Fast diffusion, mean field drifts and reverse HLS inequalities
A first example with a mean field term: phase transition and asymptotic behaviour in a *flocking* model

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June 3, 2019

Lecture 2

Intensive Week of PDEs@Cogne

The Cucker-Smale model
An introduction

- An homogeneous model
- Phase transition
- Dynamics

Key tools: linearization and an adapted (non-local) scalar product

(Xingyu Li, arXiv preprint...)

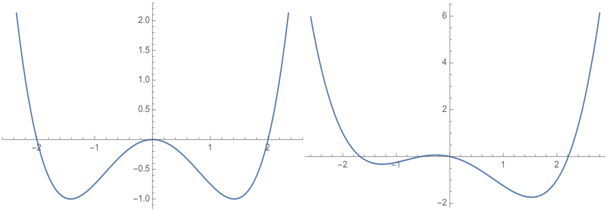
A simple version of the Cucker-Smale model

A model for bird flocking (simplified version)

$$\frac{\partial f}{\partial t} = D\,\Delta_v f + \nabla_v \cdot \left(\nabla_v \varphi(v)\, f - \mathbf{u}_f\, f\right)$$

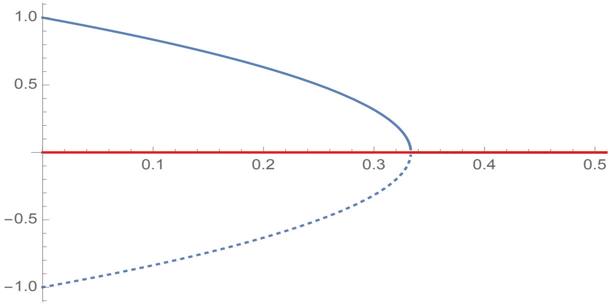
where $\mathbf{u}_f = \int v\,f\,dv$ is the average velocity
f is a probability measure

$$\varphi(v) = \frac{1}{4}\,|v|^4 - \frac{1}{2}\,|v|^2$$



(J. Tugaut, 2014)
(A. Barbaro, J. Cañizo, J.A. Carrillo, and P. Degond, 2016)

Stationary solutions: phase transition



- $d = 1$: there exists a bifurcation point $D = D_*$ such that the only stationary solution corresponds to $\mathbf{u}_f = 0$ if $D > D_*$ and there are three solutions corresponding to $\mathbf{u}_f = 0, \pm u(D)$ if $D < D_*$
- $\mathbf{u}_f = 0$ is linearly unstable if $D < D_*$

Notation: $f_\star^{(0)}$, $f_\star^{(+)}$, $f_\star^{(-)}$

Dynamics

The free energy

$$\mathcal{F}[f] := D \int_{\mathbb{R}^d} f \, \log f \, dv + \int_{\mathbb{R}^d} f \, \varphi \, dv - \frac{1}{2} \, |\mathbf{u}_f|^2$$

decays according to

$$\frac{d}{dt} \mathcal{F}[f(t,\cdot)] = - \int_{\mathbb{R}^d} \left| D \, \frac{\nabla_v f}{f} + \nabla_v \varphi - \mathbf{u}_f \right|^2 f \, dv$$

- $d = 1$: if $\mathcal{F}[f(t=0,\cdot)] < \mathcal{F}[f_\star^{(0)}]$ and $D < D_*$, then

$$\mathcal{F}[f(t,\cdot)] - \mathcal{F}\left[f_\star^{(\pm)}\right] \leq C \, e^{-\lambda \, t}$$

- $d = 1$: λ is the eigenvalue of the linearized problem at $f_\star^{(\pm)}$ in the weighted space $\mathrm{L}^2\left((f_\star^{(\pm)})^{-1}\right)$ with scalar product

$$\langle f, g \rangle_{\pm} := D \int_{\mathbb{R}} f \, g \left(f_\star^{(\pm)}\right)^{-1} dv - \mathbf{u}_f \, \mathbf{u}_g$$

The Cucker-Smale model
Results and proofs

An homogenous Cucker-Smale model

$$\frac{\partial f}{\partial t} = D\,\Delta f + \nabla \cdot \Big((v - \mathbf{u}_f)\, f + \alpha\, v\, \big(|v|^2 - 1\big)\, f \Big)$$

Here $t \geq 0$ denotes the time variable, $v \in \mathbb{R}^d$ is the velocity variable

$$\mathbf{u}_f(t) = \frac{\int_{\mathbb{R}^d} v\, f(t,v)\, dv}{\int_{\mathbb{R}^d} f(t,v)\, dv} \quad \text{is the } \textit{mean velocity}$$

(J. Tugaut), (A. Barbaro, J. Canizo, J. Carrillo, P. Degond)

Theorem (X. Li)

Let $d \geq 1$ and $\alpha > 0$. There exists a critical $D_* > 0$ such that

(i) $D > D_*$: *only one stable stationary distribution with* $\mathbf{u}_f = \mathbf{0}$

(ii) $D < D_*$: *one instable isotropic stationary distribution with* $\mathbf{u}_f = \mathbf{0}$ *and a continuum of stable non-negative non-symmetric polarized stationary distributions (unique up to a rotation)*

Any stationary solution can be written as

$$f_{\mathbf{u}}(v)=\frac{e^{-\frac{1}{D}\left(\frac{1}{2}\,|v-\mathbf{u}|^2+\frac{\alpha}{4}\,|v|^4-\frac{\alpha}{2}\,|v|^2\right)}}{\int_{\mathbb{R}^d}e^{-\frac{1}{D}\left(\frac{1}{2}\,|v-\mathbf{u}|^2+\frac{\alpha}{4}\,|v|^4-\frac{\alpha}{2}\,|v|^2\right)}\,dv}$$

where $\mathbf{u}=(u_1,..u_d)\in\mathbb{R}^d$ solves $\int_{\mathbb{R}^d}(\mathbf{u}-v)\,f_{\mathbf{u}}(v)\,dv=0$
Up to a rotation, $\mathbf{u}=(u,0,...0)=u\,e_1$ is given by

$$\mathcal{H}(u)=0$$

where

$$\mathcal{H}(u):=\int_{\mathbb{R}^d}(v_1-u)\,e^{-\frac{1}{D}(\varphi_\alpha(v)-u\,v_1)}\,dv\quad\text{and}\quad\varphi_\alpha(v):=\tfrac{\alpha}{4}\,|v|^4+\tfrac{1-\alpha}{2}\,|v|^2$$

A technical observation

$$\mathcal{H}(u) = \alpha \int_{\mathbb{R}^d} \left(1 - |v|^2\right) v_1 \, e^{-\frac{1}{D}(\varphi_\alpha(v) - u\, v_1)} \, dv$$

because (integrate on $\mathbb{R}^d$)

$$-D \frac{\partial}{\partial v_1} \left(e^{-\frac{1}{D}(\varphi_\alpha(v) - u\, v_1)} \right) = \left(v_1 - u + \alpha \left(|v|^2 - 1 \right) v_1 \right) e^{-\frac{1}{D}(\varphi_\alpha(v) - u\, v_1)}$$

(integrate on $\mathbb{R}^d$) and

$$\mathcal{H}'(u) = \frac{\alpha}{D} \int_{\mathbb{R}^d} \left(1 - |v|^2\right) v_1^2 \, e^{-\frac{1}{D}(\varphi_\alpha(v) - u\, v_1)} \, dv \,, \; \mathcal{H}'(0) = \frac{\alpha}{D} \, |\mathbb{S}^{d-1}| \left(j_{d+1} - j_{d+3} \right)$$

where $j_d(D) := \int_0^\infty s^d \, e^{-\frac{1}{D}\varphi_\alpha(s)} \, ds$ + elementary manipulations

$$j_{n+5} - 2\, j_{n+3} + j_{n+1} = \int_0^\infty s^{n+1} \left(s^2 - 1 \right)^2 e^{-\frac{\varphi_\alpha}{D}} \, ds > 0$$

$$\alpha\, j_{n+5} + (1 - \alpha)\, j_{n+3} = \int_0^\infty s^{n+2} \, \varphi_\alpha' \, e^{-\frac{1}{D}\varphi_\alpha} \, ds = (n+2)\, D\, j_{n+1}$$

The bifurcation point D_*

If $d = 1$, let us consider a continuous positive function ψ on $\mathbb{R}^+$ such that the function $s \mapsto \psi(s)\, e^{s^2}$ is integrable and define

$$H(u) := \int_0^{+\infty} \left(1 - s^2\right)\, \psi(s)\, \sinh(s\, u)\, ds \quad \forall\, u \geq 0$$

For any $u > 0$, $H''(u) < 0$ if $H(u) \leq 0$. As a consequence, H changes sign at most once on $(0, +\infty)$

If $d \geq 2$, consider a series expansion

Lemma

$\mathcal{H}(u) = 0$ has as a solution $u = u(D) > 0$ if and only if $D < D_*$ and $\lim_{D \to (D_*)_-} u(D) = 0$

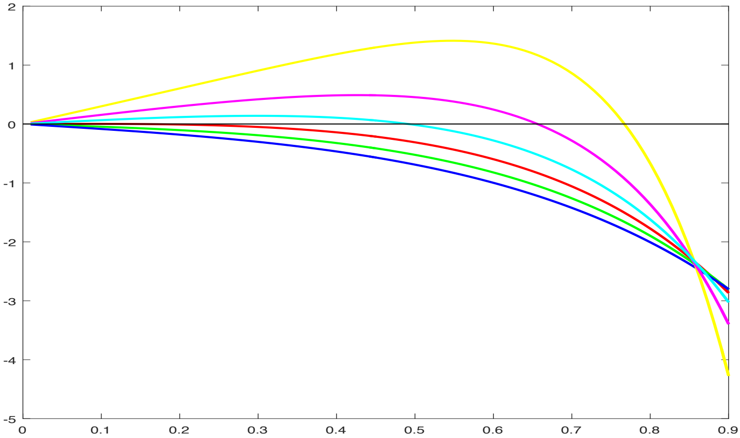


Figure: Plot of $u \mapsto \mathcal{H}(u)$ when $d = 2$, $\alpha = 2$, and $D = 0.2, 0.25, \ldots 0.45$

Relative entropy and related quantities

- *Free energy*

$$\mathcal{F}[f] := D \int_{\mathbb{R}^d} f \, \log f \, dv + \int_{\mathbb{R}^d} f \, \varphi_\alpha \, dv - \frac{1}{2} \, |\mathbf{u}_f|^2$$

- *Relative entropy* with respect to a stationary solution $f_{\mathbf{u}}$

$$\mathcal{F}[f] - \mathcal{F}[f_{\mathbf{u}}] = D \int_{\mathbb{R}^d} f \, \log \left(\frac{f}{f_{\mathbf{u}}} \right) dv - \frac{1}{2} \, |\mathbf{u}_f - \mathbf{u}|^2$$

- *Relative Fisher information*

$$\mathcal{I}[f] := \int_{\mathbb{R}^d} \left| D \, \frac{\nabla f}{f} + \alpha \, v \, |v|^2 + (1 - \alpha) \, v - \mathbf{u}_f \right|^2 f \, dv$$

- *Non-equilibrium Gibbs state*

$$G_f(v) := \frac{e^{-\frac{1}{D} \left(\frac{1}{2} \, |v - \mathbf{u}_f|^2 + \frac{\alpha}{4} \, |v|^4 - \frac{\alpha}{2} \, |v|^2 \right)}}{\int_{\mathbb{R}^d} e^{-\frac{1}{D} \left(\frac{1}{2} \, |v - \mathbf{u}_f|^2 + \frac{\alpha}{4} \, |v|^4 - \frac{\alpha}{2} \, |v|^2 \right)} \, dv}$$

Gibbs state vs. stationary solution

$\mathcal{F}[f]$ is a Lyapunov function in the sense that

$$\frac{d}{dt}\mathcal{F}[f(t,\cdot)] = -\,\mathcal{I}[f(t,\cdot)]$$

where $\mathcal{F}[f] - \mathcal{F}[f_{\mathbf{u}}] = D\int_{\mathbb{R}^d} f\,\log\left(\frac{f}{f_{\mathbf{u}}}\right)dv - \frac{1}{2}\,|\mathbf{u}_f - \mathbf{u}|^2$ and

$$\mathcal{I}[f] = D^2\int_{\mathbb{R}^d}\left|\nabla\log\left(\frac{f}{G_f}\right)\right|^2 f\,dv$$

$\frac{d}{dt}\mathcal{F}[f(t,\cdot)] = 0$ if and only if $f = G_f$ is a stationary solution

Stability and coercivity

$$Q_{1,\mathbf{u}}[g] := \lim_{\varepsilon\to0} \frac{2}{\varepsilon^2}\,\mathcal{F}\big[f_{\mathbf{u}}(1+\varepsilon\,g)\big] = D\int_{\mathbb{R}^d} g^2\,f_{\mathbf{u}}\,dv - D^2\,|\mathbf{v}_g|^2$$

$$\text{where } \mathbf{v}_g := \tfrac{1}{D}\int_{\mathbb{R}^d} v\,g\,f_{\mathbf{u}}\,dv$$

$$Q_{2,\mathbf{u}}[g] := \lim_{\varepsilon\to0} \frac{1}{\varepsilon^2}\,\mathcal{I}\big[f_{\mathbf{u}}\,(1+\varepsilon\,g)\big] = D^2\int_{\mathbb{R}^d} |\nabla g - \mathbf{v}_g|^2\,f_{\mathbf{u}}\,dv$$

Stability: $Q_{1,\mathbf{u}} \ge 0$ *?*

Coercivity: $Q_{2,\mathbf{u}} \ge \lambda\,Q_{1,\mathbf{u}}$ *for some* $\lambda > 0$ *?*

Stability of the isotropic stationary solution

$$Q_{1,\mathbf{0}}[g] = D \int_{\mathbb{R}^d} g^2 \, f_{\mathbf{0}} \, dv - D^2 \, |\mathbf{v}_g|^2$$

We consider the space of the functions $g \in \mathrm{L}^2(f_{\mathbf{0}} \, dv)$ such that

$$\int_{\mathbb{R}^d} g \, f_{\mathbf{0}} \, dv = 0$$

Lemma (X. Li)

$Q_{1,\mathbf{0}}$ is a nonnegative quadratic form if and only if $D \geq D_*$ and

$$Q_{1,\mathbf{0}}[g] \geq \eta(D) \int_{\mathbb{R}^d} g^2 \, f_{\mathbf{0}} \, dv$$

for some explicit $\eta(D) > 0$ if $D > D_*$

Stability of the polarized stationary solution

Corollary (X. Li)

$\mathcal{F}$ has a unique nonnegative minimizer with unit mass, $f_{\mathbf{0}}$, if $D \geq D_*$. Otherwise, if $D < D_*$, we have

$$\min \mathcal{F}[f] = \mathcal{F}[f_{\mathbf{u}}] < \mathcal{F}[f_{\mathbf{0}}]$$

for any $u \in \mathbb{R}^d$ such that $|\mathbf{u}| = u(D)$.

The minimum is taken on $\mathrm{L}^1_+\big(\mathbb{R}^d, (1+|v|^4)\,dv\big)$ such that $\int_{\mathbb{R}^d} f\,dv = 1$

Corollary (X. Li)

Let $D < D_*$, $|\mathbf{u}| = u(D) \neq 0$. Then

$$Q_{1,\mathbf{u}}[g] \geq 0$$

Hint: $f_{\mathbf{u}}$ minimizes the free energy

A coercivity result

Poincaré inequality: if $\int_{\mathbb{R}^d} h\, f_{\mathbf{u}}\, dv = 0$

$$\int_{\mathbb{R}^d} |\nabla h|^2 f_{\mathbf{u}}\, dv \geq \Lambda_D \int_{\mathbb{R}^d} |h|^2 f_{\mathbf{u}}\, dv$$

Let $f \in \mathrm{L}^1(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} f\, dv = 1$, $g = (f - f_{\mathbf{u}})/f_{\mathbf{u}}$ and let $\mathbf{u}[f] = \frac{u(D)}{|\mathbf{u}_f|}\, \mathbf{u}_f$ if $D < D_*$ and $\mathbf{u}_f \neq \mathbf{0}$. Otherwise take $\mathbf{u}[f] = \mathbf{0}$

Proposition (X. Li)

Let $d \geq 1$, $\alpha > 0$, $D > 0$. If $\mathbf{u} = \mathbf{0}$, then

$$Q_{2,\mathbf{u}}[g] \geq \mathcal{C}_D\, Q_{1,\mathbf{u}}[g]$$

Otherwise, if $|\mathbf{u}| = u(D) \neq 0$ for some $D \in (0, D_*)$, then

$$Q_{2,\mathbf{u}}[g] \geq \mathcal{C}_D \left(1 - \kappa(D)\right) \frac{(\mathbf{v}_g \cdot \mathbf{u})^2}{|\mathbf{v}_g|^2\, |\mathbf{u}|^2}\, Q_{1,\mathbf{u}}[g]$$

Recall that $\mathbf{v}_g := \frac{1}{D}\int_{\mathbb{R}^d} (v-\mathbf{u})\, g\, f_{\mathbf{u}}\, dv$

$$Q_{2,\mathbf{u}}[g] \geq \mathcal{C}_D \left(1-\kappa(D)\right) \frac{(\mathbf{v}_g \cdot \mathbf{u})^2}{|\mathbf{v}_g|^2\,|\mathbf{u}|^2}\, Q_{1,\mathbf{u}}[g]$$

$\kappa(D) < 1$ and as a special case, if $\mathbf{u} = \mathbf{u}[f]$, then

$$Q_{2,\mathbf{u}}[g] \geq \mathcal{C}_D \left(1-\kappa(D)\right) Q_{1,\mathbf{u}}[g]$$

Apply Poincaré to $h(v)=g(v)-(v-\mathbf{u})\cdot\mathbf{v}_g$

$$\begin{aligned}\frac{1}{D^2}\,Q_{2,\mathbf{u}}[g]&=\int_{\mathbb{R}^d}|\nabla g-\mathbf{v}_g|^2\,f_{\mathbf{u}}\,dv\\ &\ge\Lambda_D\int_{\mathbb{R}^d}\big(g^2+|\mathbf{v}_g\cdot(v-\mathbf{u})|^2-2\,\mathbf{v}_g\cdot(v-\mathbf{u})\,g\big)\,f_{\mathbf{u}}\,dv\\ &=\Lambda_D\left[\int_{\mathbb{R}^d}|g|^2\,f_{\mathbf{u}}\,dv+\int_{\mathbb{R}^d}|\mathbf{v}_g\cdot(v-\mathbf{u})|^2\,f_{\mathbf{u}}\,dv-2\,D\,|\mathbf{v}_g|^2\right]\end{aligned}$$

Lemma (X. Li)

Assume that $d \geq 1$, $\alpha > 0$ *and* $D > 0$.

(i) *In the case* $\mathbf{u} = \mathbf{0}$, *we have that* $\int_{\mathbb{R}^d} |v|^2 f_{\mathbf{0}}\, dv > d\, D$ *if and only if* $D < D_*$

(ii) *In the case* $d \geq 2$, $D \in (0, D_*)$ *and* $\mathbf{u} \neq \mathbf{0}$, *we have that*

$$\int_{\mathbb{R}^d} \big|(v - \mathbf{u}) \cdot \mathbf{u}\big|^2 f_{\mathbf{u}}\, dv < D\, |\mathbf{u}|^2$$

$$\int_{\mathbb{R}^d} \big|(v - \mathbf{u}) \cdot \mathbf{w}\big|^2 f_{\mathbf{u}}\, dv = D\, |\mathbf{w}|^2 \quad \forall\, w \in \mathbb{R}^d \quad \textit{such that} \quad \mathbf{u} \cdot \mathbf{w} = 0$$

$$\frac{1}{D} \int_{\mathbb{R}^d} \big|(v - \mathbf{u}) \cdot \mathbf{w}\big|^2 f_{\mathbf{u}}\, dv = \kappa(D)\, (\mathbf{w} \cdot \mathbf{e})^2 + |\mathbf{w}|^2 - (\mathbf{w} \cdot \mathbf{e})^2 \quad \forall\, \mathbf{w} \in \mathbb{R}^d$$

High noise: convergence to the isotropic solution

Theorem (X. Li)

For any $d \geq 1$ and any $\alpha > 0$, if $D > D_*$, then for any solution f with nonnegative initial datum $f_{\rm in}$ of mass 1 such that $\mathcal{F}[f_{\rm in}] < \infty$, there is a positive constant C such that, for any time $t > 0$,

$$0 \leq \mathcal{F}[f(t,\cdot)] - \mathcal{F}[f_{\mathbf{0}}] \leq C\, e^{-\mathcal{C}_D\, t}$$

An exponential rate of convergence for radially symmetric solutions

Logarithmic Sobolev inequality

$$\int_{\mathbb{R}^d} \left|\nabla \log \left(\frac{f}{f_0}\right)\right|^2 f\,dv \geq \mathcal{K}_0 \int_{\mathbb{R}^d} f\,\log\left(\frac{f}{f_0}\right) dv = \mathcal{F}[f] - \mathcal{F}[f_0] \qquad (1)$$

Proposition (X. Li)

A solution $f \in C^0\left(\mathbb{R}^+, \mathrm{L}^1(\mathbb{R}^d)\right)$ of with radially symmetric initial datum $f_{\rm in} \in \mathrm{L}^1_+(\mathbb{R}^d)$ such that $\mathcal{F}[f_{\rm in}] < \infty$. Then

$$0 \leq \mathcal{F}[f(t,\cdot)] - \mathcal{F}[f_0] \leq C\,e^{-\lambda\,t}$$

for some $\lambda > 0$

The Gibbs state and the stationary solution coincide

Continuity and convergence of the velocity average

Proposition (X. Li)

Let $\alpha > 0$, $D > 0$ and consider a solution $f \in C^0\left(\mathbb{R}^+, \mathrm{L}^1(\mathbb{R}^d)\right)$ with initial datum $f_{\mathrm{in}} \in \mathrm{L}^1_+(\mathbb{R}^d)$ such that $\mathcal{F}[f_{\mathrm{in}}] < \infty$. Then $t \mapsto \mathbf{u}_f(t)$ is a Lipschitz continuous function on $\mathbb{R}^+$ such that $\lim_{t\to+\infty} \mathbf{u}_f(t) = \mathbf{0}$ if $D \geq D_*$ and $\lim_{t\to+\infty} |\mathbf{u}_f(t)| = u$ with either $u = 0$ or $u = u(D)$ if $D \in (0, D_*)$

$$\frac{d\mathbf{u}_f}{dt} = -\,\alpha \int_{\mathbb{R}^d} v\left(|v|^2 - 1\right) f\, dv$$

- *Csiszár-Kullback inequality*

$$\int_{\mathbb{R}^d} f\, \log\left(\frac{f}{G_f}\right) dv \geq \frac{1}{4}\, \|f - G_f\|^2_{\mathrm{L}^1(\mathbb{R}^d)}$$

$$\int_{\mathbb{R}^d} v\,(f - G_f)\, dv = \mathbf{u}_f - \int_{\mathbb{R}^d} v\, G_f\, dv = \int_{\mathbb{R}^d} (\mathbf{u}_f - v)\, G_f\, dv = -\,\frac{\mathcal{H}(u_f)}{\mathcal{C}(u_f)}$$

A non-local scalar product for the linearized evolution operator

In terms of $f = f_{\mathbf{0}}\,(1+g)$ the evolution equation is

$$f_{\mathbf{0}}\,\frac{\partial g}{\partial t} = D\,\nabla\cdot\Big((\nabla g - \mathbf{v}_g)\,f_{\mathbf{0}} - \mathbf{v}_g\,g\,f_{\mathbf{0}}\Big)$$

with $\mathbf{v}_g = \frac{1}{D}\int_{\mathbb{R}^d} v\,g\,f_{\mathbf{0}}\,dv$ and $Q_{1,\mathbf{0}}[g] = \langle g, g\rangle$ where

$$\langle g_1, g_2\rangle := D\int_{\mathbb{R}^d} g_1\,g_2\,f_{\mathbf{0}}\,dv - D^2\,\mathbf{v}_{g_1}\cdot\mathbf{v}_{g_2}$$

is a *scalar product* on the space $\mathfrak{X} := \big\{g \in \mathrm{L}^2(f_{\mathbf{0}}\,dv)\,:\,\int_{\mathbb{R}^d} g\,f_{\mathbf{0}}\,dv = 0\big\}$

$$\frac{\partial g}{\partial t} = \mathcal{L}\,g - \mathbf{v}_g\cdot\Big(D\,\nabla g - (v + \nabla\varphi_\alpha)\,g\Big)$$

$$\mathcal{L}\,g := D\,\Delta g - (v + \nabla\varphi_\alpha)\cdot(\nabla g - \mathbf{v}_g)$$

Lemma (X. Li)

Assume that $D > D_*$ *and* $\alpha > 0$*. The norm* $g \mapsto \sqrt{\langle g, g \rangle}$ *is equivalent to the standard norm on* $\mathrm{L}^2(f_{\mathbf{0}}\,dv)$ *according to*

$$\eta(D) \int_{\mathbb{R}^d} g^2\, f_{\mathbf{0}}\, dv \le \langle g, g \rangle \le D \int_{\mathbb{R}^d} g^2\, f_{\mathbf{0}}\, dv \quad \forall\, g \in \mathcal{X}$$

The linearized operator $\mathcal{L}$ *is self-adjoint on* $\mathcal{X}$ *and*

$$-\langle g, \mathcal{L}\, g \rangle = Q_{2,\mathbf{0}}[g]$$

The scalar product $\langle\cdot,\cdot\rangle$ is well adapted to the linearized evolution operator in the sense that a solution of the *linearized equation*

$$\frac{\partial g}{\partial t} = \mathcal{L}\, g$$

with initial datum $g_0 \in \mathcal{X}$ is such that

$$\frac{1}{2}\,\frac{d}{dt}\,Q_{1,\mathbf{0}}[g] = \frac{1}{2}\,\frac{d}{dt}\langle g, g\rangle = \langle g, \mathcal{L}\, g\rangle = -\,Q_{2,\mathbf{0}}[g]$$

and has exponential decay. According to Proposition 3.4, we know that

$$\langle g(t,\cdot), g(t,\cdot)\rangle = \langle g_0, g_0\rangle\, e^{-2\,\mathcal{C}_D\,t} \quad \forall\, t \ge 0$$

Proof of the exponential rate of convergence

$$\frac{\partial g}{\partial t} = \mathcal{L}\, g - \mathbf{v}_g \cdot \Big(D\, \nabla g - (v + \nabla \varphi_\alpha)\, g \Big)$$

A Grönwall estimate

$$\frac{1}{2}\,\frac{d}{dt}\, Q_{1,\mathbf{0}}[g] + \; Q_{2,\mathbf{0}}[g] = D^2\, \mathbf{v}_g \cdot \int_{\mathbb{R}^d} g\, (\nabla g - \mathbf{v}_g)\, f_{\mathbf{0}}\, dv$$

based on

$$\frac{d}{dt}\, Q_{1,\mathbf{0}}[g] \le -\, 2\, \mathcal{C}_D \left(1 - |\mathbf{u}_f(t)|\, \sqrt{\tfrac{\mathcal{C}_D}{\eta(D)}} \right) Q_{1,\mathbf{0}}[g]$$

We know that $\lim_{t\to+\infty} |\mathbf{u}_f(t)| = 0$, which proves that

$$\limsup_{t\to+\infty} e^{2\,(\mathcal{C}_D - \varepsilon)\, t}\, Q_{1,\mathbf{0}}[g(t,\cdot)] < +\infty$$

for any $\varepsilon \in (0, \mathcal{C}_D)$

Symmetric and non-symmetric stationary states

Without limitation on D but without rates...

Lemma (X. Li)

For any $d \geq 1$ and any $\alpha > 0$, if $D < D_*$, then for any solution $f \in C^0\big(\mathbb{R}^+, \mathrm{L}^1(\mathbb{R}^d)\big)$ with initial datum $f_{\rm in} \geq 0$ of mass 1 such that $\mathcal{F}[f_{\rm in}] < \mathcal{F}[f_{\mathbf{0}}]$. Then $\lim_{t\to+\infty} |\mathbf{u}_f(t)| = u(D)$ and $\lim_{t\to+\infty} \mathcal{F}[f(t,\cdot)] = \mathcal{F}[f_{\mathbf{u}}]$ for some $\mathbf{u} \in \mathbb{R}^d$ such that $|\mathbf{u}| = u(D)$ and

$$f(t+n,\cdot) \longrightarrow f_{\mathbf{u}} \quad in \quad \mathrm{L}^1(\mathbb{R}^+ \times \mathbb{R}^d) \quad as \quad n \to +\infty$$

if $\lim_{t\to+\infty} \mathbf{u}_f(t) = \mathbf{u}$

An exponential rate of convergence for partially symmetric solutions in the polarized case

Proposition (X. Li)

Let $\alpha > 0$, $D > 0$ *and consider a solution* $f \in C^0\left(\mathbb{R}^+, \mathrm{L}^1(\mathbb{R}^d)\right)$ *with initial datum* $f_{\rm in} \in \mathrm{L}^1_+(\mathbb{R}^d)$ *such that* $\mathcal{F}[f_{\rm in}] < \mathcal{F}[f_{\mathbf{0}}]$ *and* $\mathbf{u}_{f_{\rm in}} = (u, 0 \ldots 0)$ *for some* $u \neq 0$. *We further assume that* $f_{\rm in}(v_1, v_2, \ldots v_{i-1}, v_i, \ldots) = f_{\rm in}(v_1, v_2, \ldots v_{i-1}, -v_i, \ldots)$ *for any* $i = 2$, $3, \ldots d$. *Then*

$$0 \leq \mathcal{F}[f(t,\cdot)] - \mathcal{F}[f_{\mathbf{u}}] \leq C\, e^{-\lambda\, t} \quad \forall\, t \geq 0$$

holds with $\lambda = \mathcal{C}_D\left(1 - \kappa(D)\right) > 0$

Without symmetry assumption, the question of the rate of convergence to a solution / to the set of polarized solutions sis still open

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Thank you for your attention !