzmann equation with polynomial perturbation near Maxwellian

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The Vlasov-Poisson-Boltzmann equation

The Vlasov-Poisson Boltzmann equation (VPB)

$$\partial_t F + v \cdot \nabla_x F - \nabla_x \phi \cdot \nabla_v F = Q(F,F)$$

where

$$-\Delta\phi = \int_{\mathbb{R}^3} F(v)dv - 1, \quad \int_{\mathbb{T}^3} \phi(t,x)dx = 0$$

and $F(t,x,v) \geq 0$, $t > 0$ for $x \in \mathbb{T}^3$, $v \in \mathbb{R}^3$. Q denotes the Boltzmann collision operator

$$Q(f,g)(v) := \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(|v-v_*|,\sigma)(g(v'_*)f(v') - g(v_*)f(v))d\sigma dv_*$$

where (v, v_*) and (v', v'_*) are the velocities of particles before and after elastic collision given by

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \quad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma$$

obviously

$$v + v_* = v' + v'_*, \quad |v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2$$

Boltzmann collision kernel B

Define the deviation angle θ through

$$\cos\theta := \frac{v - v_*}{|v - v_*|} \cdot \sigma.$$

B takes the form

$$B(v - v_*, \sigma) = |v - v_*|^\gamma b(\frac{v - v_*}{|v - v_*|} \cdot \sigma)$$

where the angular function b satisfies

$$\sin\theta b(\cos\theta) \sim \theta^{-1-2s}$$

we focus on the strong singularity case

$$-3 < \gamma \le 1, \quad \frac{1}{2} \le s < 1$$

and we can suppose without losing generality that $0 \le \theta \le \frac{\pi}{2}$

Landau kernel

We also study the Landau case, where the kernel

$$Q(g,f)(v) = \nabla_v \cdot \int_{\mathbb{R}^3} \Phi(v-v_*)(g(v_*)\nabla_v f(v) - f(v)\nabla_{v_*} g(v_*))dv_*$$

with

$$\Phi(u) = |u|^{\gamma+2}\left(I - \frac{u \otimes u}{|u|^2}\right), \quad -3 \leq \gamma \leq 1.$$

Landau case can be seen as $s = 1$.

Maxwellian and the perturbation

Suppose that $\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} F dx dv = 1$. The equilibrium is the Maxwellian-type

$$\mu = (2\pi)^{-3/2} e^{-\frac{|v|^2}{2}}$$

define $F = \mu + f$. Then

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_v f + (\nabla_x \phi \cdot v)\mu = Lf + Q(f, f) \tag{1}$$

where

$$Lf = Q(f, \mu) + Q(\mu, f), \quad -\Delta\phi = \int_{\mathbb{R}^3} f(x, v) dv, \quad \int_{\mathbb{T}^3} \phi(x) dx = 0$$

The global existence, uniqueness, and large time behaviour have been proved by Y. Guo in the space $H^4_{x,v}(\mu^{-1/2})$. We extend it to the $H^2_{x,v}(\langle v\rangle^k)$ case.

Background

For the exponential weight $H^4_{x,v}(\mu^{-1/2})$ case, the Landau equation is proved by Y. Guo, the Boltzmann case is proved by P. Gressman, R. Strain. The Vlasov-Poisson-Boltzmann/Landau case is proved by Y. Guo and T. Yang et al.
The polynomial weight case is proved by the semigroup method, which is first initiated by C. Mouhot and developed and extended by M. Gualdani, S. Mischler, C. Mouhot for the cutoff Boltzmann equation.
For the polynomial weight case $H^2_xL^2_v(\langle v\rangle^k)$, the Landau equation is proved by K. Carrapatoso I. Tristani, K. Wu for the hard potential case and K. Carrapatoso, S. Mischler for the soft potential case.
For the non-cutoff Boltzmann equation, F. Hérau, D. Tonon, I. Tristani proved the hard potential case and C.Cao, L.He and J.Ji proved the soft potential case recently.
A natural question is how to extend it to the VPB equation.

Basic propositions

Conservation law:

$$\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} f(v)dvdx = 0, \quad \int_{\mathbb{T}^3}\int_{\mathbb{R}^3} vf(v)dvdx = 0,$$

$$\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} |v|^2 f(v)dvdx + \int_{\mathbb{T}^3} |\nabla_x \phi(t,x)|^2 dx = 0$$

Continuity equation:

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3} f(v)dv + \nabla_x \cdot \int_{\mathbb{R}^3} vf(v)dv = 0.$$

null space of L:

$$\ker(L) = \text{span}\{\mu, v_1\mu, v_2\mu, v_3\mu, |v|^2\mu\}.$$

projection onto $\ker(L)$:

$$Pf = \left(\int_{\mathbb{R}^3} fdv\right)\mu + \sum_{i=1}^{3}\left(\int_{\mathbb{R}^3} v_i fdv\right) v_i\mu + \left(\int_{\mathbb{R}^3} \frac{|v|^2-3}{6} fdv\right)(|v|^2-3)\mu.$$

Weighted function and functional space

For $k > 0$ large enough, choose the weight function

$$w(\alpha, \beta) = \langle v \rangle^{k+a|\alpha|+b|\beta|+c}$$

$a, b, c \in \mathbb{R}$ need to be chosen carefully. The functional spaces with the norms

$$X_k := \{f \in L^2_{x,v} : C_{|\alpha|,|\beta|} w(\alpha, \beta) \partial^\alpha_\beta f \in L^2_{x,v}, |\alpha| + |\beta| \leq 2\}$$

$$Y_k(\bar{Y}_k) := \{f \in L^2_{x,v} : C_{|\alpha|,|\beta|} w(\alpha, \beta) \partial^\alpha_\beta f \in L^2_x H^s_{\gamma/2}(L^2_x L^2_{\gamma/2}), |\alpha| + |\beta| \leq 2\}$$

with the constants $C_{|\alpha|,|\beta|}$
If the initial data f_0 of (1) satisfies $||f_0||_{X_k}$ small enough, then we have the global existence and convergence to the Maxwellian

Main result

Theorem (C.Cao, D.Deng, X.Li)

There exist constants $k_0 \geq 14$ and $\varepsilon > 0$, such that for all $k \geq k_0$, if the initial data f_0 of (1) *satisfies $||f_0||_{X_k} < \varepsilon$, then* (1) *has a unique global solution $f \in L^\infty([0,\infty); X_k)$ that satisfies $\mu + f \geq 0$ and*

$$||f(t)||_{H^2_x L^2_v} \lesssim e^{-\lambda t} ||f_0||_{X_k} \quad \text{if} \quad \gamma \in [0,1]$$

and

$$||f(t)||_{H^2_x L^2_v} \lesssim \langle t \rangle^{-\lambda} ||f_0||_{X_k} \quad \text{if} \quad \gamma \in (-3,0)$$

for some constant $\lambda > 0$.

The similar result holds for the Vlasov-Poisson-Landau equation with $\gamma \in [-3, 1]$.

Sketch of the proof

The proof combines works by Y. Guo and the semigroup method introduced by M. P. Gualdani, S. Mischler, and C. Mouhot

Notice that $LPf = 0$, so we need to treat the terms Pf and $(I-P)f$ separately.

Difference from the Boltzmann case: the conservation law is not the same so we need to estimate Pf.

We can prove that the term $\nabla_x Pf$ can be controlled by $(I-P)f$ term by the method of Y.Guo, and from Poincaré inequality, we can also get the estimate of Pf(notice that $x \in \mathbb{T}^3$)

For the upper bound of term $(I-P)f$, we use semigroup method: define the norm

$$|||f|||^2 := \|f\|^2_{L^2_k} + \eta \int_0^{+\infty} \|S_L(\tau)(I-P)f\|^2_{L^2_v} d\tau$$

and calculate its time deriavitive.

The requirement $s \geq \frac{1}{2}$

The time deriative of the semigroup related norm will produce the term

$$\int_0^\infty (S_L(\tau)\nabla_x\phi(x)\cdot\nabla_v f, S_L(\tau)f)d\tau$$

we can't integrate by parts to estimate it, because the operators $S_L(t)$ and ∇_v don't commute. After assuming $s \geq \frac{1}{2}$, we can prove that

$$\begin{aligned}\|S_L(t)\nabla_x\phi(x)\cdot\nabla_v f\|_{L^2_v} &\lesssim t^{-1/2}e^{-\lambda t}|\nabla_x\phi(x)|\|\nabla_v f\|_{H^{-s}_{12}} \\ &\lesssim t^{-1/2}e^{-\lambda t}|\nabla_x\phi(x)|\|f\|_{H^s_{12}}\end{aligned}$$

we can also prove that

$$\|S_L(t)f\|_{H^s} \leq \|f\|_{H^s_k}, \quad \|S_L(t)f\|_{H^{-s}} \leq \|f\|_{H^{-s}_k}.$$

so this term can be finally estimated.

Preliminaries on the Boltzmann operator

Lemma (L.He)

Let $w_1, w_2 \in \mathbb{R}$, $a, b \in [0, 2s]$ *with* $w_1 + w_2 = \gamma + 2s$ *and* $a + b = 2s$.
Then for any smooth functions f, g, h *we have*
(1) if $\gamma + 2s > 0$, *then*

$$|(Q(g,h), f)_{L^2_v}| \lesssim (\|g\|_{L^1_{\gamma+2s+(-w_1)^+(-w_2)^+}} + \|g\|_{L^2})\|h\|_{H^a_{w_1}}\|f\|_{H^b_{w_2}},$$

(2) if $\gamma + 2s = 0$, *then*

$$|(Q(g,h), f)_{L^2_v}| \lesssim (\|g\|_{L^1_{w_3}} + \|g\|_{L^2})\|h\|_{H^a_{w_1}}\|f\|_{H^b_{w_2}},$$

where $w_3 = \max\{\delta, (-w_1)^+ + (-w_2)^+\}$, *with* $\delta > 0$ *sufficiently small.*
(3) if $-1 < \gamma + 2s < 0$ *we have*

$$|(Q(g,h), f)_{L^2_v}| \lesssim (\|g\|_{L^1_{w_4}} + \|g\|_{L^2_{-(\gamma+2s)}})\|h\|_{H^a_{w_1}}\|f\|_{H^b_{w_2}}$$

where $w_4 = \max\{-(\gamma + 2s), \gamma + 2s + (-w_1)^+ + (-w_2)^+\}$

Preliminaries on the Boltzmann operator

Lemma (C.Cao, L.He, J.Ji)

Suppose that $-3 < \gamma \leq 1$. *For any* $k \geq 14$, *and smooth functions* g, h, *we have*

$$|(Q(h,\mu), g\langle v\rangle^{2k})| \leq 2\gamma_3 \|b(\cos\theta)\sin^{k-\frac{3+\gamma}{2}}\frac{\theta}{2}\|_{L^1_\theta}\|h\|_{L^2_{k+\gamma/2}}\|g\|_{L^2_{k+\gamma/2}} + C_k\|h\|_{L^2}\|g\|_{L^2}$$

for some constant $C_k > 0$, *where* γ_3 *is the constant such that*

$$\int_{\mathbb{R}^3} \mu(v)|v-v_*|^\gamma dv \leq \gamma_3 \langle v_*\rangle^\gamma$$

holds. Moreover, for any $|\beta| \leq 2$ *we have*

$$|(Q(h,\partial_\beta\mu), g\langle v\rangle^{2k})| \leq C_k\|h\|_{L^2_{k+\gamma/2}}\|g\|_{L^2_{k+\gamma/2}}$$

Preliminaries on the Boltzmann operator

Lemma (C.Cao, L.He, J.Ji)

Suppose that $-3 < \gamma \leq 1, \gamma + 2s > -1$ *and* $G = \mu + g \geq 0$. *Then if*

$$G \geq 0, \quad \|G\|_{L^1} \geq 1/2, \quad \|G\|_{L^1_2} + \|G\|_{L\log L} \leq 4.$$

we have

$$(Q(G,f), f\langle v\rangle^{2k}) \leq -\frac{\gamma_2}{12}\|b(\cos\theta)(1-\cos^{2k-3-\gamma}\frac{\theta}{2})\|_{L^1_\theta}\|f\|^2_{L^2_{k+\gamma/2}} + C_k\|f\|^2_{L^2}$$
$$-\gamma_1\|f\|^2_{H^s_{k+\gamma/2}} + C_k\|f\|_{L^2_{14}}\|g\|_{H^s_{k+\gamma/2}}\|f\|_{H^s_{k+\gamma/2}} + C_k\|g\|_{L^2_{14}}\|f\|^2_{H^s_{k+\gamma/2}}$$

for some constants $\gamma_1, \gamma_2, C_k > 0$, *where* γ_2 *is the constant such that*

$$\gamma_2\langle v\rangle^\gamma \leq \int_{\mathbb{R}^3}|v-v_*|^\gamma\mu_*dv_*.$$

Upper bound for the Boltzmann operator

Lemma (C.Cao, L.He, J.Ji)

For any smooth functions f, g, h and $k \geq 12$, we have

$$(Q(f,g), h\langle v\rangle^{2k}) \lesssim \|f\|_{L^2_{14}}\|g\|_{H^s_{k+\gamma/2+2s}}\|h\|_{H^s_{k+\gamma/2}}+\|g\|_{L^2_{14}}\|f\|_{H^s_{k+\gamma/2}}\|h\|_{H^s_{k+\gamma/2}}$$

In particular by duality we have

$$\|Q(f,g)\|_{H^{-s}_{k-\gamma/2}} \lesssim \|f\|_{L^2_{14}}\|g\|_{H^s_{k+\gamma/2+2s}} + \|g\|_{L^2_{14}}\|f\|_{H^s_{k+\gamma/2}}.$$

Basic estimates for the linearized Boltzmann operator

There exist constants $C_1, C_2, C_3 > 0$, such that for any smooth functions f, g, h and any $k > 12$ large, there exists a constant $C_k > 0$, such that
(1)

$$|(Q(f,\mu), g)_{X_k}| \leq C_1 \|f\|_{\bar{Y}_k} \|g\|_{\bar{Y}_k} + C_k \sum_{|\alpha|+|\beta|\leq 2} \|\partial^\alpha_\beta f\|_{L^2_{x,v}} \|\partial^\alpha_\beta g\|_{L^2_{x,v}}$$

(2) If $G = \mu + g \geq 0$,

$$\begin{aligned}(Q(\mu+g, f), f)_{X_k} \leq & -C_2 \|f\|^2_{\bar{Y}_k} - C_3 \|f\|^2_{Y_k} + C_k \sum_{|\alpha|+|\beta|\leq 2} \|\partial^\alpha_\beta f\|^2_{L^2_{x,v}} \\ & + C_k \|f\|_{X_k} \|g\|_{Y_k} \|f\|_{Y_k} + C_k \|g\|_{X_k} \|f\|_{Y_k} \|f\|_{Y_k}\end{aligned}$$

Basic estimates for the Vlasov-Poisson part

(3)

$$|(-v \cdot \nabla_x f, f)_{X_k}| \leq \frac{C_2}{4} \|f\|_{Y_k}^2$$

(4)

$$(\nabla_x \phi \cdot v\mu, f)_{X_k} \leq C_k \|g\|_{H_x^2 L_2^2} \|f\|_{H_x^2 L_v^2}$$

(5)

$$(\nabla_x \phi \cdot \nabla_v f, f)_{X_k} \leq C_k \|g\|_{Y_k} \|f\|_{Y_k} \|f\|_{X_k}.$$

Strategy of proving the estimates

The estimate about the Boltzmann operator is similar as C.Cao, L.He and J.Ji.
Notice that here we need to choose the weighted function w carefully to let it satisfy the interpolation inequalities, which are used in the estimates.
Moreover, the VPB equation has the $\nabla_x \phi \nabla_v f$ term, so we need to also calculate the $v-$derivative term.
For the Vlasov-Poisson term, we use Sobolev inequality give the estimate about the $\nabla_x \phi$. Finally, after choosing fit constant $C_{\alpha,\beta}$, we get the results above.

Local existence

The main sketch is: define the series of functions $\{F_n\}$ as follows:

$$F_0(t,x,v)=\mu, \quad (\partial_t + v\cdot\nabla_x - \nabla_x\phi_n\cdot\nabla_v)F_{n+1} = Q(F_n, F_{n+1})$$

$$-\Delta\phi_n = \int_{\mathbb{R}^3} F_n dv - 1$$

then the series of functions $f_n := F_n - \mu$ satisfies

$$f_0 = 0, \quad (\partial_t + v\cdot\nabla_x - \nabla_x\phi_n\cdot\nabla_v)f_{n+1} + \nabla_x\phi_n\cdot v\mu$$
$$= Q(f_n,\mu) + Q(\mu, f_{n+1}) + Q(f_n, f_{n+1}), \quad -\Delta\phi_n = \int_{\mathbb{R}^3} f_n dv$$

and we take the limit as $n\to\infty$ and use the fixed point theorem to prove the result.

Upper bounds for the energy

Define

$$E(f) = \|f\|_{X_k}^2, \quad D(f) = \|f\|_{Y_k}^2$$

then for f as the solution to (1), for any $t \geq 0$,

$$E(f) + \int_0^t D(f)ds \lesssim E(f_0) + \int_0^t \sum_{|\alpha| \leq 2} \|\partial^\alpha f\|_{L^2_{x,v}}^2 ds + \int_0^t D(f)E(f)ds.$$

for the case $\partial^\alpha_\beta f, |\beta| \geq 1$, we can use the interpolation to let it be controlled by $||f||_{Y_k}^2$, so there is only $\partial^\alpha f$ remaining.

Upper bounds for $\nabla_x Pf$ (abc inequality, Y.Guo)

We rewrite the equation (1) as

$$\partial_t f + v \cdot \nabla_x f + (\nabla_x \phi \cdot v)\mu - Lf = N(f), \quad -\Delta\phi = \int_{\mathbb{R}^3} f(t,x,v)dv$$

here $N(f) := \nabla_x \phi \cdot \nabla_v f + Q(f,f)$ is the nonlinear part. For any $m \in \mathbb{N}$,

$$\sum_{|\alpha|=m} \left(||\nabla_x \partial^\alpha Pf||^2_{L^2_x L^2_v} + ||\partial^\alpha \nabla_x \phi||^2 \right) - \frac{dG(t)}{dt}$$

$$\lesssim \sum_{|\alpha|=m} ||\nabla_x \partial^\alpha (I-P)f||^2_{L^2_x L^2_{10}} + \sum_{|\alpha|=m} ||\partial^\alpha (I-P)f||^2_{L^2_x L^2_{10}} + \sum_{|\alpha|=m} ||\partial^\alpha N_{||}||^2_{L^2_x}$$

Upper bounds for $\nabla_x Pf$ (abc inequality, Y. Guo)

where

$$G^2(t) \lesssim \sum_{|\alpha|=m} ||\partial^\alpha f||^2_{L^2_x L^2_{10}} ||\partial^\alpha \nabla_x Pf||^2_{L^2_{x,v}}$$

and $\partial^\alpha N_{||}$ is the L^2_v projection of $\partial^\alpha N(f)(t,x,v)$ onto the subspace generated by the basis $\{\mu,\ v_i\mu,\ v_iv_j\mu,\ v_i|v|^2\mu\},\quad 1 \le i,j \le 3.$ Moreover

$$||(I-P)\nabla_x \partial^\alpha f||^2_{L^2_x H^s_{k+\gamma/2}}$$

$$\gtrsim \varepsilon \sum_{|\alpha|=m} \left(||\nabla_x \partial^\alpha f||^2_{L^2_x H^s_{k+\gamma/2}} - ||(I-P)\partial^\alpha f||^2_{L^2_x L^2_{10}} - ||\partial^\alpha N(f)_{||}||^2 \right) - \varepsilon \frac{dG}{dt}$$

for $\varepsilon > 0$ small enough. This allows us to use $(I-P)f$ term to estimate $\nabla_x Pf$.

Upper bounds for $(I-P)f$

For any $|\alpha|\leq 2$, $\epsilon>0$ small, $t\geq 0$ and $k\geq 20$ large, there exist constants $c_1, C_k, M_k>0$ and $0<\eta\leq\frac{C_k}{\epsilon}$, s.t.

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{|||\partial^\alpha f|||^2}{2}+M_k\|\nabla_x\partial^\alpha\phi\|_{L^2_x}^2\right)+c_1\|(I-P)\partial^\alpha f\|^2_{L^2_xH^s_{k+\gamma/2}}\\
\leq&\epsilon\|P\partial^\alpha f\|^2_{L^2_{x,v}}+\frac{C_k}{\epsilon}\sqrt{E(f)}\sum_{|\alpha'|\leq\alpha}\|\partial^{\alpha'}f\|^2_{L^2_xH^s_{k+\gamma/2}}\rightsquigarrow \text{Lower order}\\
+&C_k\sqrt{E(f)}\|\partial^\alpha f\|_{L^2_xH^s_{k+\gamma/2}}\sum_{|\alpha'|\leq|\alpha|-1}\|\partial^{\alpha'}f\|_{L^2_xH^s_{k+\gamma/2+5s}}\rightsquigarrow \text{Lower order}
\end{aligned}
$$

Upper bounds for $(I-P)f$

Denote $k(\alpha) = k + 10s - 5|\alpha|s$, and

$$\|f\|_{Z_k}^2 = \sum_{|\alpha|\le 2} \|\partial^\alpha f \langle v\rangle^{k(\alpha)}\|_{L_x^2 L_v^2}^2 + \eta \int_0^{+\infty} \|S_L(\tau)(I-P)f\|_{L_x^2 L_v^2}^2 d\tau$$

then after summing $|\alpha| \le 2$, there exists $0 < \eta \le \frac{C_k}{\epsilon}$, such that

$$\frac{d}{dt}\left(\frac{\|f\|_{Z_k}^2}{2} + M_k \sum_{|\alpha|\le 2} \|\nabla_x \partial^\alpha \phi\|_{L_x^2}^2\right) + c_1 \sum_{|\alpha|\le 2} \|(I-P)\partial^\alpha f \langle v\rangle^{k(\alpha)}\|_{L_x^2 H_{\gamma/2}^s}^2$$

$$\le \epsilon \sum_{|\alpha|\le 2} \|P\partial^\alpha f\|_{L_{x,v}^2}^2 + \frac{C_k}{\epsilon}\sqrt{E(f)} \sum_{|\alpha|\le 2} \|\partial^\alpha f \langle v\rangle^{k(\alpha)}\|_{L_x^2 H_{\gamma/2}^s}^2 \rightsquigarrow \text{Lower order.}$$

Upper bounds for Pf

Define $k(\alpha) = k + 10s - 5|\alpha|s$. For any $k \geq 14$ large, $E(f) \leq M$ and M is very small, c_1 is defined above, then there exist constants $c_2, c_3 > 0$ and a function $G(t)$ such that

$$
\begin{aligned}
& c_1 \sum_{|\alpha|\leq 2} \|(I-P)\partial^\alpha f \langle v\rangle^{k(\alpha)}\|^2_{L^2_x H^s_{\gamma/2}} \\
& \geq c_2 \sum_{|\alpha|\leq 2} \|\langle v\rangle^{k(\alpha)} \partial^\alpha f\|^2_{L^2_x H^s_{\gamma/2}} - c_3 M \sum_{|\alpha|\leq 2} \|\partial^\alpha f\|^2_{L^2_x L^2_{10}} - \frac{d}{dt} G(t)
\end{aligned}
$$

where

$$
|G(t)| \leq \frac{1}{4} \sum_{|\alpha|\leq 2} \|\partial^\alpha f\|^2_{L^2_x L^2_{10}}.
$$

Proof of the main theorem

As mentioned before, we show that there exists $M > 0$ small enough, such that if $E(f_0) \leq M$, then we have the global existence
Denote

$$T_* := \sup_{t>0} \left\{ E(f) + \int_0^t D(f)(s)ds \leq M \right\}$$

we know from local existence that $T_* > 0$. Next, we show that $T_* = \infty$.
Choose M satisfies

$$CM^{\frac{3}{2}} \leq E(f_0), \quad CE(f_0) \leq \frac{M}{2}$$

then from the lemmas above,

$$E(f) + \int_0^t D(f)ds \leq CE(f_0) + \int_0^t D(f)E(f)ds$$

Proof of the main theorem

since

$$\int_0^{T_*} D(f)(s)ds \leq M \leq 1$$

from Grönwall lemma,

$$E(f) + \int_0^t D(f)ds \leq CE(f_0) \leq \frac{M}{2}, \quad \forall 0 \leq T \leq T_*$$

this implies that $T_* = \infty$

Convergence rate

Our goal is to show the convergence of the term

$$X(t) := \sum_{|\alpha|\le 2} \|\langle v\rangle^{k(\alpha)}\partial^\alpha f\|^2_{L^2_{x,v}}$$

from the lemmas above, we can show that the norm

$$Y(t) := \frac{\|f\|^2_{Z_k}}{2} + \sum_{|\alpha|\le 2} \|\nabla_x \partial^\alpha \phi\|^2_{L^2_x} - G(t)$$

is equivalent to $X(t)$, and

$$\frac{d}{dt}Y(t) + \frac{c_2}{4}\sum_{|\alpha|\le 2} \|\langle v\rangle^{k(\alpha)}\partial^\alpha f\|^2_{L^2_x L^2_{\gamma/2}} \le 0$$

the case $\gamma \ge 0$ can be directly deduced from Gronwall's inequality
the case $\gamma < 0$, we need to interpolate to get the result about the convergence.

Future works and open problem

Since we have recently proved it for the Vlasov-Poisson-Boltzmann equation, we plan to move to the Vlasov-Maxwell-Boltzmann system, which writes

$$\partial_t F + v \cdot \nabla_x F + (E + v \times B) \cdot \nabla_v F = Q(F, F)$$

with

$$\partial_t E - \nabla_x \times B = -\int_{\mathbb{R}^3} vF(v)dv,$$

$$\partial_t B + \nabla_x \times E = 0, \quad \nabla_x \cdot B = 0, \quad \nabla_x \cdot E = \int_{\mathbb{R}^3} Fdv$$

the Vlasov-Poisson system is a special case correspond to $B = 0$ in the Vlasov-Maxwell system.

We only prove it for the strong singularity case, how to prove it for the weak singularity case $s \in (0, \frac{1}{2})$.

Thank you for the attention!