Incompressible Navier-Stokes-Fourier limit from the Landau equation

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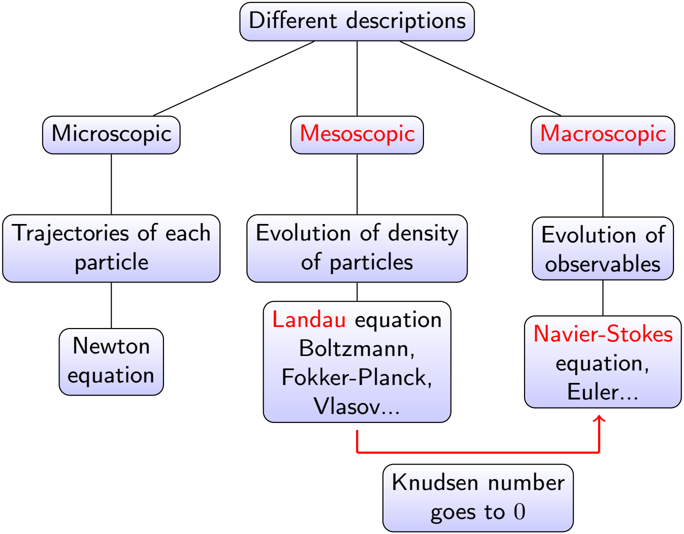
In a particle system (gas)



The Landau equation

- Landau (1936) : Kinetic model in plasma physics that describes the evolution of the density function $f(t,x,v)$, $t \in \mathbb{R}^+$ the time, $x \in \mathbb{T}^3$ the position and $v \in \mathbb{R}^3$ the velocity.

Landau equation

$$\partial_t f + v \cdot \nabla_x f = Q(f,f)$$

- Q is the Landau collision operator (bilinear operator and acts only on variable v) :

$$Q(f,g)(v) = \nabla_v \cdot \left\{ \int_{\mathbb{R}^3} a(v-v_\star)[g(v_\star)\nabla_v f(v) - f(v)\nabla_v g(v_\star)] \, \mathrm{d}v_\star \right\}$$

  - $a_{i,j}(z) = |z|^{\gamma+2} \underbrace{\left(\delta_{ij} - \frac{z_i z_j}{|z|^2}\right)}_{\text{projection onto } z^\perp}$

- $\gamma \in [-2,1] \leftrightarrow$ hard potentials, Maxwellian molecules and moderately soft potentials
- $\gamma \in (-3,-2) \leftrightarrow$ very soft potentials
- $\gamma = -3 \leftrightarrow$ Coulombian potential

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Remark

For $g(v) = \mu(v) = (2\pi)^{-3/2} e^{-|v|^2/2}$, *we have*

- $Q(f,\mu)(v) = \nabla_v \cdot \{(a *_v \mu)\nabla_v f - (b *_v \mu) f\}, \quad b_i(v) = \sum_j \partial_j a_{ij}(v)$
- *It looks like the Fokker-Planck operator :*

$$L_{FP}(f) = \nabla_v \cdot \{\nabla_v f + v f\}$$

Basic properties

- Conservation of mass, momentum and energy :

$$\int_{\mathbb{R}^3} Q(f,f) \begin{pmatrix} 1 \\ v_i \\ |v|^2 \end{pmatrix} \mathrm{d}v = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

- Entropy of the system :

$$H(f) := \int_{\mathbb{T}^3 \times \mathbb{R}^3} f \, \mathsf{log} f \, \mathrm{d}x \mathrm{d}v$$

- Entropy dissipation :

$$D(f) := -\int_{\mathbb{R}^3} Q(f,f) \, \mathsf{log} f \, \mathrm{d}v \geqslant 0$$

- H-Theorem

$$\frac{\mathrm{d}}{\mathrm{d}t} H(t) = -\int_{\mathbb{T}^3} D(f) \, \mathrm{d}x \leqslant 0$$

- $D(f) = 0 \iff f$ (local Maxwellian)
- $Q(\mu, \mu) = 0$

The normalized Maxwellian

$$\mu(v) = (2\pi)^{-3/2} e^{-|v|^2/2}$$

The rescaled Landau equation

- The model :

$$\begin{cases} \partial_t f + v\cdot\nabla_x f = \frac{1}{\varepsilon}Q(f,f), & (t,x,v)\in\mathbb{R}^+\times\mathbb{T}^3\times\mathbb{R}^3 \\ f_{|t=0} = f_0, \end{cases}$$

- The Knudsen number : $1/\varepsilon$ is the average number of collisions for each particle per unit time
- Rescaling and perturbation :
  - $f_\varepsilon(t,x,v) = f\left(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon},v\right)$
  - $f_\varepsilon(t,x,v) = \mu + \varepsilon\mu^{1/2}g_\varepsilon(t,x,v)$
- The perturbative system :

$$\begin{cases} \partial_t g_\varepsilon + \frac{1}{\varepsilon}v\cdot\nabla_x g_\varepsilon - \frac{1}{\varepsilon^2}\mathcal{L}g_\varepsilon = \frac{1}{\varepsilon}\Gamma(g_\varepsilon,g_\varepsilon) \\ g_{\varepsilon\,|t=0} = g_{\varepsilon,0} \end{cases} \tag{1}$$

- $\Gamma(g_1,g_2) = \mu^{-1/2}Q(\mu^{1/2}g_1,\mu^{1/2}g_2)$
- $\mathcal{L}$ is the homogeneous linearized Landau operator :
  - $\mathcal{L}$ acts only in variable v
  - $\mathcal{L}$ is self-adjoint in $L^2(\mathbb{R}^3_v)$
  - $\mathcal{L}$ is a negative operator $L^2(\mathbb{R}^3_v)$

- $\mathcal{L} = \underbrace{\mathcal{L}_1}_{\text{diffusion part}} + \underbrace{\mathcal{L}_2}_{\text{compact part}}$
- $\mathcal{L}_1 g = \Gamma(\sqrt{\mu}, g), \quad \mathcal{L}_2 g = \Gamma(g, \sqrt{\mu})$
- The diffusion part $\mathcal{L}_1$:

$$\mathcal{L}_1 g = \nabla_v \cdot [\mathbf{A}(v)\nabla_v g] - \Big(\mathbf{A}(v)\frac{v}{2}\cdot\frac{v}{2}\Big)g + \nabla_v \cdot \Big[\mathbf{A}(v)\frac{v}{2}\Big]g$$

  - ⋆ $\mathbf{A}(v) = (\overline{a}_{ij}(v))_{1\leqslant i,j\leqslant 3}$ is a symmetric matrix with

$$\overline{a}_{ij} = a_{ij} *_v \mu$$

- The compact part $\mathcal{L}_2$:

$$\mathcal{L}_2 g = -\mu^{-1/2}\partial_i \Big\{\mu\Big[a_{ij} *_v \Big\{\mu^{1/2}\Big[\partial_j g + \frac{v_j}{2} g\Big]\Big\}\Big]\Big\}$$

- $\mathcal{N}(\mathcal{L}) = \text{Span}\left\{\sqrt{\mu}, v_1\sqrt{\mu}, v_2\sqrt{\mu}, v_3\sqrt{\mu}, |v|^2\mu\right\}$

Notations

- The Lesbegue spaces : $L^2_v = L^2(\mathbb{R}^3)$, $L^2_{x,v} = L^2(\mathbb{T}^3 \times \mathbb{R}^3)$
- For $s \in \mathbb{N}$, H^s_x is the usual Sobolev space on $\mathbb{T}^3$
- $H^1_{v,\star}$-norm defined by

$$\|g\|^2_{H^1_{v,\star}} := \|\langle v\rangle^{\frac{\gamma}{2}+1} g\|^2_{L^2_v} + \|\langle v\rangle^{\frac{\gamma}{2}} P_v \nabla_v g\|^2_{L^2_v} + \|\langle v\rangle^{\frac{\gamma}{2}+1}(I-P_v)\nabla_v g\|^2_{L^2_v},$$

where $\langle v\rangle = (1+|v|^2)^{1/2}$ and P_v is the projection on v, i.e. $P_v\eta = \left(\eta \cdot \frac{v}{|v|}\right)\frac{v}{|v|}$

- From [Mouhot-Strain][1], we have :

$$(-\mathcal{L}g, g)_{L^2_v} \geqslant C_\gamma \|g\|^2_{H^1_{v,\star}}, \quad \forall g \in \mathcal{N}(\mathcal{L})^\perp$$

- $|\!|\!| \cdot |\!|\!|$ defined by

$$|\!|\!| g |\!|\!|^2 = \sum_{|\alpha|\leqslant 3} \int_{\mathbb{T}^3} \|\partial^\alpha_x g\|^2_{H^1_{v,\star}} \mathrm{d}x$$

- $g = \Pi_0 g + (I - \Pi_0)g$; Π_0 is the orthogonal projection to $\mathcal{N}$
- The fluid/macroscopic part $\Pi_0 g$:

$$\Pi_0 g(t,x,v) = \{a(t,x) + v\cdot b(t,x) + |v|^2 c(t,x)\}\sqrt{\mu}$$

- $(I-\Pi_0)g$: the kinetic or microscopic part

1. Mouhot, C., and Strain, R. Spectral gap and coercivity estimates for linearized Boltzmann collision operators without angular cutoff. Journal de Mathématiques Pures et Appliquées 87(05 2007), 515–5

Uniform estimate and global solutions

We introduce the following energy functional and dissipation :

- $\mathcal{E}^2(g) = \|g\|^2_{H^3_x L^2_v}$
- $\mathcal{D}(g) = \|\!|(I - \Pi_0)g|\!\|$
- $\mathcal{C}(g) = \|\nabla_x \Pi_0 g\|_{H^2_x L^2_v}$

$$\int_{\mathbb{T}^3 \times \mathbb{R}^3} g_{\varepsilon,0}(x,v)\mu^{1/2}(v) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} \mathrm{d}x\mathrm{d}v = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (IC)$$

Theorem

There exists $M_0 > 0$ such that : for $\varepsilon \in (0,1)$ $g_{\varepsilon,0}$ satisfies (IC) and $\|g_{\varepsilon,0}\|_{H^3_x L^2_v} \leqslant M_0$, then the Cauchy problem (1) admits a unique global solution

$$g_\varepsilon \in L^\infty([0,\infty); H^3_x L^2_v)$$

with the global energy estimate

$$\sup_{t \geqslant 0} \mathcal{E}^2(t) + C_0 \int_0^\infty \frac{1}{\varepsilon^2} \mathcal{D}^2(t)\mathrm{d}t + C_0 \int_0^\infty \mathcal{C}^2(t)\mathrm{d}t \leqslant C_0' \, \mathcal{E}^2(0)$$

where $C_0, C_0' > 0$ are independent of ε.

Strategy of the proof

- We show the existence of local solution on $[0, T]$, $T > 0$.
- We introduce the following macroscopic energy

$$E^2 := \mathcal{E}^2 + \eta\varepsilon\Big(\sum_{|\alpha|\leqslant 2} (\partial^\alpha r, \nabla_x \partial^\alpha (a,b,c))_{L^2_x} + (\partial^\alpha b, \nabla_x \partial^\alpha a)_{L^2_x} \Big), \eta > 0$$

  - $r = ((I - \Pi_0)g, e)_{L^2_v}$
  - $e \in \mathrm{Span}\{v_i|v|^2\mu^{1/2}, v_i^2\mu^{1/2}, v_iv_j\mu^{1/2}, v_i\mu^{1/2}, \mu^{1/2}\}$, for $i,j = 1,2,3$ (so-called 13-moments of Grad [Guo] [2])
- We choose $\eta > 0$ small such that :

$$\frac{\mathrm{d}}{\mathrm{d}t}E^2 + c_0\Big(\frac{1}{\varepsilon^2}\mathcal{D}^2 + \mathcal{C}^2\Big) \leqslant c_1 E\left\{\frac{1}{\varepsilon}\mathcal{D}^2 + \mathcal{C}^2\right\}, c_0, c_1 > 0$$

- Using that $\|g_{\varepsilon,0}\|_{H^3_x L^2_v} \leqslant M_0$, to obtain

$$E^2(T) + c_2\int_0^T \left\{\frac{1}{\varepsilon^2}\mathcal{D}^2 + \mathcal{C}^2\right\}\mathrm{d}t \leqslant E^2(0), \quad T > 0$$

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2. Guo, Y. The Boltzmann equation in the whole space. Indiana University mathematics journal (2004), 1081–109

Limit to fluid incompressible Navier-Stokes-Fourier

The hydrodynamical limit of (1) as ε goes to zero is the incompressible Navier-Stokes-Fourier system associated with the Boussinesq equation which writes

$$\begin{cases} \partial_t u + u \cdot \nabla_x u + \nabla_x p = \nu \Delta_x u, \\ \partial_t \theta + u \cdot \nabla_x \theta = \kappa \Delta_x \theta, \\ \nabla_x \cdot u = 0, \\ \rho + \theta = 0. \end{cases}$$

In this system :

- θ : temperature
- ρ : density
- p : pressure
- u : velocity vector field
- ν : viscocity
- κ : heat conductivity

- The coefficients ν and κ are determined by $\mathcal{L}$

Local conservation laws

- The fluid variables :

$$\rho_\varepsilon = (g_\varepsilon, \sqrt{\mu})_{L^2_v}\,, \quad u_\varepsilon = (g_\varepsilon, v\sqrt{\mu})_{L^2_v}\,, \quad \theta_\varepsilon = \left(g_\varepsilon, \left(\frac{|v|^2}{3} - 1\right)\sqrt{\mu}\right)_{L^2_v}$$

Local conservation laws

$$\begin{cases} \partial_t \rho_\varepsilon + \frac{1}{\varepsilon}\nabla_x \cdot u_\varepsilon = 0 \\ \partial_t u_\varepsilon + \frac{1}{\varepsilon}\nabla_x(\rho_\varepsilon + \theta_\varepsilon) + \frac{1}{\varepsilon}\nabla_x \cdot \big(A(v)\sqrt{\mu}, g_\varepsilon\big)_{L^2_v} = 0 \\ \partial_t \theta_\varepsilon + \frac{2}{3}\frac{1}{\varepsilon}\nabla_x \cdot u_\varepsilon + \frac{2}{3}\frac{1}{\varepsilon}\nabla_x \cdot \big(B(v)\sqrt{\mu}, g_\varepsilon\big)_{L^2_v} = 0 \end{cases}$$

where $A = v \otimes v - \frac{|v|^2}{3}\,\mathrm{Id}$ and $B = v\left(\frac{|v|^2}{2} - \frac{5}{2}\right)$

Theorem

There exists $M_0 > 0$ *such that : if* $g_{\varepsilon,0}$ *satisfies*

1) (IC), $g_{\varepsilon,0} \in H^3_x L^2_v$, $\|g_{\varepsilon,0}\|_{H^3_x L^2_v} \leqslant M_0$
2) *There exist* $\rho_0(x)$, $\theta_0(x) \in H^3_x$ *and* $u_0(x) \in H^3_x$ *such that* $g_{\varepsilon,0} \underset{\varepsilon \to 0}{\longrightarrow} g_0$ *strongly in* $H^3_x L^2_v$, *where*

$$g_0(x,v) = \rho_0(x)\sqrt{\mu}(v) + u_0(x) \cdot v\sqrt{\mu}(v) + \theta_0(x)\Big(\frac{|v|^2}{2} - \frac{3}{2}\Big)\sqrt{\mu}(v).$$

Then, as $\varepsilon \longrightarrow 0$,

- $g_\varepsilon \rightharpoonup \rho\sqrt{\mu} + u \cdot v\sqrt{\mu} + \theta(\frac{|v|^2}{2} - \frac{3}{2})\sqrt{\mu}$ *weakly-⋆ in* $L^\infty([0,\infty); H^3_x L^2_v)$.
- $(\rho, u, \theta) \in C(\mathbb{R}^+; H^2_x) \cap L^\infty(\mathbb{R}^+; H^3_x)$ *is a solution of the incompressible Navier-Stokes-Fourier equation with initial data*

$$u|_{t=0} = \mathcal{P}u_0(x), \quad \theta|_{t=0} = \frac{3}{5}\theta_0(x) - \frac{2}{5}\rho_0(x),$$

where $\mathcal{P}u_0(x)$ *is the divergence-free part of* $u_0(x)$.

1) Some existing results of weak convergence in the framework of strong solutions :

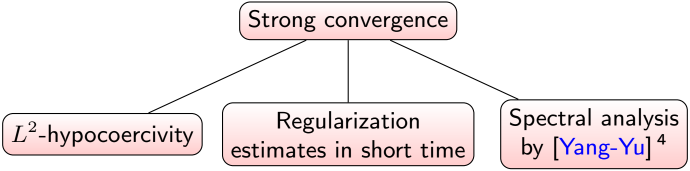
- From Boltzmann equation without cutoff to incompressible Navier-Stokes : [Jiang-Xu-Zhao, 2018]
- From Boltzmann equation with cutoff to incompressible Navier-Stokes : [Briant, 2015], [Briant-Merino-Mouhot, 2019], [Guo, 2006]

2) Some existing results of weak convergence in the framework of weak solutions :

- The renormalized solutions for the Boltzmann equation (from DiPerna-Lions) and the Leray solutions for the Navier-Stokes equations : Bardos, Golse, Levermore, Lions, Masmoudi, Saint-Raymond...

Remark on strong convergence

- In [Carrapatoso-Rachid-Tristani, 2021] [3], we have improve the above result :



3. K. Carrapatoso, M. Rachid, and I. Tristani, Regularization estimates and hydrodynamical limit for the Landau equation, 2021, arXiv : 2107.12044

4. T. Yang and H. Yu, Spectrum analysis of some kinetic equations, Arch. Ration. Mech. Anal. 222.2 (2016), pp. 731–768

Strategy of the proof

- From the global energy estimate we have :
  - $\sup_{t\geqslant 0} \|g_\varepsilon(t)\|^2_{H^3_x L^2_v} \leqslant C \qquad (N_1)$
  - $\int_0^\infty \|\|(I-\Pi_0)g_\varepsilon\|\|^2 \mathrm{d}t \leqslant C\varepsilon^2 \qquad (N_2)$
- By using (N_1), (N_2), we show the weak-$\star$ convergence of $(g_\varepsilon)_\varepsilon$ to

$$g(t,x,v) = \rho\sqrt{\mu} + u\cdot v\sqrt{\mu} + \theta\Big(\frac{|v|^2}{2} - \frac{3}{2}\Big)\sqrt{\mu}$$

- We show the following convergences

$$\rho_\varepsilon \longrightarrow \rho \text{ weakly-} \star \text{ in } L^\infty(\mathbb{R}^+; H^3_x)$$
$$u_\varepsilon \longrightarrow u \text{ weakly-} \star \text{ in } L^\infty(\mathbb{R}^+; H^3_x)$$
$$\theta_\varepsilon \longrightarrow \theta \text{ weakly-} \star \text{ in } L^\infty(\mathbb{R}^+; H^3_x)$$

- We use Aubin-Lions Theorem to prove

$$\mathcal{P}u_\varepsilon \longrightarrow u \text{ strongly in } C(\mathbb{R}^+; H^2_x)$$
$$\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon \longrightarrow \theta \text{ strongly in } C(\mathbb{R}^+; H^2_x)$$

- We obtain the proof by using the previous limits and local conservation laws

Conclusion and Perspective

Conclusion :

- We have obtained a weak convergence to the Navier-Stokes solutions from the Landau equation.

Perspective :

- What about the limit of Vlasov-Poisson-Landau ?

Thanks for your attention !

Mohamad Rachid. Incompressible Navier-Stokes-Fourier limit from the Landau equation. Kinetic & Related Models, 2021, 14 (4) : 599-638. doi : 10.3934/krm.2021017