OPTIMAL FUNCTIONAL INEQUALITIES AND NONLINEAR FLOWS

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ABSTRACT. Le cours portera sur des inégalités fonctionnelles liées à l'inégalité de Sobolev, les constantes optimales qui leur sont associées, et les stratégies pour obtenir des inégalités dites améliorées. Le principal outil du cours consistera à utiliser des flots linéaires ou non-linéaires définis par une équation aux dérivées partielles d'évolution (en pratique une équation de diffusion ou de dérive-diffusion). Il s'agira plus précisément d'exhiber une fonctionnelle (entropie généralisée, énergie libre) associée à l'inégalité, qui se comporte de manière monotone au cours de l'évolution, c'est-à-dire joue le rôle d'une fonctionnelle de Lyapunov.

On s'intéressera aux questions de structure (quand est-ce que l'équation d'évolution est un flot gradient de l'entropie pour une notion de distance bien choisie ? quand est-ce que le flot correspond à une structure de dualité ?), aux questions d'optimalité (quand est-ce qu'une version améliorée de l'inégalité garantit que le cas d'égalité dans l'inégalité est atteint dans le régime asymptotique ?). Par ailleurs, on introduira des situations géométriques variées (cas de variétés dont la courbure n'est pas de signe constant) et on fera le lien avec les méthodes de rigidité utilisées pour les équations semi-linéaires elliptiques et des estimations spectrales pour des opérateurs de Schrödinger. Dans tous les problèmes évoqués ci-dessus, on s'intéressera à des flots dont la monotonie est garantie non pas par une condition de positivité ponctuelle mais par une quantité intégrale, typiquement une condition de positivité d'une valeur propre.

The page of this course is available at
https://www.ceremade.dauphine.fr/∼dolbeaul/Courses/Cimi-2014/index.pdf
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1. Gradient flows and functional inequalities

1.1. An introduction.

1.2. The Ornstein-Uhlenbeck equation: a nonlocal criterion for some interpolation inequalities.

See [20, Section 1, pp. 478–484] or

https://www.ceremade.dauphine.fr/∼dolbeaul/Preprints/Publi/0752.pdf
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1.3. The gradient flow interpretation.

See [21, Section 2, pp. 3189–3192] or

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Also see [1] for a general reference.

1.4. An introduction to the fast diffusion equation and to the related Gagliardo-Nirenberg inequalities.

See [25] for a general reference on the fast diffusion equation / porous media equation and [7] for the relation of the fast diffusion equation with a class of Gagliardo-Nirenberg inequalities. See [2, chapter 6] for a general presentation of the above class of Gagliardo-Nirenberg inequalities within the framework of Sobolev inequalities.

2. Fast diffusion equation: rates, linearization, duality

2.1. Best matching self-similar profiles and improved rates of convergence in nonlinear diffusion equations.

https://www.ceremade.dauphine.fr/∼dolbeaul/Courses/Cimi-2014/FDE.pdf
(login: Jean , password: Dolbeault)

See [22, 23].

2.2. Gagliardo-Nirenberg inequalities for systems and Lieb-Thirring inequalities.

https://www.ceremade.dauphine.fr/∼dolbeaul/Courses/Cimi-2014/LT.pdf
(login: Jean , password: Dolbeault)

See [17, 19].

2.3. Sharp asymptotics for the subcritical Keller-Segel model.

https://www.ceremade.dauphine.fr/∼dolbeaul/Courses/Cimi-2014/KS.pdf
(login: Jean , password: Dolbeault)

See [9, 5].

3. Flows on the sphere and on compact manifolds, and improved interpolation inequalities

3.1. Sobolev and Hardy-Littlewood-Sobolev inequalities: duality, flows. The limit case of Moser-Trudinger-Onofri and logarithmic Hardy-Littlewood-Sobolev inequalities.

https://www.ceremade.dauphine.fr/∼dolbeaul/Courses/Cimi-2014/SobHLS.pdf
(login: Jean , password: Dolbeault)

[6, 3, 4, 8, 18, 10, 24].

3.2. Interpolation inequalities (Gagliardo-Nirenberg, Sobolev and Onofri inequalities): rigidity results, nonlinear flows and applications.

https://www.ceremade.dauphine.fr/∼dolbeaul/Courses/Cimi-2014/GNS.pdf
(login: Jean , password: Dolbeault)

See [15, 11, 14, 13, 16, 10, 12].

4. Concluding remarks

• *The goal: improving functional inequalities, what for ?*

1. to be able to identify all extremal functions,
2. to get stability results,
3. to explain how linearized estimates can be used in nonlinear problems,
4. to explain faster convergence rates than expected from classical functional inequalities, for solutions to nonlinear evolution equations.

Recall that "best constants correspond to the worst possible case" and so it makes sense to quantify how far one is from the worst case or, for evolution problem, what can be expected for the speed of convergence when one is away from the worst case in terms of convergence. For instance, when convergence is faster than exponential, it is interesting to evaluate for how long one has to wait before entering the asymptotic regime which can be, for instance, exponential.

• *The tools:*

1. An adapted nonlinear flow which monotonically explores the energy landscape (you can see eventually it as a gradient flow but this is not the point),
2. Linearization methods around that attractor, even if it is not known explicitly, as it is the case for the sub-critical Keller-Segel model; by identifying relevant modes in the asymptotic regime, one can prove linear stability properties or get improved asymptotic rates or improved quantities (relative entropies with respect to best matching profiles) in order to measure convergence at the nonlinear level, and even get improved speeds of convergence using the generators of the transformations associated with the lowest relevant modes,
3. The remainder terms in the entropy – entropy production terms: they are the quantities that have to be studied to get improved functional inequalities.

• *A bonus.* In some cases, the stationarity when applying the flow can be discarded and provides for free a *rigidity* result, thus showing that the unique critical points are the limiting points for the evolution by the flow. This provides a method for proving uniqueness which is of interest by itself, and explains why a global result can be expected, at leafs in some range of the parameters.

• *The challenges.*

1. Improvements have been obtained by crude estimates so far, in interpolation inequalities of Gagliardo-Nirenberg-Sobolev type; they are themselves not optimal and deserve further study, with applications to the interplay between linearized problems and nonlinear ones, and estimates of how long it takes to enter in the asymptotic regime.
2. Essentially nothing has been done for systems, while there are potentially important applications in mathematical biology (chemotaxis, coupled reaction-diffusion systems, etc) and in physics (stability of matter).

Warning: the references list indicates a selection of a few textbooks and of some research papers they were used for the preparation of this course. The reader who wants to have a more reasonable account of the existing literature is invited to refer to the references quoted inside each of them.

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