Explicit decay rates for discrete velocity BGK models

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linear BGK models[1]

$$\partial_t f(v,x,t) + v\partial_x f(v,x,t) = \mathrm{Q}_{\mathrm{relax}} f(v,x,t)$$
$$:= \sigma \left[M_T(v) \int_{\mathbb{R}} f(x,v,t)dv - f(x,v,t) \right]$$

Idea: Replace nonlinear Boltzmann collision by linear relaxation term.

- $f(v,x,t)$ particle density at velocity $v \in \mathbb{R}$, position $x \in \Omega$, time $t \geq 0$.
- $M_T(v)$ is the normalized Maxwellian at temperature T.
- Prescribed relaxation rate $\sigma > 0$.
- Conservation of mass and momentum.

[1] named after the three physicists Bhatnagar, Gross, and Krook

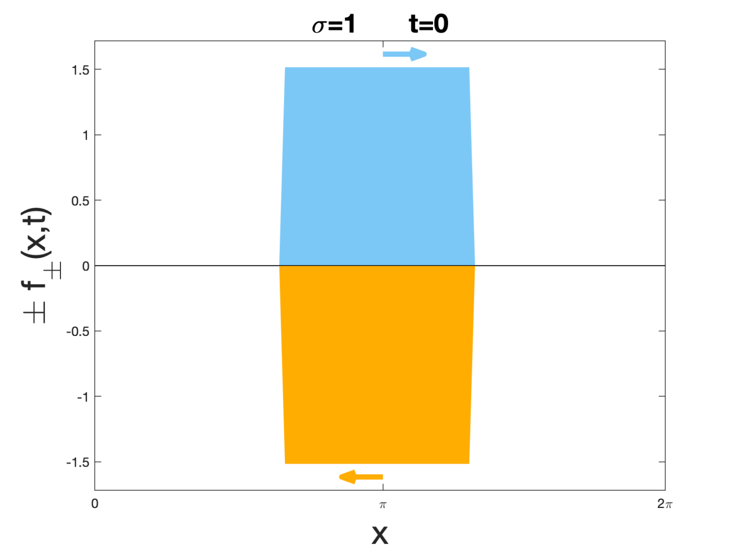
Further simplification: From $v \in \mathbb{R}$ to $v \in \{-1, +1\}$:

Goldstein–Taylor Model for $x \in \mathbb{T}^1$:

$$\begin{aligned} \partial_t f_+(x,t) + \partial_x f_+(x,t) &= \frac{\sigma(x)}{2}(f_-(x,t) - f_+(x,t)), \\ \partial_t f_-(x,t) - \partial_x f_-(x,t) &= -\frac{\sigma(x)}{2}(f_-(x,t) - f_+(x,t)), \\ f_\pm(x,0) &= f_{\pm,0} \in L^2(\mathbb{T}) \end{aligned} \tag{GT}$$

- $f_\pm(\cdot, t)$ probability density of particles with $v = \pm 1$.
- Relaxation coefficient $\sigma(x) > 0$.

Equation exhibits **hypocoercive dynamics**



Global (normalized) equilibrium: $(f_+^\infty, f_-^\infty)^T = (\frac{1}{2}, \frac{1}{2})^T$

Overview

Goals:

- Analyze the long-time behavior of the GT model for **non-homogeneous relaxation** $\sigma(x)$.
- Obtain **explicit decay rates**.
- Extend results to **multi-velocity** BGK setting.

Strategy:

- First **homogeneous relaxation:**
  Mode-by-Mode Lyapunov functional for lin. ODEs.
- **Non-homogeneous relaxation:**
  Functional via pseudo-differential operator.

Constant Relaxation $\sigma > 0$

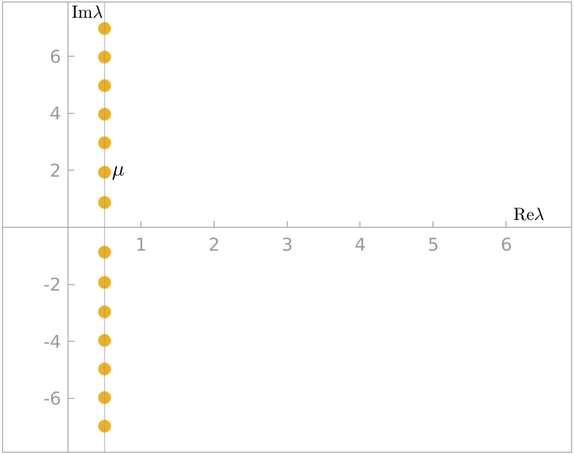
Reformulate equation in mass density and flux density

$$u := f_+ + f_-, \qquad w := f_+ - f_-.$$

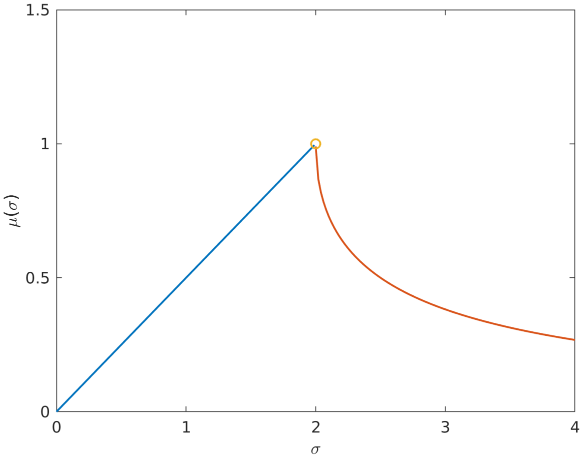
Goldstein–Taylor Model in Fourier Space:

$$\partial_t \begin{pmatrix} \hat{u}_k(t) \\ \hat{w}_k(t) \end{pmatrix} = -\underbrace{\begin{pmatrix} 0 & ik \\ ik & \sigma \end{pmatrix}}_{\mathrm{C}_k(\sigma)} \underbrace{\begin{pmatrix} \hat{u}_k(t) \\ \hat{w}_k(t) \end{pmatrix}}_{\hat{y}_k(t)}, \quad k \in \mathbb{Z}.$$

Spectrum of $\mathrm{C}_k(\sigma)$ determines decay behaviour of solutions.

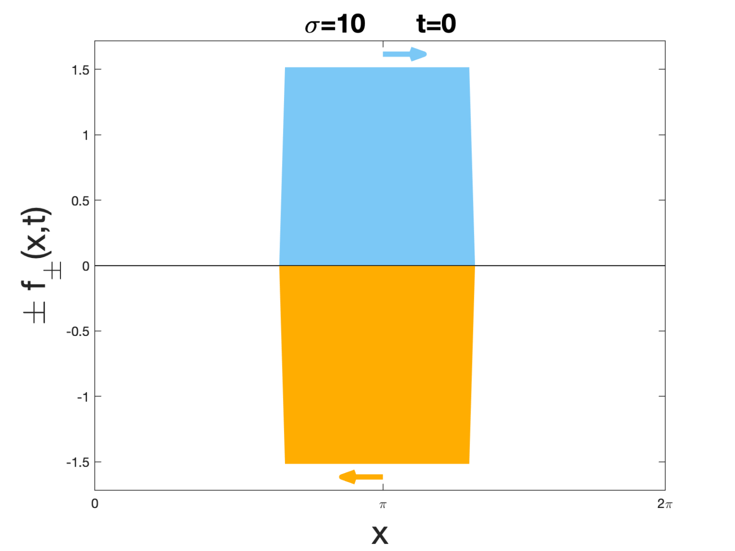


Spectrum of $\mathsf{C}_k(\sigma)$ for Fourier modes $k \in \mathbb{Z} \setminus \{0\}$.



uniform-in-k spectral gap

$$\mu(\sigma) := \min_{k\in\mathbb{Z}} \mu_k(\sigma) = \mu_1(\sigma) = \mathrm{Re}\left(\frac{\sigma}{2} - \sqrt{\frac{\sigma^2}{4} - 1}\right) \in (0, 1].$$



σ=10 t=0
1.5
1
0.5
0
-0.5
-1
-1.5
± f±(x,t)
0
π
2π
x

Hypocoercivity method

Plan: Construct mode-by-mode Lyapunov functional for Goldstein-Taylor model on $\mathbb{T}^1$

$$\partial_t \begin{pmatrix} \hat{u}_k(t) \\ \hat{w}_k(t) \end{pmatrix} = -\underbrace{\begin{pmatrix} 0 & ik \\ ik & \sigma \end{pmatrix}}_{\mathbf{C}_k(\sigma)} \underbrace{\begin{pmatrix} \hat{u}_k(t) \\ \hat{w}_k(t) \end{pmatrix}}_{\hat{y}_k(t)}, \quad k \in \mathbb{Z}.$$

- **Difficulty:** Matrices $\mathbf{C}_k$ are **not** Hermitian.

$$\frac{d}{dt}|\hat{y}_k|_2^2 = -\hat{y}_k^H(\mathbf{C}_k^H + \mathbf{C}_k)\hat{y}_k \leq 0.$$

- $\implies$ Euclidean norm needs to be modified to catch (sharp) decay rates.

Lyapunov matrix inequalities

Lemma 1 (Arnold, Erb '14)

Let $\mathsf{C} \in \mathbb{C}^{n\times n}$ *be* ***positive stable****, i.e.*
$\mu_{\mathsf{C}} := \min\{\operatorname{Re}\lambda \mid \lambda \in \sigma(\mathsf{C})\} > 0\,.$

1. *If all* $\lambda_\mu \in \{\lambda \in \sigma(\mathsf{C}) \,|\, \operatorname{Re}\lambda = \mu\}$ *are* ***not defective***
   (i.e. geometric = algebraic multiplicity)
   $\Rightarrow \quad \exists\, \mathsf{P} \in \mathbb{C}^{n\times n},\ \mathsf{P} > 0 \ : \quad \mathsf{PC} + \mathsf{C}^H\mathsf{P} \geq 2\mu_{\mathsf{C}}\mathsf{P}\,.$

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2. *If (at least) one* λ_μ *is* ***defective*** $\Rightarrow$
   $\forall \varepsilon > 0 \quad \exists\, \mathsf{P} = \mathsf{P}(\varepsilon) > 0 \ : \quad \mathsf{PC} + \mathsf{C}^H\mathsf{P} \geq 2(\mu_{\mathsf{C}} - \varepsilon)\mathsf{P}\,.$

The Matrix P can be constructed explicitly.

Case $\sigma \in (0, 2)$:

For each Fourier mode $\hat{y}_k = (\hat{u}_k, \hat{w}_k)^T$, $k \in \mathbb{Z} \setminus \{0\}$, define the **modified norm** (constructed according to Lemma 1)

$$|\hat{y}_k|_{P_k}^2 := \hat{y}_k^H P_k \hat{y}_k, \qquad \text{with} \qquad P_k := \begin{pmatrix} 1 & \frac{\sigma}{2ik} \\ -\frac{\sigma}{2ik} & 1 \end{pmatrix} > 0.$$

$$\frac{d}{dt}|\hat{y}_k|_{\mathrm{P}_k}^2 = -\hat{y}_k(\mathrm{C}_k^H \mathrm{P}_k + \mathrm{P}_k \mathrm{C}_k)\hat{y}_k \leq -\sigma\, \hat{y}_k^H \mathrm{P}_k \hat{y}_k$$
$$\implies |\hat{y}_k(t)|_{\mathrm{P}_k}^2 \leq e^{-\sigma t}|\hat{y}_k(0)|_{\mathrm{P}_k}^2.$$

$|\cdot|_{P_k}^2$ is Lyapunov functional with sharp decay rate for each mode $k \in \mathbb{Z} \setminus \{0\}$.

Case $\sigma \in (0,2)$: Spatial Lyapunov functional

Idea: Recast functional in position space

$$\sum_{k\in\mathbb{Z}} \|(\hat{u}_k - \hat{u}_k^\infty, \hat{w}_k)^T\|_{P_k}^2 = \sum_{k\in\mathbb{Z}} |\hat{u}_k - \hat{u}_k^\infty|^2 + |\hat{w}_k|^2 - \sigma \operatorname{Re}(\overline{\hat{w}_k} \frac{\hat{u}_k - \hat{u}_k^\infty}{ik})$$
$$= \frac{1}{2\pi} \int_0^{2\pi} \Big((u - u^\infty)^2 + w^2 - \sigma w \partial_x^{-1}(u - u^\infty) \Big) dx$$
$$=: E_\sigma[u - u^\infty, w],$$

where $u^\infty \equiv 1$.

Spatial Entropy Functional

For parameter $\theta \in (0,2)$:

$$E_{\theta}[u,w] := \|u\|_{L^2}^2 + \|w\|_{L^2}^2 - \frac{\theta}{2\pi}\int_0^{2\pi} w\partial_x^{-1} u dx,$$

where

$$(\partial_x^{-1}u)(x) := \int_0^x u dx + c(u) \quad \text{with} \quad c(u) := -\frac{1}{2\pi}\int_0^{2\pi}\int_0^x u dy dx.$$

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Norm bounds:

$$(1-\tfrac{\theta}{2})\|(u-u^\infty, w)\|_{L^2}^2 \le E_\theta[u-u^\infty, w] \le (1+\tfrac{\theta}{2})\|(u-u^\infty, w)\|_{L^2}^2.$$

Decay estimates for constant $\sigma > 0$

Lemma 2 (Arnold, Einav, Signorello, W.)

Let $(u, w)^T$ be a solution to (GT) with constant $\sigma > 0$.

(i) *For $\sigma \in (0, 2)$*

$$E_\sigma[u(t) - u^\infty, w(t)] \leq E_\sigma[u(0) - u^\infty, w(0)]e^{-\sigma t}.$$

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$$E_{\sigma}[u(t) - u^{\infty}, w(t)] \leq E_{\sigma}[u(0) - u^{\infty}, w(0)]e^{-\sigma t}.$$

(ii) $\sigma = 2$, *with* $\theta_{\varepsilon} := \frac{2(2-\varepsilon^2)}{2+\varepsilon^2}$,

$$E_{\theta_{\varepsilon}}[u(t) - u^{\infty}, w(t)] \leq E_{\theta_{\varepsilon}}[u(0) - u^{\infty}, w(0)]e^{-2(1-\varepsilon)t}.$$

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$$E_{\theta_{\varepsilon}}[u(t)-u^{\infty}, w(t)] \le E_{\theta_{\varepsilon}}[u(0)-u^{\infty}, w(0)]e^{-2(1-\varepsilon)t}.$$

(iii) *For $\sigma > 2$ with* $\mu = \frac{\sigma}{2} - \sqrt{\frac{\sigma^2}{4} - 1}$,

$$E_{\frac{4}{\sigma}}[u(t)-u^{\infty}, w(t)] \le E_{\frac{4}{\sigma}}[u(0)-u^{\infty}, w(0)]e^{-2\mu t}.$$

Space-dependent relaxation $\sigma(x)$

Theorem 3 (Arnold, Einav, Signorello, W.)

Let $(u,w)^T$ be a solution to (GT) with $u_0, w_0 \in L^2(\mathbb{T})$ and

$$0 < \sigma_{\min} := \inf_{x\in\mathbb{T}} \sigma(x) \le \sup_{x\in\mathbb{T}} \sigma(x) =: \sigma_{\max} < \infty.$$

Then, for $\theta^* = \min\{\sigma_{\min}, \frac{4}{\sigma_{\max}}\}$ exists an explicit decay rate $\alpha^*(\sigma_{\min}, \sigma_{\max}) > 0$ such that

$$E_{\theta^*}[u(t) - u^\infty, w(t)] \le e^{-\alpha^* t} E_{\theta^*}[u_0 - u^\infty, w_0].$$

- **Proof:** perturbative approach
- Decay rate $\alpha^*(\sigma_{\min}, \sigma_{\max})$ is not sharp.

Discussion on GT result

[Bernard, Salvarani '13]:

- ⊕ For $\sigma \in L^1(\mathbb{T})$, $\sigma(x) \geq 0$ sharp decay rate

$$\alpha = \min\{\sigma_{\text{avg}}, D(0)\},$$

  where $D(0)$ is the spectral gap of Telegrapher's equation.
- ⊖ Method restricted to two velocities.
- ⊖ Rate in general not explicit.

Our approach:

- Extends to multi-velocity models with $\sigma(x)$.
- Extends to $x \in \mathbb{R}$.

Connection to [DMS], method for $\sigma \in (0, 2)$

Idea: Optimize "twist" operator A for each mode $k \in \mathbb{Z}$.

The **projection on local-in-x equilibria**

$$\Pi_k := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

"Twist" operator

$$A_k := \left(I + (T_k \Pi)^* T_k \Pi\right)^{-1} (T_k \Pi)^* = \begin{pmatrix} 0 & -\frac{i k}{1+k^2} \\ 0 & 0 \end{pmatrix}.$$

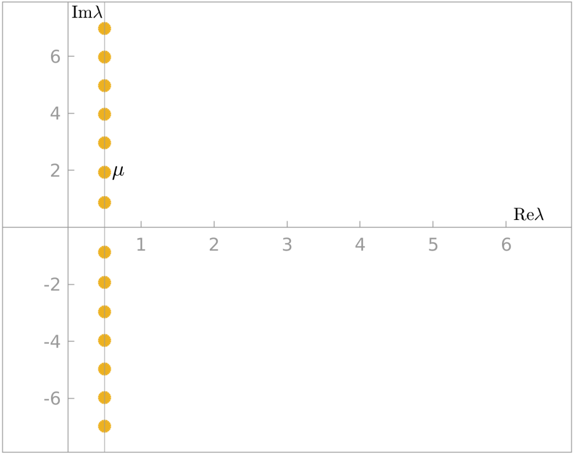
For $\delta_k(\sigma) := \frac{\sigma (1+k^2)}{2 k^2}$, the modal Lyapunov functional is given as

$$H_k(\delta)[\hat{y}_k] := \frac{1}{2} |\hat{y}_k|_2^2 + \delta_k(\sigma) \, \mathrm{Re}\left(\hat{y}_k^H A_k \hat{y}_k\right)$$
$$= \frac{1}{2} |\hat{y}_k|^2_{\mathbf{P}_k(\sigma)}.$$

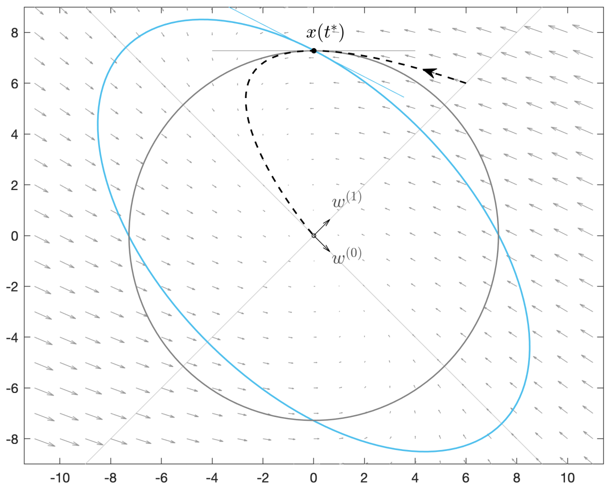
$\implies$ Sharp decay rates as $\mathbf{P}_k$ satisfies matrix inequality.

Thank you for your attention.

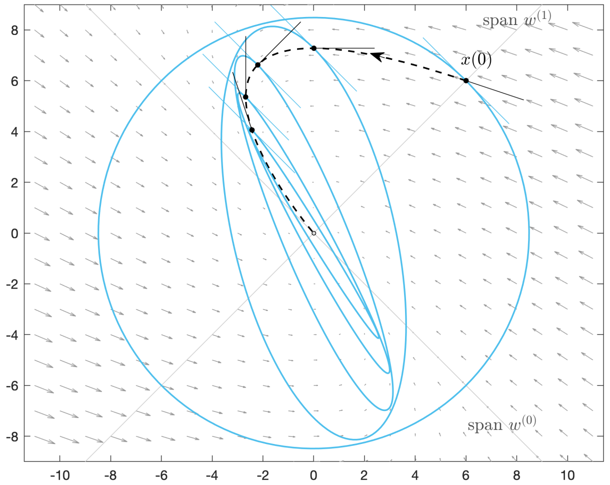
Arnold, A., Einav, A., Signorello, B., W., T.: Large time convergence of the non-homogeneous Goldstein–Taylor equation (2020).
Arnold, A., Dolbeault, J., Schmeiser, C. and W., T.: Sharpening of decay rates in Fourier based hypocoercivity methods (2021)
Bernard, É., Salvarani, F.: Optimal estimate of the spectral gap for the degenerate Goldstein–Taylor model. (2013).
Bouin, E., Dolbeault, J., Mischler, S., Mouhot, C., Schmeiser, C.: Hypocoercivity without confinement. (2020).



Eigenvalues of $A_k(\sigma = 1)$ for Fourier modes $k \in \mathbb{Z} \setminus \{0\}$.



$x(t^*)$
$w^{(1)}$
$w^{(0)}$
8
6
4
2
0
-2
-4
-6
-8
-10
-8
-6
-4
-2
0
2
4
6
8
10



span $w^{(1)}$
$x(0)$
span $w^{(0)}$
8
6
4
2
0
-2
-4
-6
-8
-10
-8
-6
-4
-2
0
2
4
6
8
10