STABILITY IN FUNCTIONAL INEQUALITIES

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Obtaining explicit stability estimates in classical functional inequalities like the Sobolev inequality has been an essentially open question 30 years, after the celebrated but non-constructive result [1] of G. Bianchi and H. Egnell in 1991. Recently, new methods have emerged which provide some clues on these fascinating questions. The goal of the course is to give a general overview on stability in some fundamental functional inequalities and introduce methods that can be used to obtain explicit estimates.

Program

(1) The Sobolev inequality and the non-constructive stability result of Bianchi–Egnell using concentration-compactness methods [1, 13]
(2) Duality and stability in Hardy-Littlewood-Sobolev inequalities [7]
(3) Entropy methods on the Euclidean space [14, 3]
(4) Stability results for Gagliardo-Nirenberg inequalities on the Euclidean space [2, 3]
(5) Stability results on the sphere and on the Gaussian space seen as an infinite dimensional limit of spheres [11, 8, 10, 6, 4, 5]
(6) A constructive stability result for the Sobolev and the logarithmic Sobolev inequalities [9]

Course 1. (26/3/2024)
(1) Some history of the Sobolev inequality. One can refer to R. Frank's course given in Cetraro [13] for a detailed overview and many references. One of the issues is to identify all optimal functions. The entropy methods (see [3, Chapters 1 & 2]) give an alternative strategy to symmetrization methods. The concentration-compactness methods (see for instance [18, Chapter 4]) is the key tool in the result [1] of G. Bianchi and H. Egnell but the proof is non-constructive.
(2) The duality between Sobolev and Hardy-Littlewood-Sobolev (HLS) inequalities is known for instance from [15]. It can be used as in [7] to produce a stability result in (HLS). Again the proof is non-constructive. A simpler, constructive result can be obtained in a weaker norm as in [12], which moreover identifies the optimal functions for the optimal stability result using an improved version of the inequality based on a nonlinear flow.

Course 2. (28/3/2024)
(3) An introduction to Rényi entropy powers and relative entropies for the fast diffusion flow associated to a family of Gagliardo-Nirenberg inequalities: see [3, Chapter 2] for details. How to connect the Gagliardo-Nirenberg inequalities with Sobolev's inequality by Bakry's trick is detailed in [3, Section 1.2.1.2]. As a consequence, one can use as in [17] the result of Bianchi–Egnell (conveniently adapted, also see [16]) to produce a non-constructive stability result for the Gagliardo-Nirenberg inequalities.
(4) Stability results for Gagliardo-Nirenberg inequalities on the Euclidean space: the strategy [2].

Course 3. (2/4/2024)
(4) Stability results for Gagliardo-Nirenberg inequalities on the Euclidean space (continued) the method is detailed, with the spectral expansion in the large time asymptotics [2, Chapter 2], regularity results to control the threshold time [3, Chapters 3 & 4], and a backward-in-time argument

based on the *carré du champ method*. Stability result are obtained, where the distance to the optimal functions is measured by the relative Fisher information in the subcritical case [2, Chapter 5] and in the critical case of the logarithmic Sobolev inequality [2, Chapter 6]. The method is constructive and explains how a global problem can be reduced to a local analysis, to the price of a decay condition on the tails of the function.

(5) Note treated

(6) Detailed results will be exposed in the Workshop *Nonlinear Analysis: Geometric, Variational and Dispersive aspects* on April 5, 2024. In the case of the logarithmic Sobolev inequality, stability results in strong norms can be achieved (see [4]) using moment conditions that control the behaviour of the tails of the functions.

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