Stability in Gagliardo-Nirenberg-Sobolev inequalities

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References and collaborators

Stability in Gagliardo-Nirenberg-Sobolev inequalities.
M. Bonforte, J. Dolbeault, B. Nazaret, and N. S., **161 pages**
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Outline of the talk

Part I: Functional inequalities and stability

▷ *Critical Sobolev inequality: best constant and optimizers*

▷ *The stability issue: a question raised by Brezis and Lieb*

▷ *A family of Gagliardo-Nirenberg-(Sobolev) inequalities*

Part II: A constructive result for the subcritical Gagliardo-Nirenberg inequality

▷ *Stability for Gagliardo-Nirenberg*

▷ *Ideas and strategy of the proof: entropy methods and parabolic regularity*

Part III: The Sobolev's inequality

▷ *Why Sobolev's inequality is different?*

▷ *Constructive stability for Sobolev's inequality*

Part I: functional inequalities and stability

Sobolev's inequality

Let $d \geq 3$, the *Sobolev inequality* states

$$\|\nabla f\|_2 \geq \mathsf{S}_d \, \|f\|_{2\,p^\star} \quad \text{for any} \quad f \in W^{1,2}(\mathbb{R}^d)$$

where $p^\star = d/(d-2)$ and $\|f\|_p = \left(\int_{\mathbb{R}^d} |f|^p \, dx\right)^{\frac{1}{p}}$

▷ The optimal constant S_d has been computed by Aubin and Talenti (1976) (but also previous contribution by Rodemich (1966)) and it is achieved on

$$\mathsf{S}_d = \frac{\|\nabla g\|_2}{\|g\|_{2\,p^\star}} \quad \text{where} \quad \mathsf{g}(x) = (1+|x|^2)^{-\frac{d-2}{2}}$$

▷ By scaling, homogeneity and translations, the constant is achieved also on the manifold

$$\mathcal{M} = \{\mathsf{g}_{\lambda,\mu,y}(x) := \mu\, \lambda^{-\frac{d}{2^\star}}\, \mathsf{g}\left(\frac{x-y}{\lambda}\right) : (\lambda,\mu,y) \in (0,\infty) \times \mathbb{R} \times \mathbb{R}^d\}$$

Stability: a question raised by Brezis and Lieb (1985)

Is there a natural way to bound from below

$$\delta_S[f] = \|\nabla f\|_2^2 - \mathsf{S}_d^2 \, \|f\|_{2p^\star}^2$$

in terms of a "distance" to the manifold of the optimal Aubin-Talenti functions?

In other words: *assume that $\delta_S[u]$ is small, can we prove that u is close (in some topology) to a Aubin-Talenti function?*

▷ [Bianchi-Egnell (1991)] There is a positive constant C such that

$$\delta_S[f] \geq \mathsf{C} \inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|_{L^2(\mathbb{R}^d)}^2 ,$$

and $\mathcal{M}$ is the manifold of Aubin-Talenti functions. The constant C is *NOT* known!

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Stability: improvements for the Sobolev inequality

Since Bianchi-Egnell's result several improvements have been obtained

▷ [Cianchi-Fusco-Maggi-Pratelli (2009)] and [Figalli-Maggi-Pratelli,2013] there are constants α and κ and $f \mapsto \lambda(f)$ such that

$$\|\nabla f\|^2_{L^2(\mathbb{R}^d)} \geq \mathsf{S}_d \, \left(1 + \kappa \, \lambda(f)^\alpha\right) \|f\|^2_{L^{2^\star}(\mathbb{R}^d)} \,, \quad \lambda(f) = \inf_{h \in \mathcal{M}} \frac{\|f - h\|^{2^\star}_{L^{2^\star}(\mathbb{R}^d)}}{\|f\|^{2^\star}_{L^{2^\star}(\mathbb{R}^d)}}$$

▷ [Dolbeault-Jankowiak (2009)] Assume that $d \geq 3$ and let $q = \frac{d+2}{d-2}$. There exists a constant $\mathcal{C}$ with $1 < \mathcal{C} \leq 1 + \frac{4}{d}$ such that

$$\delta_S[f] \geq \frac{\mathcal{C}}{\|f\|^{2(2^\star-2)}_{L^{2^\star}(\mathbb{R}^d)}} \left(\|f^q\|^2_{L^{\frac{2d}{d+2}}(\mathbb{R}^d)} - \mathsf{S}_d \int_{\mathbb{R}^d} |f|^q \, (-\Delta)^{-1} |f|^q \, dx \right)$$

▷ Other contibutions: [Figalli-Neumayer, 2018], [Neumayer 2020], [Figalli, Ru-Ya Zhang 2020]

The problem of obtaining ***constructive*** *results is widely open!*

Stability of related families of inequalities

▷ Literature on stability of Sobolev type inequalities is huge:
– Weak $L^{2^*/2}$-remainder term in bounded domains [Brezis, Lieb, 1985]
– Fractional versions and $(-\Delta)^s$ [Lu, Wei, 2000] [Gazzola, Grunau, 2001] [Bartsch, Weth, Willem, 2003] [Chen, Frank, Weth, 2013]
– Symmetrization [Cianchi, Fusco, Maggi, Pratelli, 2009] and [Figalli, Maggi, Pratelli, 2010] ▷

Literature on stability of related inequalities
– Hardy-Littlewood-Sobolev: [Chen, Frank, Weth, 2013], [Carlen-Figalli,2013]
– Log-Sobolev: [Fathi, Indrei, Ledoux, 2014], [Eldan, Lehec, Shenfeld 2020], [Indrei, Kim, 2019], [Kim 2021]

▷ Literature on stability of inequalities in geometry
– [Frank, 2021], [Engelstein-Neumayer-Spolaor, 2021]

The Gagliardo-Nirenberg-Sobolev inequalities we consider

We consider the inequalities

$$\|\nabla f\|_2^{\vartheta}\ \|f\|_{p+1}^{1-\vartheta} \geq \mathcal{C}_{\mathrm{GNS}}(p)\ \|f\|_{2\,p} \tag{GNS}$$

$$\theta = \frac{d\,(p-1)}{(d+2-p\,(d-2))\,p}\,,\quad p\in(1,+\infty) \text{ if } d=1 \text{ or } 2\,,\quad p\in(1,p^*] \text{ if } d\geq 3\,,\quad p^* = \frac{d}{d-2}$$

Theorem (del Pino, Dolbeault (2002))

Equality case in (GNS) *is achieved if and only if*

$$f\in\mathcal{M} := \left\{g_{\lambda,\mu,y}\ :\ (\lambda,\mu,y)\in(0,+\infty)\times\mathbb{R}\times\mathbb{R}^d\right\}$$

Aubin-Talenti functions: $g_{\lambda,\mu,y}(x) := \mu\,\mathsf{g}\left((x-y)/\lambda\right)$ *where* $\quad \mathsf{g}(x) = \left(1+|x|^2\right)^{-\frac{1}{p-1}}$

$\triangleright$ When $d\geq 3$ and $p = p^\star = d/(d-2)$ we shall rename $\mathcal{C}_{\mathrm{GNS}}(p^\star)^{-1} = \mathsf{S}_d$

Stability for Gagliardo-Nirenberg inequalities?

Let us consider the (homogeneous/scale invariant) deficit functional

$$\delta_{HGNS}[f] = \|\nabla f\|_2^{\vartheta}\ \|f\|_{p+1}^{1-\vartheta} - \mathcal{C}_{GNS}\,\|f\|_{2p} \geq 0$$

where for $d \geq 3$, $1 < p \leq \frac{d}{d-2} = p^\star$, and for $d = 1, 2$ $1 < p < \infty = p^\star$.

Recall that $\delta_{HGNS}[\mathsf{g}] = 0$ if $\mathsf{g}(x) = (1 + |x|^2)^{-\frac{1}{p-1}}$

(Q): If $\delta_{HGNS}[f]$ is small, in what sense, if any, is f close to g?

$\triangleright$ [Figalli-Carlen 2010] In $\mathbb{R}^2$ with $p = 3$. Let $f \in W^{1,2}(\mathbb{R}^2)$ be a nonnegative function such that $\|f\|_{L^6(\mathbb{R}^2)} = \|\mathsf{g}\|_{L^6(\mathbb{R}^2)}$ (g being (an) Aubin-Talenti profile). Then there exist universal constants $K_1, \delta_1 > 0$ such that, whenever $\delta_{HGN} \leq \delta_1$

$$\sqrt{\delta_{HGNS}}[f] \geq K_1 \inf_{\lambda>0, x_0\in\mathbb{R}^2} \|f^6 - \lambda^2\, \mathsf{g}^6_{\lambda,x_0}\|_{L^1(\mathbb{R}^2)}$$

$\triangleright$ Other results by [Seuffert 2017], [Nguyen 2019]

$\triangleright$ Another improvement is due to [Dolbeault-Toscani]

Part II: a constructive stability result

Preliminaries: non scale invariant form of the inequality

The non scale invariant *deficit functional*

$$\delta_{GNS}[f] := a\,\|\nabla f\|_2^2 + b\,\|f\|_{p+1}^{p+1} - \mathcal{K}_{\mathrm{GNS}}\,\|f\|_{2p}^{2p\gamma} \geq 0$$

where $a = \frac{1}{2}(p-1)^2$, $b = 2\,\frac{d-p\,(d-2)}{p+1}$, $\mathcal{K}_{\mathrm{GNS}} = \|\mathsf{g}\|_{L^{2p}(\mathbb{R}^d)}^{2p\,(1-\gamma)}$ and $\gamma = \frac{d+2-p\,(d-2)}{d-p\,(d-4)}$

▷ Take $f_\lambda(x) := \lambda^{\frac{d}{2p}} f(\lambda x)$ then

$$\delta_{GNS}[f_\lambda] = a\,\lambda^\alpha \|\nabla f\|_{L^2(\mathbb{R}^d)}^2 + b\,\lambda^{-\beta}\|f\|_{L^{p+1}(\mathbb{R}^d)}^{p+1} - \mathcal{K}_{\mathrm{GNS}}\|f\|_{L^{2p}(\mathbb{R}^d)}$$

Optimizing in λ one finds $\|\nabla f\|_2^\vartheta\,\|f\|_{p+1}^{1-\vartheta} \geq \mathcal{C}_{GNS}(p)\,\|f\|_{2p}$.

Preliminaries: Entropy and Fisher information

▷ The *Relative entropy* recall that $\mathsf{g}^{1-p} = 1 + |x|^2$

$$\mathcal{F}[f|\mathsf{g}] := \frac{2\,p}{1-p} \int_{\mathbb{R}^d} \left(|f|^{p+1} - \mathsf{g}^{p+1} - \tfrac{1+p}{2\,p}\, \mathsf{g}^{1-p} \left(|f|^{2\,p} - \mathsf{g}^{2\,p} \right) \right) dx$$

▷ *A Csiszár-Kullback inequality*. There exists a constant $C_p > 0$ such that

$$\| |f| - \mathsf{g} \|_{L^{2p}}^{2p} \le \left\| |f|^{2\,p} - \mathsf{g}^{2\,p} \right\|_{L^1} \le C_p \sqrt{\mathcal{F}[f|\mathsf{g}]} \quad \text{if} \quad \|f\|_{L^{2\,p}(\mathbb{R}^d)} = \|\mathsf{g}\|_{L^{2\,p}(\mathbb{R}^d)}$$

▷ The *Relative Fisher information*

$$\mathcal{J}[f|g] := \frac{p+1}{p-1} \int_{\mathbb{R}^d} \left| (p-1)\, \nabla |f| + |f|^p\, \nabla g^{1-p} \right|^2 dx$$

▷ We can rewrite the $\delta_{GNS}[f]$ as

$$\frac{p+1}{p-1}\, \delta_{GNS}[f] = \mathcal{J}[f|g] - 4\, \mathcal{F}[f|\mathsf{g}]$$

Constructive stability for Gagliardo-Nirenberg

The *relative entropy*

$$\mathcal{F}[f|\mathsf{g}] := \frac{2\,p}{1-p}\int_{\mathbb{R}^d}\left(|f|^{p+1}-\mathsf{g}^{p+1}-\frac{1+p}{2\,p}\,\mathsf{g}^{1-p}\left(|f|^{2\,p}-\mathsf{g}^{2\,p}\right)\right)dx$$

The *deficit functional*

$$\delta_{GNS}[f] = a\,\|\nabla f\|_2^2 + b\,\|f\|_{p+1}^{p+1} - \mathcal{K}_{\rm GN}\,\|f\|_{2p}^{2\,p\,\gamma} \geq 0$$

Theorem (M. Bonforte, J. Dolbeault, B. Nazaret, N.S.)

Let $d \geq 1$*,* $p \in (1, p^\star)$*,* $A > 0$*. There is a (computable)* $\mathcal{C} > 0$ *such that*

$$\delta_{GNS}[f] = \mathcal{J}[f|g] - 4\,\mathcal{F}[f|\mathsf{g}] \geq \mathcal{C}\,\mathcal{F}[f|\mathsf{g}]$$

for any $f \in \mathcal{W} := \left\{f \in \mathrm{L}^1(\mathbb{R}^d, (1+|x|)^2\,\mathrm{d}x)\ :\ \nabla f \in \mathrm{L}^2(\mathbb{R}^d, \mathrm{d}x)\right\}$ *such that*

$$\|f\|_{2p} = \|\mathsf{g}\|_{2p}\,, \quad \int_{\mathbb{R}^d} x\,|f|^{2\,p}\,dx = 0 \quad \sup_{r>0} r^{\frac{d-p\,(d-4)}{p-1}} \int_{|x|>r} |f|^{2\,p}\,dx \leq A$$

▷ As a consequence we get $\delta_{GNS}[f] \geq \mathcal{C}\,\||f| - \mathsf{g}\|_{2p}^{4p}$ and $\delta_{GNS}[f] \geq \|\nabla|f| - \nabla\mathsf{g}\|_2^8$

A general stability result

Theorem (M. Bonforte, J. Dolbeault, B. Nazaret, N. S.)

Let $d \geq 1$ and $p \in (1, p^*)$. For any $f \in \mathcal{W}$, such that $A[f] < \infty$, $E[f] < \infty$, we have

$$\|\nabla f\|^{\theta}_{L^2(\mathbb{R}^d)}\,\|f\|^{1-\theta}_{L^{p+1}(\mathbb{R}^d)} - \mathcal{C}_{\mathrm{GNS}}\,\|f\|_{L^{2p}(\mathbb{R}^d)} \geq \mathfrak{S}[f]\,\mathsf{E}[f]$$

The constant $\mathfrak{S}[f]$ is computable!

$$\lambda[f] := \left(\frac{2\,d\,\kappa[f]^{p-1}}{p^2-1}\,\frac{\|f\|_{p+1}^{p+1}}{\|\nabla f\|_2^2}\right)^{\frac{2p}{d-p\,(d-4)}} \quad \kappa[f] := \frac{\mathcal{M}^{\frac{1}{2p}}}{\|f\|_{2p}}$$

$$\mathsf{A}[f] := \frac{\mathcal{M}}{\lambda[f]^{\frac{d-p\,(d-4)}{p-1}}\,\|f\|_{2p}^{2p}}\,\sup_{r>0} r^{\frac{d-p\,(d-4)}{p-1}} \int_{|x|>r} |f(x+x_f)|^{2p}\,dx$$

$$\mathsf{E}[f] := \frac{2p}{1-p}\int_{\mathbb{R}^d}\left(\frac{\kappa[f]^{p+1}}{\lambda[f]^{d\,\frac{p-1}{2p}}}\,|f|^{p+1} - \mathsf{g}^{p+1} - \frac{1+p}{2p}\,\mathsf{g}^{1-p}\left(\frac{\kappa[f]^{2p}}{\lambda[f]^2}\,|f|^{2p} - \mathsf{g}^{2p}\right)\right)dx$$

Idea of the proof in the non-critical case

Stability by the fast diffusion flow

$$(\mathrm{CP})\quad \begin{cases} \partial_t u = \Delta u^m & \text{in } (0,\infty)\times\mathbb{R}^d\,,\\ u\,(0,x) = u_0(x) & \text{in } \mathbb{R}^d\,,\end{cases}$$

Parameters and main features:

- m in the range $$m_1 := \frac{d-1}{d} \le m < 1\,,\quad \text{with}\quad d \ge 3\,,\quad u_0 \in \mathrm{L}^1_+(\mathbb{R}^d)$$
- Existence and uniqueness in $\mathrm{L}^1_{\mathrm{loc}}$ are settled, solutions are C^∞, see **Herrero-Pierre** '85.
$$\int_{\mathbb{R}^d} u(t,x)\,\mathrm{d}x = \int_{\mathbb{R}^d} u_0(x)\,\mathrm{d}x \qquad \int_{\mathbb{R}^d} x\,u(t,x)\,\mathrm{d}x = \int_{\mathbb{R}^d} x\,u_0(x)\,\mathrm{d}x \quad \forall t>0\,.$$
- (CP) admits the self-similar solution (called **Barenblatt**)

$$\mathcal{B}_M(t,x) = \frac{t^{\frac{1}{1-m}}}{\left[b_0\,\frac{t^{2\vartheta}}{M^{2\vartheta(1-m)}} + b_1|x|^2\right]^{\frac{1}{1-m}}} = t^{-d\vartheta}\,\mathsf{B}_M(x\,t^{-\vartheta}),$$

where $\vartheta^{-1} = 2 - d(1-m) > 0$, and

$$\mathsf{B}_M(x) = \left[\frac{b_0}{M^{2\vartheta(1-m)}} + b_1|x|^2\right]^{\frac{1}{m-1}}$$

Self-similar variables: entropy-entropy production method

Let $d \geq 3$, $(d-1)/d < m < 1$, the a Fokker-Planck type equation

$$\frac{\partial v}{\partial t} + \nabla \cdot \left[v \left(\nabla v^{m-1} - 2\,x \right) \right] = 0$$

can be obtained from $u_t = \Delta u^m$ and admits a stationary solution

$$\mathcal{B}(x) = \left(1 + |x|^2\right)^{-\frac{1}{1-m}} \text{ “} = g^{2p}(x)\text{”}$$

when $p = \frac{1}{2m-1}$. A Lyapunov functional [Ralston,Newman 1984]

Generalized entropy or Free energy

$$\mathcal{F}[v] := -\frac{1}{m} \int_{\mathbb{R}^d} \left(v^m - \mathcal{B}^m - m\,\mathcal{B}^{m-1}\,(v - \mathcal{B}) \right) \mathrm{d}x$$

Entropy production is measured by the Generalized Fisher information

$$\frac{d}{dt}\mathcal{F}[v] = -\,\mathcal{I}[v]\,, \quad \mathcal{I}[v] := \int_{\mathbb{R}^d} v \left| \nabla v^{m-1} + 2\,x \right|^2 \mathrm{d}x$$

The entropy - entropy production inequality

Recall $\mathcal{B}(x) := \left(1+|x|^2\right)^{-\frac{1}{1-m}}$,

$$I[v] = \int_{\mathbb{R}^d} v\left|\nabla v^{m-1} + 2\,x\right|^2 \mathrm{d}x\,,\; F[v] = \int_{\mathbb{R}^d} \left(\frac{\mathcal{B}^m}{m} - \frac{v^m}{m} + |x|^2\,(v-\mathcal{B})\right) \mathrm{d}x$$

By del Pino, Dolbeault (2002) we have

$$\mathcal{I}[v] - 4\,\mathcal{F}[v] = C_p\left(a\,\|\nabla f\|^2_{L^2(\mathbb{R}^d)} + b\,\|f\|^{p+1}_{L^{p+1}(\mathbb{R}^d)} - \mathcal{K}_{\mathrm{GNS}}\,\|f\|^{2\,p\,\gamma}_{L^{2\,p}(\mathbb{R}^d)}\right) = \frac{p+1}{p-1}\,\delta_{GNS}[f]$$

where $C_p > 0$, $\gamma = \gamma(p,d) > 0$, $p = \frac{1}{2\,m-1}$, $\frac{d-1}{d} < m < 1$ and $v = |f|^{2\,p}$.

Also, if v solves $\frac{\partial v}{\partial \tau} + \nabla\cdot\left[v\left(\nabla v^{m-1} - 2\,x\right)\right] = 0$

$$\frac{d}{dt}\left(\mathcal{I}[v(t)] - 4\,\mathcal{F}[v(t)]\right) \leq 0$$

Our goal: $\mathcal{I}[v(t)] - (4+\eta)\mathcal{F}[v(t)] \geq 0$ along the flow

The asymptotic time layer improvement: ideas

In the *large time asymptotic* (where $v \sim \mathcal{B}$) we can "linearize" the quantities $\mathcal{F}[v]$ and $\mathcal{I}[v]$ we get

$$\mathcal{F}[v] \sim \mathsf{F}[g] \quad \text{and} \quad \mathcal{I}[v] \sim \mathsf{I}[g]\,.$$

where $g = v\mathcal{B}^{m-2} - \mathcal{B}^{m-1}$ and

$$\mathsf{F}[g] := \frac{m}{2}\int_{\mathbb{R}^d} |g|^2\,\mathcal{B}^{2-m}\,dx \quad \text{and} \quad \mathsf{I}[g] := m\,(1-m)\int_{\mathbb{R}^d} |\nabla g|^2\,\mathcal{B}\,dx$$

In [Blanchet, Bonforte, Dolbeault, Grillo and Vazquez, 2009] the authors proved that

$$\mathsf{I}[g] \geq 4\,\alpha\,\mathsf{F}[g] \quad \text{where} \quad \alpha = 2 - d\,(1-m)$$

and $\alpha > 1$ if $m \in ((d-1)/d, 1)$, which is an improvement with respect to

$$\mathcal{I}[v] \geq 4\mathcal{F}[v]\,.$$

If $(1-\varepsilon)\,\mathcal{B} \leq v \leq (1+\varepsilon)\,\mathcal{B}$ then $\mathcal{Q}[v] := \frac{\mathcal{I}[v]}{\mathcal{F}[v]} \geq 4 + \eta$

The initial time layer improvement: backward estimate

Consider, $\mathcal{Q}[v] := \frac{\mathcal{I}[v]}{\mathcal{F}[v]}$, by the *Entropy-Entropy production inequality* we have that

$$\frac{d\mathcal{Q}}{dt} \leq \mathcal{Q}\,(\mathcal{Q}-4)$$

Lemma

Assume that $m > m_1$ and v is a solution to R-FDE with nonnegative initial datum v_0. If for some $\eta > 0$ and $T > 0$, we have $\mathcal{Q}[v(T,\cdot)] \geq 4+\eta$, then

$$\mathcal{Q}[v(t)] \geq 4 + \frac{4\,\eta\,e^{-4\,T}}{4+\eta-\eta\,e^{-4\,T}} \quad \forall\, t \in [0,T]$$

Q) : when can we gurantee that $Q[v(t_\star)] \geq 4+\eta$ for some $\eta > 0$?

or equivalently that $\|\frac{v(t_\star)}{\mathcal{B}} - 1\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \leq \varepsilon$

The strategy

Choose $\varepsilon > 0$*, small enough*

Get a threshold time $t_\star(\varepsilon)$

0 $\qquad$ $t_\star(\varepsilon)$ $\qquad$ t

Backward estimate

by entropy methods

Forward estimate

based on a spectral gap

Initial time layer

Asymptotic time layer

Uniform convergence in relative error: statement

Theorem (M. Bonforte, J. Dolbeault, B. Nazaret, N. S. (2020) and M. Bonforte, N.S (2019))

Under the current assumptions, let $\varepsilon \in (0, 1/2)$, small enough, $A > 0$, and $G > 0$. There exists an explicit time $t_\star \geq 0$ such that, if v is a solution of

$$\frac{\partial v}{\partial t} + \nabla \cdot \left[v \left(\nabla v^{m-1} - 2\,x \right) \right] = 0$$

with nonnegative initial datum $v_0 \in \mathrm{L}^1(\mathbb{R}^d)$ satisfying

$$\sup_{r>0} r^{\frac{2-d\,(1-m)}{(1-m)}} \int_{|x|>r} v_0\,dx \leq A < \infty \qquad (\mathrm{H}_A)$$

$\int_{\mathbb{R}^d} v_0\,dx = \int_{\mathbb{R}^d} \mathcal{B}\,dx = M$ *and* $\mathcal{F}[u_0] \leq G$, *then*

$$\sup_{x\in\mathbb{R}^d} \left| \frac{v(t,x)}{\mathcal{B}(x)} - 1 \right| \leq \varepsilon \quad \forall\, t \geq T := \log\{\tau_\star \frac{1 + A^{1-m} + G^{\frac{\alpha}{2}}}{\varepsilon^{\mathsf{a}}}\}$$

Back to Improved entropy-entropy production inequality

Theorem

Let $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/2, 1)$ if $d = 1$, $A > 0$ and $G > 0$. Then there is a positive number ζ such that

$$\mathcal{I}[v] \geq (4 + \zeta)\, \mathcal{F}[v]$$

for any nonnegative function $v \in \mathrm{L}^1(\mathbb{R}^d)$ such that $\mathcal{F}[v] = G$, $\int_{\mathbb{R}^d} v\, dx = M$, $\int_{\mathbb{R}^d} x\, v\, dx = 0$ and v satisfies (H_A)

We have the *asymptotic time layer estimate*

$$(1 - \varepsilon)\, \mathcal{B} \leq v(t, \cdot) \leq (1 + \varepsilon)\, \mathcal{B} \quad \forall\, t \geq T$$

and, as a consequence, the initial time layer estimate

$$\mathcal{I}[v(t, .)] \geq (4 + \zeta)\, \mathcal{F}[v(t, .)] \quad \forall\, t \in [0, T]\,, \quad \text{where} \quad \zeta = \frac{4\, \eta\, e^{-4\,T}}{4 + \eta - \eta\, e^{-4\,T}}$$

Part III: The Sobolev inequality

Why Sobolev is different?

The scaklee invariant deficit functional:

$$\delta_{HGNS}[f] = \|\nabla f\|_2^{\vartheta}\ \|f\|_{p+1}^{1-\vartheta} - \mathcal{C}_{GNS}\,\|f\|_{2p} \geq 0$$

where for $d \geq 3$, $1 < p \leq \frac{d}{d-2} = p^\star$, and for $d = 1, 2$ $1 < p < \infty = p^\star$ and $\delta_{HGNS}[f] = 0$ iff

$$f \in \{g_{\lambda,\mu,y}(x) := \mu\,\mathsf{g}\left((x-y)/\lambda\right)\} \quad \text{where} \quad \mathsf{g}(x) = \left(1 + |x|^2\right)^{-\frac{1}{p-1}}.$$

The non scale invariant deficit functional:

$$\delta_{GNS}[f] := a\ \|\nabla f\|_2^2 + b\ \|f\|_{p+1}^{p+1} - \mathcal{K}_{\mathrm{GN}}\,\|f\|_{2p}^{2\,p\,\gamma} = 0$$

where $a = \frac{1}{2}\,(p-1)^2$, $b = 2\,\frac{d-p\,(d-2)}{p+1}$, $\mathcal{K}_{\mathrm{GN}} = \|\mathsf{g}\|_{L^{2p}(\mathbb{R}^d)}^{2\,p\,(1-\gamma)}$ and $\gamma = \frac{d+2-p\,(d-2)}{d-p\,(d-4)}$ iff $f = g_{\lambda,\mu,y}$ such that

$$\lambda[f]\,\mu[f]^{\frac{p-1}{d-p\,(d-4)}} = \sqrt{\frac{d\,(p-1)}{d+2-p\,(d-2)}}\,.$$

$$\boxed{\|\nabla f\|_2^2 - \mathsf{S}_d\ \|f\|_{2\,p^\star}^2 = C(\mathcal{I}[f] - 4\,\mathcal{F}[f])}$$

Constructive Stability for the Sobolev's inequality

The *relative entropy*

$$\mathcal{F}[f|\mathsf{g}] := \tfrac{2\,p}{1-p}\int_{\mathbb{R}^d}\left(|f|^{p+1}-\mathsf{g}^{p+1}-\tfrac{1+p}{2\,p}\,\mathsf{g}^{1-p}\left(|f|^{2\,p}-\mathsf{g}^{2\,p}\right)\right)dx=\int_{\mathbb{R}^d}\left(\mathsf{g}^{2\,\frac{d-1}{d-2}}-f^{2\,\frac{d-1}{d-2}}\right)dx$$

Theorem

Let $d\geq 3$ and $A>0$. Then for any nonnegative function $f\in\mathcal{W}_{p^\star}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d}(1,x,|x|^2)\,f^{2^*}\,dx=\int_{\mathbb{R}^d}(1,x,|x|^2)\,\mathsf{g}\,dx \quad \textit{and} \quad \sup_{r>0} r^d\int_{|x|>r} f^{2\,p^\star}\,\mathrm{d}x\leq A\,,$$

we have

$$\|\nabla f\|_2^2-\mathsf{S}_d^2\,\|f\|_{2\,p^\star}^2\geq\mathcal{C}_\star(A)\,\mathcal{F}[f|g]$$

The stability constant is $\mathcal{C}_\star(A)=\mathfrak{C}_\star\left(1+A^{1/(2\,d)}\right)^{-1}$ where $\mathfrak{C}_\star>0$ depends only on d.

▷ As a consequence we get $\delta_{GNS}[f]\geq\mathcal{C}\,\||f|-\mathsf{g}\|_{2p}^{4p}$ and $\delta_{GNS}[f]\geq\|\nabla|f|-\nabla\mathsf{g}\|_2^8$

The Sobolev case-I

Let us consider the *fast diffusion equation*, which we recall for convenience:

$$\frac{\partial v}{\partial t} + \nabla \cdot \left(v \, \nabla v^{m-1} \right) = 2 \, \nabla \cdot (x \, v) \,, \quad v(t=0,\cdot) = v_0 \,. \tag{FDE}$$

For any $x \in \mathbb{R}^d$, let us consider the Barenblatt profile $\mathcal{B}_\lambda$ defined by

$$\mathcal{B}_\lambda(x) = \lambda^{-\frac{d}{2}} \, \mathcal{B}\left(\frac{x}{\sqrt{\lambda}} \right) \quad \text{where} \quad \mathcal{B}(x) = \left(1 + |x|^2 \right)^{1/(m-1)} \,.$$

We are interested in the specific choice of λ corresponding to

$$\lambda(t) := \frac{1}{\mathcal{K}_\star \, \mathfrak{R}(t)^2} \int_{\mathbb{R}^d} |x|^2 \, v(t,x) \, dx \quad \text{with} \quad \mathcal{K}_\star := \int_{\mathbb{R}^d} |x|^2 \, \mathcal{B} \, dx \,,$$

where v solves (FDE) and $t \mapsto \mathfrak{R}(t)$ is obtained by solving

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \left(\frac{1}{\mathcal{K}_\star} \int_{\mathbb{R}^d} |x|^2 \, v \, dx \right)^{-\frac{d}{2}\,(m-m_c)} - 1 \,, \quad \tau(0) = 0 \quad \text{and} \quad \mathfrak{R}(t) = e^{2\,\tau(t)} \quad \forall \, t \geq 0 \,.$$

With these definitions, let us consider the change of variables

$$v(t,x) = \frac{1}{\mathfrak{R}(t)^d} \, w\left(t + \tau(t), \frac{x}{\mathfrak{R}(t)} \right) \quad \forall \, (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d \,.$$

The Sobolev Case-II

▷Let us consider the change of variables

$$v(t,x) = \frac{1}{\mathfrak{R}(t)^d}\, w\left(t+\tau(t),\, \frac{x}{\mathfrak{R}(t)}\right) \quad \forall\,(t,x)\in\mathbb{R}^+\times\mathbb{R}^d\,.$$

▷ If w is obtained by the above change of variables, then w solves

$$\frac{\partial w}{\partial s} + \lambda_\star(s)^{\frac{d}{2}\,(m-m_c)}\,\nabla\cdot\left(w\,\nabla w^{m-1}\right) = 2\,\nabla\cdot(x\,w)\,, \quad w(t=0,\cdot) = v_0\,,$$

where the function $t\mapsto s(t) := t+\tau(t)$ is monotone increasing on $\mathbb{R}^+$, $\lambda_\star$ is defined by

$$\lambda_\star\big(s(t)\big) = \lambda(t) \quad \forall\, t\geq 0$$

and the function $\mathcal{B}_\star(s,x) := \mathcal{B}_{\lambda_\star(s)}(x)$ is such that for all $s\geq 0$

$$\int_{\mathbb{R}^d}\left(1,x,|x|^2\right) w(s,x)\,dx = \int_{\mathbb{R}^d}\left(1,x,|x|^2\right)\mathcal{B}_\star(s,x)\,dx$$

▷Same strategy as before but considering the best matching quantities

$$\mathcal{F}[w,\mathcal{B}_\star] \quad \text{and} \quad \mathcal{I}[w,\mathcal{B}_\star]$$

Thank you for your attention!

Thank you EFI!