Fast diffusion, mean field drifts and reverse HLS inequalities
Critical and subcritical inequalities: Flows, linearization and entropy methods

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Lecture 1

Intensive Week of PDEs@Cogne

Outline

▷ **Without weights: Gagliardo-Nirenberg inequalities and fast diffusion flows – *Entropies***

- The Bakry-Emery method on the sphere
- The fast diffusion equation on the Euclidean space
- Rényi entropy powers
- Self-similar variables and relative entropies
- Equivalence of the methods ?
- The role of the spectral gap

▷ **Symmetry breaking and linearization**

▷ **With weights: Caffarelli-Kohn-Nirenberg inequalities and weighted nonlinear flows**

- Towards a parabolic proof
- Large time asymptotics and spectral gaps
- Optimality cases

Collaborations

Collaboration with...

M.J. Esteban and M. Loss (symmetry, critical case)
M.J. Esteban, M. Loss and M. Muratori (symmetry, subcritical case)
M. Bonforte, M. Muratori and B. Nazaret (linearization and large time asymptotics for the evolution problem)
M. del Pino, G. Toscani (nonlinear flows and entropy methods)
S. Filippas, A. Tertikas, G. Tarantello, M. Kowalczyk
A. Blanchet, G. Grillo, J.L. Vázquez (large time asymptotics and linearization for the evolution equations)
M. Muratori and M. Muratori (new weighted interpolation inequalities)
N. Simonov (stability, weighted inequalities and entropy – entropy production inequalities)
M. Bonforte, B. Nazaret and N. Simonov (regularity and stability)

Background references (partial)

- Rigidity methods, uniqueness in nonlinear elliptic PDE's: (B. Gidas, J. Spruck, '81), (M.-F. Bidaut-Véron, L. Véron, '91)
- Probabilistic methods (Markov processes), semi-group theory and *carré du champ* methods (Γ_2 theory): (D. Bakry, M. Emery, 1984), (Bakry, Ledoux, 1996), (Demange, 2008), (JD, Esteban, Loss, 2014 & 2015) → *D. Bakry, I. Gentil, and M. Ledoux. Analysis and geometry of Markov diffusion operators (2014)*
- Entropy methods in PDEs
  ▷ Entropy-entropy production inequalities: Arnold, Carrillo, Desvillettes, JD, Jüngel, Lederman, Markowich, Toscani, Unterreiter, Villani..., (del Pino, JD, 2001), (Blanchet, Bonforte, JD, Grillo, Vázquez) → *A. Jüngel, Entropy Methods for Diffusive Partial Differential Equations (2016)*
  ▷ Mass transportation: (Otto) → *C. Villani, Optimal transport. Old and new (2009)*
  ▷ Rényi entropy powers (information theory) (Savaré, Toscani, 2014), (Dolbeault, Toscani)

Some preliminaries

▷ The Bakry-Emery method or *carré du champ* method (or Γ_2 method)

▷ Inequalities without weights and fast diffusion equations

Figure: The Bakry-Emery method... Courtesy: Nassif Ghoussoub

An interpolation inequality on the sphere

An example of interpolation by flows and entropy methods

$$\int_{\mathbb{S}^d} |\nabla u|^2 \, d\mu \ge \frac{d}{p-2} \left[\left(\int_{\mathbb{S}^d} |u|^p \, d\mu \right)^{\frac{2}{p}} - \int_{\mathbb{S}^d} |u|^2 \, d\mu \right]$$

(W. Beckner, 1993), (M.-F. Bidaut-Véron, L. Véron,1991)
Conditions $u \in \mathrm{H}^1(\mathbb{S}^d)$, $p \ge 1$, $p \neq 2$, $p < 2^*$ if $d \ge 3$.

With $\rho = |u|^p$

$$\int_{\mathbb{S}^d} |\nabla \rho^{\frac{1}{p}}|^2 \, d\mu \ge \frac{d}{p-2} \left[\left(\int_{\mathbb{S}^d} \rho \, d\mu \right)^{\frac{2}{p}} - \int_{\mathbb{S}^d} \rho^{\frac{2}{p}} \, d\mu \right]$$

The Bakry-Emery method on the sphere

Entropy functional

$$\mathcal{E}_p[\rho] := \frac{1}{p-2}\left[\left(\int_{\mathbb{S}^d}\rho\,d\mu\right)^{\frac{2}{p}} - \int_{\mathbb{S}^d}\rho^{\frac{2}{p}}\,d\mu\right] \quad \text{if} \quad p \neq 2$$

$$\mathcal{E}_2[\rho] := \int_{\mathbb{S}^d}\rho\,\log\left(\frac{\rho}{\|\rho\|_{\mathrm{L}^1(\mathbb{S}^d)}}\right)d\mu$$

Fisher information functional

$$\mathcal{I}_p[\rho] := \int_{\mathbb{S}^d}|\nabla\rho^{\frac{1}{p}}|^2\,d\mu$$

Bakry-Emery (carré du champ) method: use the heat flow

$$\frac{\partial\rho}{\partial t} = \Delta\rho$$

and compute $\frac{d}{dt}\mathcal{E}_p[\rho] = -\,\mathcal{I}_p[\rho]$ and $\frac{d}{dt}\mathcal{I}_p[\rho] \leq -\,d\,\mathcal{I}_p[\rho]$ to get

$$\frac{d}{dt}\left(\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho]\right) \leq 0 \quad\Longrightarrow\quad \mathcal{I}_p[\rho] \geq d\,\mathcal{E}_p[\rho]$$

with $\rho = |u|^p$, if $p \leq 2^\# := \frac{2\,d^2+1}{(d-1)^2}$

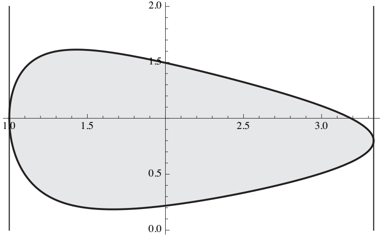
The evolution under the fast diffusion flow

To overcome the limitation $p \leq 2^{\#}$, one can consider a nonlinear diffusion of fast diffusion / porous medium type

$$\frac{\partial \rho}{\partial t} = \Delta \rho^m$$

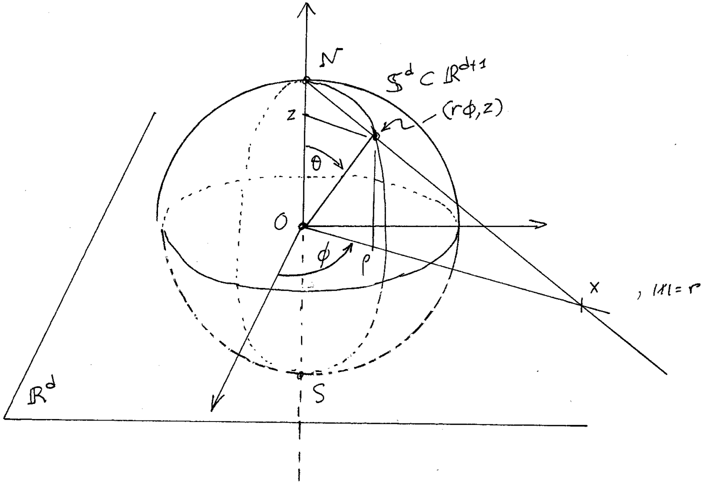
(Demange), (JD, Esteban, Kowalczyk, Loss): for any $p \in [1, 2^*]$

$$\mathcal{K}_p[\rho] := \frac{d}{dt}\Big(\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho]\Big) \leq 0$$



(p, m) admissible region, $d = 5$

Cylindrical coordinates, Schwarz symmetrization, stereographic projection...



... and the ultra-spherical operator

Change of variables $z=\cos\theta$, $v(\theta)=f(z)$, $d\nu_d:=\nu^{\frac{d}{2}-1}\,dz/Z_d$, $\nu(z):=1-z^2$

The self-adjoint *ultraspherical* operator is

$$\mathcal{L}\,f:=(1-z^2)\,f''-d\,z\,f'=\nu\,f''+\frac{d}{2}\,\nu'\,f'$$

which satisfies $\langle f_1,\mathcal{L}\,f_2\rangle=-\int_{\mathbb{R}^d}f_1'\,f_2'\,\nu\,d\nu_d$

Proposition

Let $p\in[1,2)\cup(2,2^*]$, $d\ge1$. *For any* $f\in\mathrm{H}^1([-1,1],d\nu_d)$,

$$-\langle f,\mathcal{L}\,f\rangle=\int_{\mathbb{R}^d}|f'|^2\,\nu\,d\nu_d\ge d\,\frac{\|f\|^2_{\mathrm{L}^p(\mathbb{S}^d)}-\|f\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{p-2}$$

The heat equation $\frac{\partial g}{\partial t} = \mathcal{L}\, g$ for $g = f^p$ can be rewritten in terms of f as

$$\frac{\partial f}{\partial t} = \mathcal{L}\, f + (p-1)\, \frac{|f'|^2}{f}\, \nu$$

$$-\frac{1}{2}\,\frac{d}{dt}\int_{\mathbb{R}^d} |f'|^2\,\nu\, d\nu_d = \frac{1}{2}\,\frac{d}{dt}\,\langle f, \mathcal{L}\, f\rangle = \langle \mathcal{L}\, f, \mathcal{L}\, f\rangle + (p-1)\,\left\langle \frac{|f'|^2}{f}\,\nu, \mathcal{L}\, f\right\rangle$$

$$\frac{d}{dt}\mathcal{I}[g(t,\cdot)] + 2\,d\,\mathcal{I}[g(t,\cdot)] = \frac{d}{dt}\int_{\mathbb{R}^d} |f'|^2\,\nu\, d\nu_d + 2\,d\int_{\mathbb{R}^d} |f'|^2\,\nu\, d\nu_d$$

$$= -\,2\int_{\mathbb{R}^d}\left(|f''|^2 + (p-1)\,\frac{d}{d+2}\,\frac{|f'|^4}{f^2} - 2\,(p-1)\,\frac{d-1}{d+2}\,\frac{|f'|^2\, f''}{f}\right)\nu^2\, d\nu_d$$

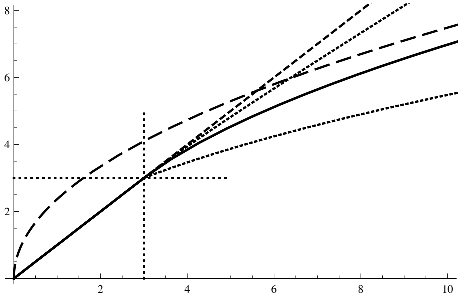
is nonpositive if

$$|f''|^2 + (p-1)\,\frac{d}{d+2}\,\frac{|f'|^4}{f^2} - 2\,(p-1)\,\frac{d-1}{d+2}\,\frac{|f'|^2\, f''}{f}$$

is pointwise nonnegative, which is granted if

$$\left[(p-1)\,\frac{d-1}{d+2}\right]^2 \le (p-1)\,\frac{d}{d+2} \iff p \le \frac{2\,d^2+1}{(d-1)^2} = 2^\# < \frac{2\,d}{d-2} = 2^*$$

Bifurcation point of view



The interpolation inequality from the point of view of bifurcations

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \lambda \, \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \geq \mu(\lambda) \, \|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}$$

Taylor expansion of $u = 1 + \varepsilon \, \varphi_1$ as $\varepsilon \to 0$ with $-\Delta \varphi_1 = d \, \varphi_1$

$$\mu(\lambda) \leq \lambda \quad \text{if (and only if)} \quad \lambda > \frac{d}{p-2}$$

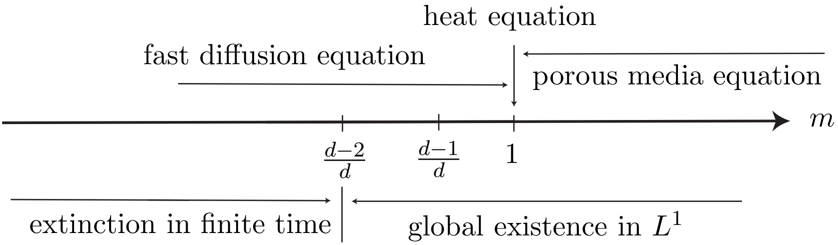
Euclidean space: Existence, classical results

Fast diffusion and porous medium equation

$$u_t = \Delta u^m \quad x \in \mathbb{R}^d\,,\ t > 0$$

Self-similar (Barenblatt) function: $\mathcal{U}(t) = O(t^{-d/(2-d(1-m))})$ as $t \to +\infty$

(Friedmann, Kamin, 1980) $\|u(t,\cdot) - \mathcal{U}(t,\cdot)\|_{L^\infty} = o(t^{-d/(2-d(1-m))})$



Existence theory, critical values of the parameter m

Inequalities without weights and fast diffusion equations

- ▷ Rényi entropy powers
- ▷ Self-similar variables and relative entropies
- ▷ Equivalence of the methods ?
- ▷ The role of the spectral gap

Rényi entropy powers and fast diffusion

▷ Rényi entropy powers, the entropy approach without rescaling: (Savaré, Toscani): scalings, nonlinearity and a concavity property inspired by information theory

▷ faster rates of convergence: (Carrillo, Toscani), (JD, Toscani)

The fast diffusion equation in original variables

Consider the nonlinear diffusion equation in $\mathbb{R}^d$, $d \geq 1$

$$\frac{\partial v}{\partial t} = \Delta v^m$$

with initial datum $v(x, t = 0) = v_0(x) \geq 0$ such that $\int_{\mathbb{R}^d} v_0 \, dx = 1$ and $\int_{\mathbb{R}^d} |x|^2 \, v_0 \, dx < +\infty$. The large time behavior of the solutions is governed by the source-type Barenblatt solutions

$$\mathcal{U}_\star(t, x) := \frac{1}{\left(\kappa \, t^{1/\mu}\right)^d} \, \mathcal{B}_\star\left(\frac{x}{\kappa \, t^{1/\mu}}\right)$$

where

$$\mu := 2 + d\,(m-1)\,, \quad \kappa := \left|\frac{2\,\mu\,m}{m-1}\right|^{1/\mu}$$

and $\mathcal{B}_\star$ is the Barenblatt profile

$$\mathcal{B}_\star(x) := \begin{cases} \left(C_\star - |x|^2\right)_+^{1/(m-1)} & \text{if } m > 1 \\ \left(C_\star + |x|^2\right)^{1/(m-1)} & \text{if } m < 1 \end{cases}$$

The Rényi entropy power F

The *entropy* is defined by

$$\mathsf{E} := \int_{\mathbb{R}^d} v^m \, dx$$

and the *Fisher information* by

$$\mathsf{I} := \int_{\mathbb{R}^d} v \, |\nabla \mathsf{p}|^2 \, dx \quad \text{with} \quad \mathsf{p} = \frac{m}{m-1} \, v^{m-1}$$

If v solves the fast diffusion equation, then

$$\mathsf{E}' = (1-m) \, \mathsf{I}$$

To compute I', we will use the fact that

$$\frac{\partial \mathsf{p}}{\partial t} = (m-1) \, \mathsf{p} \, \Delta \mathsf{p} + |\nabla \mathsf{p}|^2$$

$$\mathsf{F} := \mathsf{E}^\sigma \quad \text{with} \quad \sigma = \frac{\mu}{d\,(1-m)} = 1 + \frac{2}{1-m} \left(\frac{1}{d} + m - 1 \right) = \frac{2}{d} \, \frac{1}{1-m} - 1$$

has a linear growth asymptotically as $t \to +\infty$

The concavity property

Theorem

(Toscani-Savaré) *Assume that* $m \ge 1 - \frac{1}{d}$ *if* $d > 1$ *and* $m > 0$ *if* $d = 1$. *Then* $F(t)$ *is increasing,* $(1-m)\,\mathsf{F}''(t) \le 0$ *and*

$$\lim_{t\to+\infty} \frac{1}{t}\,\mathsf{F}(t) = (1-m)\,\sigma \lim_{t\to+\infty} \mathsf{E}^{\sigma-1}\,\mathsf{I} = (1-m)\,\sigma\,\mathsf{E}_\star^{\sigma-1}\,\mathsf{I}_\star$$

(Dolbeault-Toscani) The inequality

$$\mathsf{E}^{\sigma-1}\,\mathsf{I} \ge \mathsf{E}_\star^{\sigma-1}\,\mathsf{I}_\star$$

is equivalent to the *optimal* Gagliardo-Nirenberg inequality

$$\|\nabla w\|_2^\theta\,\|w\|_{q+1}^{1-\theta} \ge \mathsf{C}_{\rm GN}\,\|w\|_{2q}$$

if $1 - \frac{1}{d} \le m < 1$. Hint: $v^{m-1/2} = \frac{w}{\|w\|_{2q}}$, $q = \frac{1}{2\,m-1}$

The proof

Lemma

If v solves $\frac{\partial v}{\partial t} = \Delta v^m$ with $\frac{1}{d} \le m < 1$, then

$$\mathsf{I}' = \frac{d}{dt} \int_{\mathbb{R}^d} v\,|\nabla \mathsf{p}|^2\,dx = -\,2 \int_{\mathbb{R}^d} v^m \left(\|\mathrm{D}^2 \mathsf{p}\|^2 + (m-1)\,(\Delta \mathsf{p})^2 \right) dx$$

Explicit arithmetic geometric inequality

$$\|\mathrm{D}^2 \mathsf{p}\|^2 - \frac{1}{d}\,(\Delta \mathsf{p})^2 = \left\| \mathrm{D}^2 \mathsf{p} - \frac{1}{d}\,\Delta \mathsf{p}\,\mathrm{Id} \right\|^2$$

There are no boundary terms in the integrations by parts (! ?)

Remainder terms

$\mathsf{F}'' = -\,\sigma\,(1-m)\,\mathsf{R}[v]$. The *pressure variable* is $\mathsf{P} = \frac{m}{1-m}\,v^{m-1}$

$$\mathsf{R}[v] := (\sigma-1)\,(1-m)\,\mathsf{E}^{\sigma-1}\int_{\mathbb{R}^d} v^m\,\left|\Delta\mathsf{P} - \frac{\int_{\mathbb{R}^d} v\,|\nabla\mathsf{P}|^2\,dx}{\int_{\mathbb{R}^d} v^m\,dx}\right|^2 dx$$
$$+\;2\,\mathsf{E}^{\sigma-1}\int_{\mathbb{R}^d} v^m\,\left\|\,\mathrm{D}^2\mathsf{P} - \tfrac{1}{d}\,\Delta\mathsf{P}\,\mathrm{Id}\,\right\|^2 dx$$

Let

$$\mathsf{G}[v] := \frac{\mathsf{F}[v]}{\sigma\,(1-m)} = \left(\int_{\mathbb{R}^d} v^m\,dx\right)^{\sigma-1}\int_{\mathbb{R}^d} v\,|\nabla\mathsf{P}|^2\,dx$$

The Gagliardo-Nirenberg inequality is equivalent to $\mathsf{G}[v_0] \ge \mathsf{G}[v_\star]$

Proposition

$$\mathsf{G}[v_0] \ge \mathsf{G}[v_\star] + \int_0^\infty \mathsf{R}[v(t,\cdot)]\,dt$$

What's next ?

We redo the computation for the Rényi entropy power F in terms of *self-similar variables* using (in some sense) the less accurate notion of *relative entropie* but...

▷ we can justify the integrations by parts

▷ a (very nice) spectral gap appears

▷ the spectral gap explains why the Bakry-Emery method is so accurate

Self-similar variables and relative entropies

The large time behavior of the solution of $\frac{\partial v}{\partial t} = \Delta v^m$ is governed by the source-type *Barenblatt solutions*

$$v_\star(t,x) := \frac{1}{\kappa^d (\mu\, t)^{d/\mu}}\, \mathcal{B}_\star\left(\frac{x}{\kappa\,(\mu\, t)^{1/\mu}}\right) \quad \text{where} \quad \mu := 2 + d\,(m-1)$$

where $\mathcal{B}_\star$ is the Barenblatt profile (with appropriate mass)

$$\mathcal{B}_\star(x) := \left(1 + |x|^2\right)^{1/(m-1)}$$

A time-dependent rescaling: self-similar variables

$$v(t,x) = \frac{1}{\kappa^d\, R^d}\, u\left(\tau, \frac{x}{\kappa\, R}\right) \quad \text{where} \quad \frac{dR}{dt} = R^{1-\mu}\,, \quad \tau(t) := \tfrac{1}{2}\, \log\left(\frac{R(t)}{R_0}\right)$$

Then the function u solves a Fokker-Planck type equation

$$\frac{\partial u}{\partial \tau} + \nabla \cdot \Big[u \left(\nabla u^{m-1} - \, 2\, x \right) \Big] = 0$$

Free energy and Fisher information

- The function u solves a Fokker-Planck type equation

$$\frac{\partial u}{\partial \tau} + \nabla \cdot \Big[u \left(\nabla u^{m-1} - 2\, x \right) \Big] = 0$$

- (Ralston, Newman, 1984) Lyapunov functional: *Generalized entropy* or *Free energy*

$$\mathcal{F}[u] := \int_{\mathbb{R}^d} \left(-\frac{u^m}{m} + |x|^2 u \right) dx - \mathcal{F}_0$$

- Entropy production is measured by the *Generalized Fisher information*

$$\frac{d}{dt}\mathcal{F}[u] = -\mathcal{I}[u] \,, \quad \mathcal{I}[u] := \int_{\mathbb{R}^d} u \left| \nabla u^{m-1} + 2\, x \right|^2 dx$$

Relative entropy and entropy production

- *Stationary solution:* choose C such that $\|u_\infty\|_{L^1} = \|u\|_{L^1} = M > 0$

$$u_\infty(x) := \left(C + |x|^2\right)_+^{-1/(1-m)}$$

Relative entropy: Fix $\mathcal{F}_0$ so that $\mathcal{F}[u_\infty] = 0$

- *Entropy – entropy production inequality* (del Pino, J.D.)

Theorem

$d \geq 3$, $m \in [\frac{d-1}{d}, +\infty)$, $m > \frac{1}{2}$, $m \neq 1$

$$\mathcal{I}[u] \geq 4\,\mathcal{F}[u]$$

Corollary

(del Pino, J.D.) *A solution u with initial data $u_0 \in L^1_+(\mathbb{R}^d)$ such that $|x|^2\,u_0 \in L^1(\mathbb{R}^d)$, $u_0^m \in L^1(\mathbb{R}^d)$ satisfies*

$$\mathcal{F}[u(t,\cdot)] \leq \mathcal{F}[u_0]\,e^{-\,4\,t}$$

A computation on a large ball, with boundary terms

$$\frac{\partial u}{\partial \tau}+\nabla\cdot\Big[u\left(\nabla u^{m-1}-\,2\,x\right)\Big]=0\quad\tau>0\,,\quad x\in B_R$$

where B_R is a centered ball in $\mathbb{R}^d$ with radius $R>0$, and assume that u satisfies zero-flux boundary conditions

$$\left(\nabla u^{m-1}-\,2\,x\right)\cdot\frac{x}{|x|}=0\quad\tau>0\,,\quad x\in\partial B_R\,.$$

With $z(\tau,x):=\nabla\mathsf{Q}(\tau,x):=\nabla u^{m-1}-\,2\,x$, the *relative Fisher information* is such that

$$\frac{d}{d\tau}\int_{B_R}u\,|z|^2\,dx+4\int_{B_R}u\,|z|^2\,dx$$
$$+\,2\,\tfrac{1-m}{m}\int_{B_R}u^m\left(\left\|\mathrm{D}^2\mathsf{Q}\right\|^2-\,(1-m)\,(\Delta\mathsf{Q})^2\right)dx$$
$$=\int_{\partial B_R}u^m\left(\omega\cdot\nabla|z|^2\right)d\sigma\le 0\ \text{(by Grisvard's lemma)}$$

Another improvement of the GN inequalities

We recall the definitions of the *relative entropy*

$$\mathcal{E}[u] := -\frac{1}{m}\int_{\mathbb{R}^d}\left(u^m - \mathcal{B}_\star^m - m\,\mathcal{B}_\star^{m-1}\,(u - \mathcal{B}_\star)\right)dx$$

the *relative Fisher information*

$$\mathcal{I}[u] := \int_{\mathbb{R}^d} u\,|z|^2\,dx = \int_{\mathbb{R}^d} u\,\left|\nabla u^{m-1} - 2\,x\right|^2 dx$$

and $\quad \mathcal{R}[u] := 2\,\frac{1-m}{m}\int_{\mathbb{R}^d} u^m\left(\left\|\mathrm{D}^2\mathsf{Q}\right\|^2 - (1-m)\,(\Delta\mathsf{Q})^2\right)dx$

Proposition

If $1-1/d \le m < 1$ *and* $d \ge 2$, *then*

$$\mathcal{I}[u_0] - 4\,\mathcal{E}[u_0] \ge \int_0^\infty \mathcal{R}[u(\tau,\cdot)]\,d\tau$$

Entropy – entropy production and GN inequality

$$4\,\mathcal{E}[u] \le \mathcal{I}[u]$$

Rewrite it with $p = \frac{1}{2m-1}$, $u = w^{2p}$, $u^m = w^{p+1}$ as

$$\frac{1}{2}\left(\frac{2m}{2m-1}\right)^2 \int_{\mathbb{R}^d} |\nabla w|^2 dx + \left(\frac{1}{1-m} - d\right) \int_{\mathbb{R}^d} |w|^{1+p} dx - K \ge 0$$

- for some γ, $K = K_0 \left(\int_{\mathbb{R}^d} u\,dx = \int_{\mathbb{R}^d} w^{2p}\,dx\right)^\gamma$
- $w = w_\infty = v_\infty^{1/2p}$ is optimal for the *Gagliardo-Nirenberg inequality*

Theorem

[Del Pino, J.D.] *With* $1 < p \le \frac{d}{d-2}$ *(fast diffusion case) and* $d \ge 3$

$$\|w\|_{L^{2p}(\mathbb{R}^d)} \le \mathcal{C}^{\mathrm{GN}}_{p,d}\, \|\nabla w\|^{\theta}_{L^2(\mathbb{R}^d)}\, \|w\|^{1-\theta}_{L^{p+1}(\mathbb{R}^d)}$$

$$\mathcal{C}^{\mathrm{GN}}_{p,d} = \left(\frac{y(p-1)^2}{2\pi d}\right)^{\frac{\theta}{2}} \left(\frac{2y-d}{2y}\right)^{\frac{1}{2p}} \left(\frac{\Gamma(y)}{\Gamma(y-\frac{d}{2})}\right)^{\frac{\theta}{d}},\ \theta = \frac{d(p-1)}{p(d+2-(d-2)p)},\ y = \frac{p+1}{p-1}$$

Sharp rates of convergence

Assumptions on the initial datum v_0

(H1) *$V_{D_0} \le v_0 \le V_{D_1}$ for some $D_0 > D_1 > 0$*

(H2) *if $d \ge 3$ and $m \le m_*$, $(v_0 - V_D)$ is integrable for a suitable $D \in [D_1, D_0]$*

Theorem

(Blanchet, Bonforte, J.D., Grillo, Vázquez) *Under Assumptions (H1)-(H2), if $m < 1$ and $m \ne m_* := \frac{d-4}{d-2}$, the entropy decays according to*

$$\mathcal{F}[v(t,\cdot)] \le C\, e^{-2\,(1-m)\,\Lambda_{\alpha,d}\,t} \quad \forall\, t \ge 0$$

where $\Lambda_{\alpha,d} > 0$ is the best constant in the Hardy–Poincaré inequality

$$\Lambda_{\alpha,d} \int_{\mathbb{R}^d} |f|^2\, d\mu_{\alpha-1} \le \int_{\mathbb{R}^d} |\nabla f|^2\, d\mu_\alpha \quad \forall\, f \in H^1(d\mu_\alpha)$$

with $\alpha := 1/(m-1) < 0$, $d\mu_\alpha := h_\alpha\, dx$, $h_\alpha(x) := (1+|x|^2)^\alpha$

The linearized problem: exponential decay

The Hardy–Poincaré inequality

$$\Lambda_{\alpha,d}\int_{\mathbb{R}^d}|f|^2\,d\mu_{\alpha-1}\le\int_{\mathbb{R}^d}|\nabla f|^2\,d\mu_\alpha\quad\forall\,f\in H^1(d\mu_\alpha)$$

is the entropy – entropy production inequality associated with the evolution equation after linearization around the Barenblatt profile
If we consider the scalar product

$$\langle f_1,f_2\rangle=\int_{\mathbb{R}^d}f_1\,f_2\,d\mu_{\alpha-1}$$

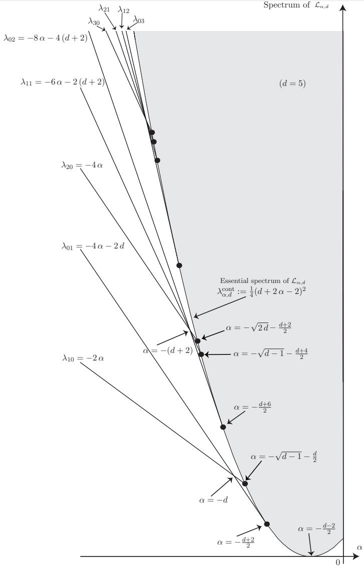
where $d\mu_{\alpha-1}=(1+|x|^2)^{\alpha-1}\,dx=h_{\alpha-1}\,dx$ and the linearized operator

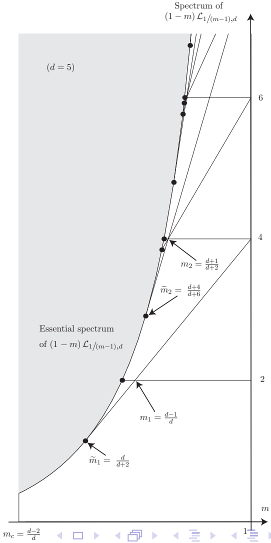
$$\mathcal{L}f:=h_{1-\alpha}\,\nabla\cdot(h_\alpha\,\nabla f)$$

then the solution of $\partial_t f=\mathcal{L}f$ is such that

$$\frac{d}{dt}\langle f,f\rangle=-\,2\,\langle\mathcal{L}f,f\rangle=-\,2\int_{\mathbb{R}^d}|\nabla f|^2\,d\mu_\alpha\le-\,2\,\Lambda_{\alpha,d}\,\langle f,f\rangle$$

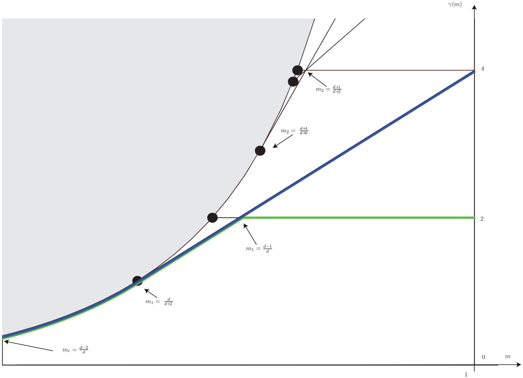
Plots ($d = 5$)


Spectrum of $\mathcal{L}_{\alpha,d}$
λ_{21}
λ_{12}
λ_{30}
λ_{03}
$\lambda_{02} = -8\,\alpha - 4\,(d+2)$
$\lambda_{11} = -6\,\alpha - 2\,(d+2)$
$(d=5)$
$\lambda_{20} = -4\,\alpha$
$\lambda_{01} = -4\,\alpha - 2\,d$
Essential spectrum of $\mathcal{L}_{\alpha,d}$
$\lambda^{\rm cont}_{\alpha,d} := \frac{1}{4}(d+2\,\alpha-2)^2$
$\alpha = -\sqrt{2\,d} - \frac{d+2}{2}$
$\alpha = -(d+2)$
$\alpha = -\sqrt{d-1} - \frac{d+4}{2}$
$\lambda_{10} = -2\,\alpha$
$\alpha = -\frac{d+6}{2}$
$\alpha = -\sqrt{d-1} - \frac{d}{2}$
$\alpha = -d$
$\alpha = -\frac{d-2}{2}$
$\alpha = -\frac{d+2}{2}$
α
0


Spectrum of
$(1-m)\,\mathcal{L}_{1/(m-1),d}$
$(d=5)$
6
4
$m_2 = \frac{d+1}{d+2}$
$\widetilde{m}_2 = \frac{d+4}{d+6}$
Essential spectrum
of $(1-m)\,\mathcal{L}_{1/(m-1),d}$
2
$m_1 = \frac{d-1}{d}$
$\widetilde{m}_1 = \frac{d}{d+2}$
m
$m_c = \frac{d-2}{d}$
1

Improved asymptotic rates

(Bonforte, J.D., Grillo, Vázquez) Assume that $m \in (m_1, 1)$, $d \geq 3$. Under Assumption (H1), if v is a solution of the fast diffusion equation with initial datum v_0 such that $\int_{\mathbb{R}^d} x\, v_0\, dx = 0$, then the asymptotic convergence holds with an improved rate corresponding to the improved spectral gap.



Higher order matching asymptotics

(J.D., G. Toscani) For some $m \in (m_c, 1)$ with $m_c := (d-2)/d$, we consider on $\mathbb{R}^d$ the fast diffusion equation

$$\frac{\partial u}{\partial \tau} + \nabla \cdot \left(u \, \nabla u^{m-1}\right) = 0$$

Without choosing R, we may define the function v such that

$$u(\tau, y + x_0) = R^{-d} \, v(t,x) \,, \quad R = R(\tau) \,, \quad t = \tfrac{1}{2} \, \log R \,, \quad x = \frac{y}{R}$$

Then v has to be a solution of

$$\frac{\partial v}{\partial t} + \nabla \cdot \left[v \left(\sigma^{\frac{d}{2}(m-m_c)} \, \nabla v^{m-1} - 2\,x \right) \right] = 0 \quad t > 0 \,, \quad x \in \mathbb{R}^d$$

with (as long as we make no assumption on R)

$$2\,\sigma^{-\frac{d}{2}(m-m_c)} = R^{\,1-d\,(1-m)} \, \frac{dR}{d\tau}$$

Refined relative entropy

Consider the family of the Barenblatt profiles

$$B_\sigma(x) := \sigma^{-\frac{d}{2}} \left(C_M + \frac{1}{\sigma}\,|x|^2\right)^{\frac{1}{m-1}} \quad \forall\, x \in \mathbb{R}^d \tag{1}$$

Note that σ is a function of t: as long as $\frac{d\sigma}{dt} \neq 0$, the Barenblatt profile B_σ *is not* a solution (it plays the role of a *local Gibbs state*) but we may still consider the relative entropy

$$\mathcal{F}_\sigma[v] := \frac{1}{m-1} \int_{\mathbb{R}^d} \left[v^m - B_\sigma^m - m\, B_\sigma^{m-1}\,(v - B_\sigma)\right] dx$$

The time derivative of this relative entropy is

$$\frac{d}{dt}\mathcal{F}_{\sigma(t)}[v(t,\cdot)] = \underbrace{\frac{d\sigma}{dt}\left(\frac{d}{d\sigma}\,\mathcal{F}_\sigma[v]\right)_{|\sigma=\sigma(t)}}_{\text{choose it } = 0} + \frac{m}{m-1}\int_{\mathbb{R}^d}\left(v^{m-1} - B_{\sigma(t)}^{m-1}\right)\frac{\partial v}{\partial t}\,dx$$

$\Longleftrightarrow$ Minimize $\mathcal{F}_\sigma[v]$ w.r.t. σ $\Longleftrightarrow$ $\int_{\mathbb{R}^d} |x|^2\, B_\sigma\, dx = \int_{\mathbb{R}^d} |x|^2\, v\, dx$

The entropy / entropy production estimate

Using the new change of variables, we know that

$$\frac{d}{dt}\mathcal{F}_{\sigma(t)}[v(t,\cdot)] = -\frac{m\,\sigma(t)^{\frac{d}{2}(m-m_c)}}{1-m}\int_{\mathbb{R}^d} v\left|\nabla\left[v^{m-1}-B_{\sigma(t)}^{m-1}\right]\right|^2 dx$$

Let $w := v/B_\sigma$ and observe that the relative entropy can be written as

$$\mathcal{F}_\sigma[v] = \frac{m}{1-m}\int_{\mathbb{R}^d}\left[w-1-\frac{1}{m}\left(w^m-1\right)\right]B_\sigma^m\,dx$$

(Repeating) define the *relative Fisher information* by

$$\mathcal{I}_\sigma[v] := \int_{\mathbb{R}^d}\left|\frac{1}{m-1}\,\nabla\left[\left(w^{m-1}-1\right)B_\sigma^{m-1}\right]\right|^2 B_\sigma\,w\,dx$$

so that $\quad \frac{d}{dt}\mathcal{F}_{\sigma(t)}[v(t,\cdot)] = -\,m\,(1-m)\,\sigma(t)\,\mathcal{I}_{\sigma(t)}[v(t,\cdot)] \quad \forall\,t>0$

When linearizing, one more mode is killed and $\sigma(t)$ scales out

Improved rates of convergence

Theorem (J.D., G. Toscani)

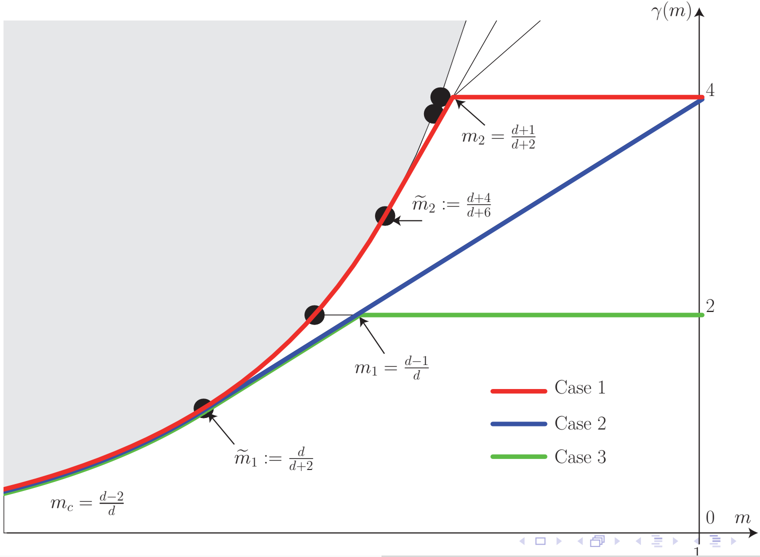
Let $m \in (\widetilde{m}_1, 1)$, $d \ge 2$, $v_0 \in L^1_+(\mathbb{R}^d)$ *such that* v_0^m, $|y|^2\, v_0 \in L^1(\mathbb{R}^d)$

$$\mathcal{F}[v(t,\cdot)] \le C\, e^{-2\,\gamma(m)\,t} \quad \forall\, t \ge 0$$

where

$$\gamma(m) = \begin{cases} \frac{((d-2)\,m-(d-4))^2}{4\,(1-m)} & \text{if } m \in (\widetilde{m}_1, \widetilde{m}_2] \\ 4\,(d+2)\,m - 4\,d & \text{if } m \in [\widetilde{m}_2, m_2] \\ 4 & \text{if } m \in [m_2, 1) \end{cases}$$

Spectral gaps and best constants



Comments

- A result by (Denzler, Koch, McCann)*Higher order time asymptotics of fast diffusion in Euclidean space: a dynamical systems approach*
- The constant C in

$$\mathcal{F}[v(t,\cdot)] \le C\, e^{-2\,\gamma(m)\,t} \quad \forall\, t \ge 0$$

can be made explicit, under additional restrictions on the initial data (Bonforte, J.D., Grillo, Vázquez) + work in progress(Bonforte, J.D., Nazaret, Simonov)

Symmetry and symmetry breaking results

- ▷ The critical Caffarelli-Kohn-Nirenberg inequality
- ▷ A family of sub-critical Caffarelli-Kohn-Nirenberg inequalities
- ▷ Linearization and spectrum

Critical Caffarelli-Kohn-Nirenberg inequality

Let $\mathcal{D}_{a,b} := \left\{ v \in \mathrm{L}^p\left(\mathbb{R}^d, |x|^{-b}\,dx\right) \,:\, |x|^{-a}\,|\nabla v| \in \mathrm{L}^2\left(\mathbb{R}^d, dx\right) \right\}$

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx \quad \forall\, v \in \mathcal{D}_{a,b}$$

holds under the conditions that $a \le b \le a+1$ if $d \ge 3$, $a < b \le a+1$ if $d = 2$, $a + 1/2 < b \le a+1$ if $d = 1$, and $a < a_c := (d-2)/2$

$$p = \frac{2\,d}{d-2+2\,(b-a)} \qquad \text{(critical case)}$$

▷ *An optimal function among radial functions:*

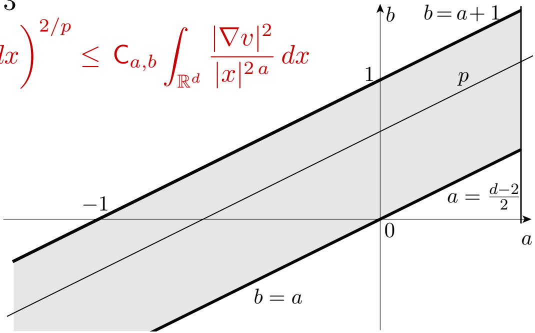
$$v_\star(x) = \left(1 + |x|^{(p-2)\,(a_c-a)}\right)^{-\frac{2}{p-2}} \quad \textit{and} \quad \mathsf{C}^\star_{a,b} = \frac{\|\,|x|^{-b}\,v_\star\,\|_p^2}{\|\,|x|^{-a}\,\nabla v_\star\,\|_2^2}$$

Question: $\mathsf{C}_{a,b} = \mathsf{C}^\star_{a,b}$ *(symmetry) or* $\mathsf{C}_{a,b} > \mathsf{C}^\star_{a,b}$ *(symmetry breaking) ?*

Critical CKN: range of the parameters

Figure: $d=3$

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx$$



$a \le b \le a+1$ if $d \ge 3$

$a < b \le a+1$ if $d=2$, $a+1/2 < b \le a+1$ if $d=1$

and $a < a_c := (d-2)/2$

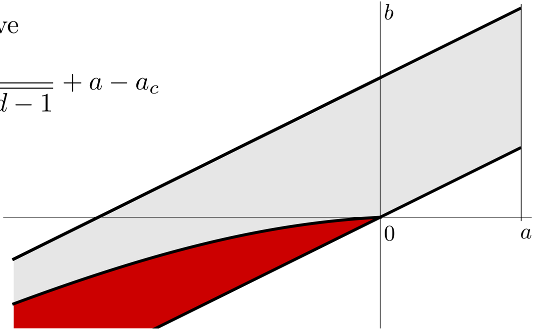
$$p = \frac{2\,d}{d-2+2\,(b-a)}$$

(Glaser, Martin, Grosse, Thirring (1976))
(Caffarelli, Kohn, Nirenberg (1984))
(F. Catrina, Z.-Q. Wang (2001))

Linear instability of radial minimizers: the Felli-Schneider curve

The Felli & Schneider curve

$$b_{\rm FS}(a) := \frac{d\,(a_c - a)}{2\sqrt{(a_c-a)^2+d-1}} + a - a_c$$



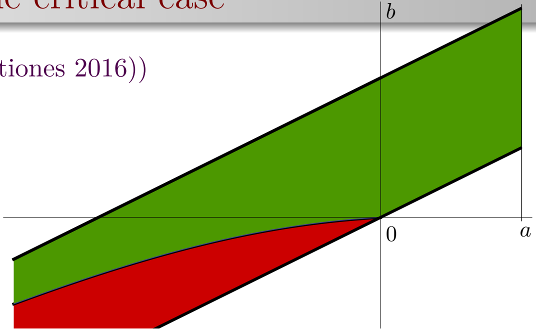
(Smets), (Smets, Willem), (Catrina, Wang), (Felli, Schneider)
The functional

$$\mathsf{C}^\star_{a,b}\int_{\mathbb{R}^d}\frac{|\nabla v|^2}{|x|^{2\,a}}\,dx - \left(\int_{\mathbb{R}^d}\frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p}$$

is linearly instable at $v = v_\star$

Symmetry *versus* symmetry breaking: the sharp result in the critical case

(JD, Esteban, Loss (Inventiones 2016))



Theorem

Let $d \geq 2$ and $p < 2^*$. If either $a \in [0, a_c)$ and $b > 0$, or $a < 0$ and $b \geq b_{\rm FS}(a)$, then the optimal functions for the critical Caffarelli-Kohn-Nirenberg inequalities are radially symmetric

The Emden-Fowler transformation and the cylinder

▷ *With an Emden-Fowler transformation, critical the Caffarelli-Kohn-Nirenberg inequality on the Euclidean space are equivalent to Gagliardo-Nirenberg inequalities on a cylinder*

$$v(r,\omega)=r^{a-a_c}\,\varphi(s,\omega)\quad\text{with}\quad r=|x|\,,\quad s=-\log r\quad\text{and}\quad\omega=\frac{x}{r}$$

With this transformation, the Caffarelli-Kohn-Nirenberg inequalities can be rewritten as the *subcritical* interpolation inequality

$$\|\partial_s\varphi\|^2_{\mathrm L^2(\mathcal C)}+\|\nabla_\omega\varphi\|^2_{\mathrm L^2(\mathcal C)}+\Lambda\,\|\varphi\|^2_{\mathrm L^2(\mathcal C)}\ge\mu(\Lambda)\,\|\varphi\|^2_{\mathrm L^p(\mathcal C)}\quad\forall\,\varphi\in\mathrm H^1(\mathcal C)$$

where $\Lambda:=(a_c-a)^2$, $\mathcal C=\mathbb R\times\mathbb S^{d-1}$ and the optimal constant $\mu(\Lambda)$ is

$$\mu(\Lambda)=\frac{1}{\mathsf C_{a,b}}\quad\text{with}\quad a=a_c\pm\sqrt\Lambda\quad\text{and}\quad b=\frac{d}{p}\pm\sqrt\Lambda$$

Linearization around symmetric critical points

Up to a normalization and a scaling

$$\varphi_\star(s,\omega)=(\cosh s)^{-\frac{1}{p-2}}$$

is a critical point of

$$\mathrm{H}^1(\mathcal{C})\ni\varphi\mapsto\|\partial_s\varphi\|^2_{\mathrm{L}^2(\mathcal{C})}+\|\nabla_\omega\varphi\|^2_{\mathrm{L}^2(\mathcal{C})}+\Lambda\,\|\varphi\|^2_{\mathrm{L}^2(\mathcal{C})}$$

under a constraint on $\|\varphi\|^2_{\mathrm{L}^p(\mathcal{C})}$

$\varphi_\star$ *is not* optimal if the Pöschl-Teller operator

$$-\partial_s^2-\Delta_\omega+\Lambda-\varphi_\star^{p-2}=-\partial_s^2-\Delta_\omega+\Lambda-\frac{1}{(\cosh s)^2}$$

has *a negative eigenvalue*

Subcritical Caffarelli-Kohn-Nirenberg inequalities

Norms: $\|w\|_{q,\gamma} := \left(\int_{\mathbb{R}^d} |w|^q \, |x|^{-\gamma} \, dx\right)^{1/q}$, $\|w\|_q := \|w\|_{q,0}$

(some) *Caffarelli-Kohn-Nirenberg interpolation inequalities* (1984)

$$\|w\|_{2p,\gamma} \le \mathsf{C}_{\beta,\gamma,p} \, \|\nabla w\|_{2,\beta}^{\vartheta} \, \|w\|_{p+1,\gamma}^{1-\vartheta} \qquad \text{(CKN)}$$

Here $\mathsf{C}_{\beta,\gamma,p}$ denotes the optimal constant, the parameters satisfy

$$d \ge 2\,, \quad \gamma-2 < \beta < \tfrac{d-2}{d}\,\gamma\,, \quad \gamma \in (-\infty, d)\,, \quad p \in (1, p_\star] \quad \text{with } p_\star := \tfrac{d-\gamma}{d-\beta-2}$$

and the exponent ϑ is determined by the scaling invariance, *i.e.*,

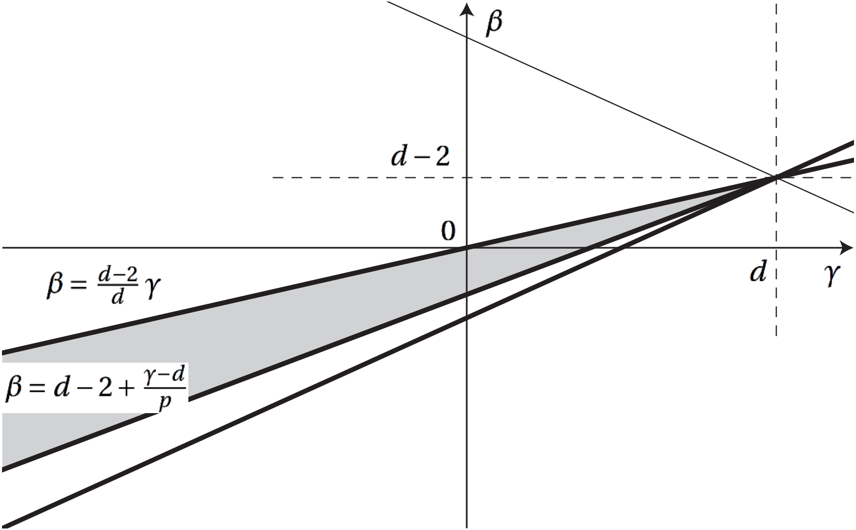
$$\vartheta = \frac{(d-\gamma)\,(p-1)}{p\,\big(d+\beta+2-2\,\gamma-p\,(d-\beta-2)\big)}$$

- Is the equality case achieved by the Barenblatt / Aubin-Talenti type function

$$w_\star(x) = \left(1 + |x|^{2+\beta-\gamma}\right)^{-1/(p-1)} \quad \forall\, x \in \mathbb{R}^d \quad ?$$

- Do we know (*symmetry*) that the equality case is achieved among radial functions?

Range of the parameters



Symmetry and symmetry breaking

(JD, Esteban, Loss, Muratori, 2016)

Let us define $\beta_{\mathrm{FS}}(\gamma) := d - 2 - \sqrt{(d-\gamma)^2 - 4\,(d-1)}$

Theorem

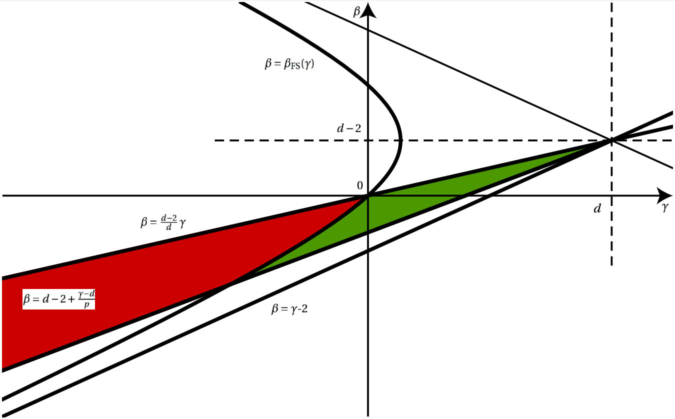
Symmetry breaking holds in (CKN) if

$$\gamma < 0 \quad \textit{and} \quad \beta_{\mathrm{FS}}(\gamma) < \beta < \frac{d-2}{d}\,\gamma$$

In the range $\beta_{\mathrm{FS}}(\gamma) < \beta < \frac{d-2}{d}\,\gamma$

$$w_\star(x) = \left(1 + |x|^{2+\beta-\gamma}\right)^{-1/(p-1)}$$

is not optimal



The grey area corresponds to the admissible cone. The light grey area is the region of symmetry, while the dark grey area is the region of symmetry breaking. The threshold is determined by the hyperbola

$$(d-\gamma)^2-(\beta-d+2)^2-4\,(d-1)=0$$

A useful change of variables

With

$$\alpha = 1 + \frac{\beta - \gamma}{2} \quad \text{and} \quad n = 2\,\frac{d-\gamma}{\beta + 2 - \gamma}\,,$$

(CKN) can be rewritten for a function $v(|x|^{\alpha-1}\,x) = w(x)$ as

$$\|v\|_{2p,d-n} \le \mathsf{K}_{\alpha,n,p}\,\|\mathfrak{D}_\alpha v\|_{2,d-n}^{\vartheta}\,\|v\|_{p+1,d-n}^{1-\vartheta}$$

with the notations $s = |x|$, $\mathfrak{D}_\alpha v = \left(\alpha\,\frac{\partial v}{\partial s}, \frac{1}{s}\,\nabla_\omega v\right)$. Parameters are in the range

$$d \ge 2\,, \quad \alpha > 0\,, \quad n > d \quad \text{and} \quad p \in (1, p_\star]\,, \quad p_\star := \frac{n}{n-2}$$

By our change of variables, $w_\star$ is changed into

$$v_\star(x) := \left(1 + |x|^2\right)^{-\frac{1}{p-1}} \quad \forall\, x \in \mathbb{R}^d$$

The symmetry breaking condition (Felli-Schneider) now reads

$$\alpha < \alpha_{\rm FS} \quad \text{with} \quad \alpha_{\rm FS} := \sqrt{\frac{d-1}{n-1}}$$

The second variation

$$\mathcal{J}[v] := \vartheta \, \log\left(\|\mathfrak{D}_\alpha v\|_{2,d-n}\right) + (1-\vartheta)\, \log\left(\|v\|_{p+1,d-n}\right) + \log \mathsf{K}_{\alpha,n,p} - \log\left(\|v\|_{2p,d-n}\right)$$

Let us define $d\mu_\delta := \mu_\delta(x)\,dx$, where $\mu_\delta(x) := (1+|x|^2)^{-\delta}$. Since $v_\star$ is a critical point of $\mathcal{J}$, a Taylor expansion at order ε^2 shows that

$$\|\mathfrak{D}_\alpha v_\star\|^2_{2,d-n}\,\mathcal{J}\big[v_\star + \varepsilon\,\mu_{\delta/2}\,f\big] = \tfrac{1}{2}\,\varepsilon^2\,\vartheta\,\mathcal{Q}[f] + o(\varepsilon^2)$$

with $\delta = \frac{2\,p}{p-1}$ and

$$\mathcal{Q}[f] = \int_{\mathbb{R}^d} |\mathfrak{D}_\alpha f|^2\,|x|^{n-d}\,d\mu_\delta - \frac{4\,p\,\alpha^2}{p-1}\int_{\mathbb{R}^d} |f|^2\,|x|^{n-d}\,d\mu_{\delta+1}$$

We assume that $\int_{\mathbb{R}^d} f\,|x|^{n-d}\,d\mu_{\delta+1} = 0$ (mass conservation)

Symmetry breaking: the proof

Proposition (Hardy-Poincaré inequality)

Let $d \geq 2$, $\alpha \in (0,+\infty)$, $n > d$ and $\delta \geq n$. If f has 0 average, then

$$\int_{\mathbb{R}^d} |\mathfrak{D}_\alpha f|^2\,|x|^{n-d}\,d\mu_\delta \geq \Lambda \int_{\mathbb{R}^d} |f|^2\,|x|^{n-d}\,d\mu_{\delta+1}$$

with optimal constant $\Lambda = \min\{2\,\alpha^2\,(2\,\delta - n), 2\,\alpha^2\,\delta\,\eta\}$ where η is the unique positive solution to $\eta\,(\eta + n - 2) = (d-1)/\alpha^2$. The corresponding eigenfunction is not radially symmetric if $\alpha^2 > \frac{(d-1)\,\delta^2}{n\,(2\,\delta-n)\,(\delta-1)}$

$\mathcal{Q} \geq 0$ iff $\frac{4\,p\,\alpha^2}{p-1} \leq \Lambda$ and symmetry breaking occurs in (CKN) if

$$2\,\alpha^2\,\delta\,\eta < \frac{4\,p\,\alpha^2}{p-1} \quad\Longleftrightarrow\quad \eta < 1$$

$$\Longleftrightarrow\quad \frac{d-1}{\alpha^2} = \eta\,(\eta + n - 2) < n - 1 \quad\Longleftrightarrow\quad \alpha > \alpha_{\rm FS}$$

Weighted nonlinear flows: Caffarelli-Kohn-Nirenberg inequalities

▷ Entropy and Caffarelli-Kohn-Nirenberg inequalities

▷ Large time asymptotics and spectral gaps

▷ Optimality cases

CKN and entropy – entropy production inequalities

When symmetry holds, (CKN) can be written as an *entropy – entropy production* inequality

$$\frac{1-m}{m}\,(2+\beta-\gamma)^2\,\mathcal{F}[v]\le\mathcal{I}[v]$$

and equality is achieved by $\mathfrak{B}_{\beta,\gamma}$. Here the *free energy* and the *relative Fisher information* are defined by

$$\mathcal{F}[v]:=\frac{1}{m-1}\int_{\mathbb{R}^d}\left(v^m-\mathfrak{B}_{\beta,\gamma}^m-m\,\mathfrak{B}_{\beta,\gamma}^{m-1}\,(v-\mathfrak{B}_{\beta,\gamma})\right)\frac{dx}{|x|^\gamma}$$

$$\mathcal{I}[v]:=\int_{\mathbb{R}^d}v\left|\nabla v^{m-1}-\nabla\mathfrak{B}_{\beta,\gamma}^{m-1}\right|^2\frac{dx}{|x|^\beta}\,.$$

If v solves the *Fokker-Planck type equation*

$$v_t+|x|^\gamma\,\nabla\cdot\left[|x|^{-\beta}\,v\,\nabla\left(v^{m-1}-|x|^{2+\beta-\gamma}\right)\right]=0\qquad\text{(WFDE-FP)}$$

then

$$\frac{d}{dt}\mathcal{F}[v(t,\cdot)]=-\,\frac{m}{1-m}\,\mathcal{I}[v(t,\cdot)]$$

Proposition

Let $m=\frac{p+1}{2\,p}$ *and consider a solution to (WFDE-FP) with nonnegative initial datum* $u_0\in\mathrm{L}^{1,\gamma}(\mathbb{R}^d)$ *such that* $\|u_0^m\|_{1,\gamma}$ *and* $\int_{\mathbb{R}^d}u_0\,|x|^{2+\beta-2\gamma}\,dx$ *are finite. Then*

$$\mathcal{F}[v(t,\cdot)]\le\mathcal{F}[u_0]\,e^{-\,(2+\beta-\gamma)^2\,t}\quad\forall\,t\ge0$$

if one of the following two conditions is satisfied:
(i) either u_0 *is a.e. radially symmetric*
(ii) or symmetry holds in (CKN)

A useful change of variables

With

$$\alpha = 1 + \frac{\beta - \gamma}{2} \quad \text{and} \quad n = 2\,\frac{d-\gamma}{\beta + 2 - \gamma}\,,$$

(CKN) can be rewritten for a function $v(|x|^{\alpha-1}\,x) = w(x)$ as

$$\|v\|_{2p,d-n} \le \mathsf{K}_{\alpha,n,p}\,\|\mathfrak{D}_\alpha v\|_{2,d-n}^{\vartheta}\,\|v\|_{p+1,d-n}^{1-\vartheta}$$

with the notations $s = |x|$, $\mathfrak{D}_\alpha v = \left(\alpha\,\frac{\partial v}{\partial s}, \frac{1}{s}\,\nabla_\omega v\right)$. Parameters are in the range

$$d \ge 2\,, \quad \alpha > 0\,, \quad n > d \quad \text{and} \quad p \in (1, p_\star]\,, \quad p_\star := \frac{n}{n-2}$$

By our change of variables, $w_\star$ is changed into

$$v_\star(x) := \left(1 + |x|^2\right)^{-1/(p-1)} \quad \forall\, x \in \mathbb{R}^d$$

The symmetry breaking condition (Felli-Schneider) now reads

$$\alpha > \alpha_{\rm FS} \quad \text{with} \quad \alpha_{\rm FS} := \sqrt{\frac{d-1}{n-1}}$$

Towards a parabolic proof

For any $\alpha \geq 1$, let $\quad \mathfrak{D}_\alpha W = \big(\alpha\,\partial_r W,\, r^{-1}\,\nabla_\omega W\big)$ so that

$$\mathfrak{D}_\alpha = \nabla + (\alpha - 1)\,\frac{x}{|x|^2}\,(x\cdot\nabla) = \nabla + (\alpha-1)\,\omega\,\partial_r$$

and define the diffusion operator L_α by

$$\mathsf{L}_\alpha = -\,\mathcal{D}_\alpha^*\,\mathfrak{D}_\alpha = \alpha^2\left(\partial_r^2 + \frac{n-1}{r}\,\partial_r\right) + \frac{\Delta_\omega}{r^2}$$

where Δ_ω denotes the Laplace-Beltrami operator on $\mathbb{S}^{d-1}$
$\frac{\partial g}{\partial t} = \mathsf{L}_\alpha g^m$ is changed into

$$\frac{\partial u}{\partial \tau} = \mathcal{D}_\alpha^*\,(u\,z)\,,\quad z := \mathfrak{D}_\alpha \mathsf{q}\,,\quad \mathsf{q} := u^{m-1} - \mathcal{B}_\alpha^{m-1}\,,\quad \mathcal{B}_\alpha(x) := \left(1+\frac{|x|^2}{\alpha^2}\right)^{\frac{1}{m-}}$$

by the change of variables

$$g(t,x) = \frac{1}{\kappa^n\,R^n}\,u\Big(\tau,\frac{x}{\kappa\,R}\Big) \quad \text{where} \quad \begin{cases} \frac{dR}{dt} = R^{1-\mu}\,, \quad R(0) = R_0 \\ \tau(t) = \frac{1}{2}\,\log\left(\frac{R(t)}{R_0}\right) \end{cases}$$

If the weight does not introduce any singularity at $x=0$...

$$\frac{m}{1-m}\,\frac{d}{d\tau}\int_{B_R} u\,|z|^2\,d\mu_n$$

$$=\int_{\partial B_R} u^m\,\big(\omega\cdot\mathfrak{D}_\alpha|z|^2\big)\,|x|^{n-d}\,d\sigma \quad (\le 0 \text{ by Grisvard's lemma})$$

$$-\,2\,\tfrac{1-m}{m}\,\big(m-1+\tfrac{1}{n}\big)\int_{B_R} u^m\,|\mathsf{L}_\alpha q|^2\,d\mu_n$$

$$-\int_{B_R} u^m\left(\alpha^4\,m_1\left|\mathsf{q}''-\tfrac{\mathsf{q}'}{r}-\tfrac{\Delta_\omega\mathsf{q}}{\alpha^2\,(n-1)\,r^2}\right|^2+\tfrac{2\,\alpha^2}{r^2}\,\left|\nabla_\omega\mathsf{q}'-\tfrac{\nabla_\omega\mathsf{q}}{r}\right|^2\right)d\mu_n$$

$$-\,(n-2)\,\big(\alpha_{\rm FS}^2-\alpha^2\big)\int_{B_R}\frac{|\nabla_\omega\mathsf{q}|^2}{r^4}\,d\mu_n$$

A formal computation that still needs to be justified (singularity at $x=0$?)

- Other potential application: the computation of Bakry, Gentil and Ledoux (chapter 6) for non-integer dimensions; weights on manifolds

[...]

Fast diffusion equations with weights: large time asymptotics

- Relative uniform convergence
- Asymptotic rates of convergence
- From asymptotic to global estimates

Here v solves the *Fokker-Planck type equation*

$$v_t + |x|^\gamma \, \nabla \cdot \Big[|x|^{-\beta} \, v \, \nabla \big(v^{m-1} - |x|^{2+\beta-\gamma} \big) \Big] = 0 \qquad \text{(WFDE-FP)}$$

Joint work with M. Bonforte, M. Muratori and B. Nazaret

Relative uniform convergence

$\zeta := 1 - \left(1 - \frac{2-m}{(1-m)\,q}\right)\left(1 - \frac{2-m}{1-m}\,\theta\right)$

$\theta := \frac{(1-m)\,(2+\beta-\gamma)}{(1-m)\,(2+\beta)+2+\beta-\gamma}$ is in the range $0 < \theta < \frac{1-m}{2-m} < 1$

Theorem

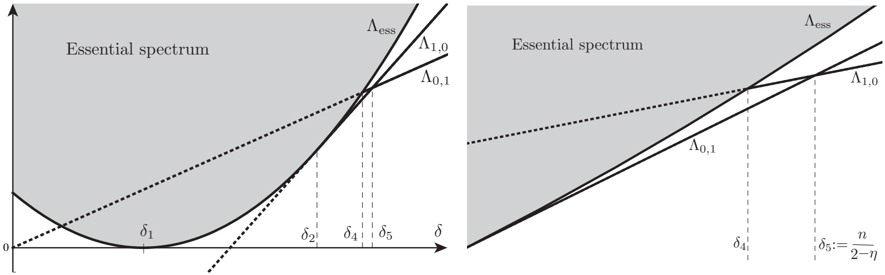
For "good" initial data, there exist positive constants $\mathcal{K}$ and t_0 such that, for all $q \in \left[\frac{2-m}{1-m}, \infty\right]$, the function $w = v/\mathfrak{B}$ satisfies

$$\|w(t) - 1\|_{\mathrm{L}^{q,\gamma}(\mathbb{R}^d)} \le \mathcal{K}\, e^{-2\,\frac{(1-m)^2}{2-m}\,\Lambda\,\zeta\,(t-t_0)} \quad \forall\, t \ge t_0$$

in the case $\gamma \in (0,d)$, and

$$\|w(t) - 1\|_{\mathrm{L}^{q,\gamma}(\mathbb{R}^d)} \le \mathcal{K}\, e^{-2\,\frac{(1-m)^2}{2-m}\,\Lambda\,(t-t_0)} \quad \forall\, t \ge t_0$$

in the case $\gamma \le 0$



The spectrum of $\mathcal{L}$ as a function of $\delta = \frac{1}{1-m}$, with $n = 5$. The essential spectrum corresponds to the grey area, and its bottom is determined by the parabola $\delta \mapsto \Lambda_{ess}(\delta)$. The two eigenvalues $\Lambda_{0,1}$ and $\Lambda_{1,0}$ are given by the plain, half-lines, away from the essential spectrum. The spectral gap determines the asymptotic rate of convergence to the Barenblatt functions

Main steps of the proof:

- Existence of weak solutions, $\mathrm{L}^{1,\gamma}$ contraction, Comparison Principle, conservation of relative mass
- Self-similar variables and the Ornstein-Uhlenbeck equation in relative variables: the ratio $w(t,x) := v(t,x)/\mathfrak{B}(x)$ solves

$$\begin{cases} |x|^{-\gamma}\, w_t = -\,\frac{1}{\mathfrak{B}}\, \nabla \cdot \Big(|x|^{-\beta}\, \mathfrak{B}\, w\, \nabla \big((w^{m-1} - 1)\, \mathfrak{B}^{m-1} \big) \Big) & \text{in } \mathbb{R}^+ \times \mathbb{R}^d \\ w(0,\cdot) = w_0 := v_0/\mathfrak{B} & \text{in } \mathbb{R}^d \end{cases}$$

- *Regularity*, relative uniform convergence (without rates) and asymptotic rates (linearization)
- The relative free energy and the relative Fisher information: linearized free energy and linearized Fisher information
- A Duhamel formula and a bootstrap

Regularity (1/2): Harnack inequality and Hölder

We change variables: $x \mapsto |x|^{\alpha-1}\,x$ and adapt the ideas of F. Chiarenza and R. Serapioni to

$$u_t + \mathsf{D}_\alpha^*\Big[\mathsf{a}\,(\mathfrak{D}_\alpha u + \mathsf{B}\,u)\Big] = 0 \quad \text{in} \quad \mathbb{R}^+\times\mathbb{R}^d$$

Proposition (A parabolic Harnack inequality)

Let $d \ge 2$, $\alpha > 0$ and $n > d$. If u is a bounded positive solution, then for all $(t_0,x_0) \in \mathbb{R}^+\times\mathbb{R}^d$ and $r > 0$ such that $Q_r(t_0,x_0) \subset \mathbb{R}^+\times B_1$, we have

$$\sup_{Q_r^-(t_0,x_0)} u \le H \inf_{Q_r^+(t_0,x_0)} u$$

The constant $H > 1$ depends only on the local bounds on the coefficients a, B *and on d, α, and $n := \frac{2\,(d-\gamma)}{\beta+2-\gamma}$*

By adapting the classical method *à la De Giorgi* to our weighted framework: Hölder regularity at the origin

Regularity (2/2): from local to global estimates

Lemma

If w is a solution of the the Ornstein-Uhlenbeck equation with initial datum w_0 bounded from above and from below by a Barenblatt profile (+ relative mass condition) = "good solutions", then there exist $\nu \in (0,1)$ and a positive constant $\mathcal{K} > 0$, depending on d, m, β, γ, C, C_1, C_2 such that:

$$\|\nabla v(t)\|_{\mathrm{L}^\infty(B_{2\lambda}\setminus B_\lambda)} \le \frac{Q_1}{\lambda^{\frac{2+\beta-\gamma}{1-m}+1}} \quad \forall\, t \ge 1\,, \quad \forall\, \lambda > 1\,,$$

$$\sup_{t\ge1} \|w\|_{C^k((t,t+1)\times B_\varepsilon^c)} < \infty \quad \forall\, k \in \mathbb{N}\,,\ \forall\, \varepsilon > 0$$

$$\sup_{t\ge1} \|w(t)\|_{C^\nu(\mathbb{R}^d)} < \infty$$

$$\sup_{\tau\ge t} |w(\tau) - 1|_{C^\nu(\mathbb{R}^d)} \le \mathcal{K} \sup_{\tau\ge t} \|w(\tau) - 1\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \quad \forall\, t \ge 1$$

Asymptotic rates of convergence

Corollary

Assume that $m \in (0,1)$, *with* $m \neq m_* := \frac{n-4}{n-2}$. *Under the relative mass condition, for any "good solution"* v *there exists a positive constant* $\mathcal{C}$ *such that*

$$\mathcal{F}[v(t)] \le \mathcal{C}\, e^{-\,2\,(1-m)\,\Lambda\,t} \quad \forall\, t \ge 0\,.$$

- With Csiszár-Kullback-Pinsker inequalities, these estimates provide a rate of convergence in $\mathrm{L}^{1,\gamma}(\mathbb{R}^d)$
- Improved estimates can be obtained using "best matching techniques"

From asymptotic to global estimates

When symmetry holds (CKN) can be written as an *entropy – entropy production* inequality

$$(2+\beta-\gamma)^2\,\mathcal{F}[v]\le\frac{m}{1-m}\,\mathcal{I}[v]$$

so that

$$\mathcal{F}[v(t)]\le\mathcal{F}[v(0)]\,e^{-\,2\,(1-m)\,\Lambda_\star\,t}\quad\forall\,t\ge0\quad\text{with}\quad\Lambda_\star:=\tfrac{(2+\beta-\gamma)^2}{2\,(1-m)}$$

Let us consider again the *entropy – entropy production* inequality

$$\mathcal{K}(M)\,\mathcal{F}[v]\le\mathcal{I}[v]\quad\forall\,v\in\mathrm{L}^{1,\gamma}(\mathbb{R}^d)\quad\text{such that}\quad\|v\|_{1,\gamma}=M\,,$$

where $\mathcal{K}(M)$ is the best constant: with $\Lambda(M):=\frac{m}{2}\,(1-m)^{-2}\,\mathcal{K}(M)$

$$\mathcal{F}[v(t)]\le\mathcal{F}[v(0)]\,e^{-\,2\,(1-m)\,\Lambda(M)\,t}\quad\forall\,t\ge0$$

Symmetry breaking and global entropy – entropy production inequalities

Proposition

- *In the symmetry breaking range of (CKN), for any $M>0$, we have*
$$0<\mathcal{K}(M)\le\frac{2}{m}\,(1-m)^2\,\Lambda_{0,1}$$
- *If symmetry holds in (CKN) then*
$$\mathcal{K}(M)\ge\frac{1-m}{m}\,(2+\beta-\gamma)^2$$

(JD, Simonov) In the whole symmetry range,

$$\mathcal{K}(M)=\frac{1-m}{m}\,(2+\beta-\gamma)^2$$

Linearization and optimality

Joint work with M.J. Esteban and M. Loss

Linearization and scalar products

With u_ε such that

$$u_\varepsilon = \mathcal{B}_\star \left(1 + \varepsilon\, f\, \mathcal{B}_\star^{1-m}\right) \quad \text{and} \quad \int_{\mathbb{R}^d} u_\varepsilon\, dx = M_\star$$

at first order in $\varepsilon \to 0$ we obtain that f solves

$$\frac{\partial f}{\partial t} = \mathcal{L}\, f \quad \text{where} \quad \mathcal{L}\, f := (1-m)\, \mathcal{B}_\star^{m-2}\, |x|^\gamma\, \mathsf{D}_\alpha^* \left(|x|^{-\beta}\, \mathcal{B}_\star\, \mathfrak{D}_\alpha f \right)$$

Using the scalar products

$$\langle f_1, f_2 \rangle = \int_{\mathbb{R}^d} f_1\, f_2\, \mathcal{B}_\star^{2-m}\, \frac{dx}{|x|^\gamma} \quad \text{and} \quad \langle\!\langle f_1, f_2 \rangle\!\rangle = \int_{\mathbb{R}^d} \mathfrak{D}_\alpha f_1 \cdot \mathfrak{D}_\alpha f_2\, \mathcal{B}_\star\, \frac{dx}{|x|^\beta}$$

$$\frac{1}{2}\, \frac{d}{dt}\, \langle f, f \rangle = \langle f, \mathcal{L}\, f \rangle = -\int_{\mathbb{R}^d} |\mathfrak{D}_\alpha f|^2\, \mathcal{B}_\star\, \frac{dx}{|x|^\beta} = -\, \langle\!\langle f, f \rangle\!\rangle$$

for any f smooth enough, and

$$\frac{1}{2}\, \frac{d}{dt} \langle\!\langle f, f \rangle\!\rangle = \int_{\mathbb{R}^d} \mathfrak{D}_\alpha f \cdot \mathfrak{D}_\alpha (\mathcal{L}\, f)\, u\, \frac{dx}{|x|^\beta} = -\, \langle\!\langle f, \mathcal{L}\, f \rangle\!\rangle$$

Linearization of the flow, eigenvalues and spectral gap

Now let us consider an eigenfunction associated with the smallest positive eigenvalue λ_1 of $\mathcal{L}$

$$-\mathcal{L}\, f_1 = \lambda_1\, f_1$$

so that f_1 realizes the equality case in the *Hardy-Poincaré inequality*

$$\langle\!\langle g, g \rangle\!\rangle = -\, \langle f, \mathcal{L}\, f \rangle \geq \lambda_1\, \|g - \bar{g}\|^2\,, \quad \bar{g} := \langle g, 1 \rangle \,/\, \langle 1, 1 \rangle$$

$$-\, \langle\!\langle g, \mathcal{L}\, g \rangle\!\rangle \geq \lambda_1\, \langle\!\langle g, g \rangle\!\rangle$$

Proof: expansion of the square :
$-\, \langle\!\langle (g - \bar{g}), \mathcal{L}\,(g - \bar{g}) \rangle\!\rangle = \langle \mathcal{L}\,(g - \bar{g}), \mathcal{L}\,(g - \bar{g}) \rangle = \|\mathcal{L}\,(g - \bar{g})\|^2$

- Key observation:

$$\lambda_1 \geq 4 \quad \Longleftrightarrow \quad \alpha \leq \alpha_{\rm FS} := \sqrt{\frac{d-1}{n-1}}$$

Symmetry breaking in CKN inequalities

- Symmetry holds in (CKN) if $\mathcal{J}[w] \ge \mathcal{J}[w_\star]$ with

$$\mathcal{J}[w] := \vartheta\,\log\big(\|\mathfrak{D}_\alpha w\|_{2,\delta}\big) + (1-\vartheta)\,\log\big(\|w\|_{p+1,\delta}\big) - \log\big(\|w\|_{2p,\delta}\big)$$

with $\delta := d - n$ and

$$\mathcal{J}[w_\star + \varepsilon\,g] = \varepsilon^2\,\mathcal{Q}[g] + o(\varepsilon^2)$$

where

$$\begin{aligned}&\tfrac{2}{\vartheta}\,\|\mathfrak{D}_\alpha w_\star\|^2_{2,d-n}\,\mathcal{Q}[g]\\ &= \|\mathfrak{D}_\alpha g\|^2_{2,d-n} + \tfrac{p\,(2+\beta-\gamma)}{(p-1)^2}\,\big[d-\gamma-p\,(d-2-\beta)\big]\int_{\mathbb{R}^d}|g|^2\,\tfrac{|x|^{n-d}}{1+|x|^2}\,dx\\ &\qquad - p\,(2\,p-1)\,\tfrac{(2+\beta-\gamma)^2}{(p-1)^2}\int_{\mathbb{R}^d}|g|^2\,\tfrac{|x|^{n-d}}{(1+|x|^2)^2}\,dx\end{aligned}$$

is a nonnegative quadratic form if and only if $\alpha \le \alpha_{\rm FS}$

- Symmetry breaking holds if $\alpha > \alpha_{\rm FS}$

Information – production of information inequality

Let $\mathcal{K}[u]$ be such that

$$\frac{d}{d\tau}\mathcal{I}[u(\tau,\cdot)] = -\,\mathcal{K}[u(\tau,\cdot)] = -\,\text{(sum of squares)}$$

If $\alpha \le \alpha_{\rm FS}$, then $\lambda_1 \ge 4$ and

$$u \mapsto \frac{\mathcal{K}[u]}{\mathcal{I}[u]} - 4$$

is a nonnegative functional

With $u_\varepsilon = \mathcal{B}_\star\,\left(1+\varepsilon\,f\,\mathcal{B}_\star^{1-m}\right)$, we observe that

$$4 \le \mathcal{C}_2 := \inf_u \frac{\mathcal{K}[u]}{\mathcal{I}[u]} \le \lim_{\varepsilon\to 0}\inf_f \frac{\mathcal{K}[u_\varepsilon]}{\mathcal{I}[u_\varepsilon]} = \inf_f \frac{\langle\langle f,\mathcal{L}\,f\rangle\rangle}{\langle\langle f,f\rangle\rangle} = \frac{\langle\langle f_1,\mathcal{L}\,f_1\rangle\rangle}{\langle\langle f_1,f_1\rangle\rangle} = \lambda_1$$

- if $\lambda_1 = 4$, that is, if $\alpha = \alpha_{\rm FS}$, then $\inf \mathcal{K}/\mathcal{I} = 4$ is achieved in the asymptotic regime as $u \to \mathcal{B}_\star$ and determined by the spectral gap of $\mathcal{L}$
- if $\lambda_1 > 4$, that is, if $\alpha < \alpha_{\rm FS}$, then $\mathcal{K}/\mathcal{I} > 4$

Symmetry in Caffarelli-Kohn-Nirenberg inequalities

If $\alpha \le \alpha_{\rm FS}$, the fact that $\mathcal K/\mathcal I \ge 4$ has an important consequence. Indeed we know that

$$\frac{d}{d\tau}\left(\mathcal I[u(\tau,\cdot)] - 4\,\mathcal F[u(\tau,\cdot)]\right) \le 0$$

so that

$$\mathcal I[u] - 4\,\mathcal F[u] \ge \mathcal I[\mathcal B_\star] - 4\,\mathcal F[\mathcal B_\star] = 0$$

This inequality is equivalent to $\mathcal J[w] \ge \mathcal J[w_\star]$, which establishes that optimality in (CKN) is achieved among symmetric functions. In other words, the linearized problem shows that for $\alpha \le \alpha_{\rm FS}$, the function

$$\tau \mapsto \mathcal I[u(\tau,\cdot)] - 4\,\mathcal F[u(\tau,\cdot)]$$

is monotone decreasing

- This explains why the method based on nonlinear flows provides the *optimal range for symmetry*

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