Fast diffusion, mean field drifts and reverse HLS inequalities
Sharp asymptotics for the subcritical Keller-Segel model

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Lecture 3

Intensive Week of PDEs@Cogne

Outline

1. An introduction
   - The super-critical range: life after blow-up
   - The subcritical range
   - Self-similar variables and a first convergence result
2. Functional framework and sharp asymptotics
   - Stationary solutions and linearization
   - Scalar product and spectrum
   - Rates of convergence for the nonlinear model
3. Extensions, consequences
   - Parabolic-parabolic models
   - Improved inequalities

Keller-Segel model: an introduction

Warning !

1. Literature is huge
2. Physics can be addressed in various ways: gravitation (Smoluchowski-Poisson) and statistics of gravitating systems, aggregation dynamics (sticky systems), biology (Patlak, Keller-Segel)
3. Standard techniques have been reinvented many times: virial estimates, cumulated mass densities, matched asymptotics

$\Longrightarrow$ some entry points in the literature

- do not specialize to radial solutions
- put emphasis on functional analysis
- insist on nonlinear evolution
- deal with the subcritical case: at least it gives some hint on how a *subcritical bubble* appears in the critical limit

The parabolic-elliptic Keller – Segel system

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u \, \nabla v) & x \in \mathbb{R}^2 \,, \; t > 0 \\ -\Delta v = u & x \in \mathbb{R}^2 \,, \; t > 0 \\ u(\cdot, t = 0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

We make the choice:

$$v(t,x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log|x-y| \, u(t,y) \, dy$$

and observe that

$$\nabla v(t,x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2} \, u(t,y) \, dy$$

Mass conservation: $\dfrac{d}{dt} \displaystyle\int_{\mathbb{R}^2} u(t,x) \, dx = 0$

Blow-up: the virial computation

Collapse (S. Childress, J.K. Percus 1981) $M = \int_{\mathbb{R}^2} n_0\,dx > 8\pi$ and $\int_{\mathbb{R}^2} |x|^2\,n_0\,dx < \infty$: blow-up in finite time

a solution u of

$$\frac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u\,\nabla v)$$

satisfies

$$\frac{d}{dt}\int_{\mathbb{R}^2} |x|^2\,u(t,x)\,dx$$

$$= -\underbrace{\int_{\mathbb{R}^2} 2\,x\cdot\nabla u\,dx}_{-4\,M} + \frac{1}{2\pi}\iint_{\mathbb{R}^2\times\mathbb{R}^2} \underbrace{\frac{2x\cdot(y-x)}{|x-y|^2}\,u(t,x)\,u(t,y)\,dx\,dy}_{\frac{(x-y)\cdot(y-x)}{|x-y|^2}\,u(t,x)\,u(t,y)\,dx\,dy}$$

$$= 4\,M - \frac{M^2}{2\pi} < 0 \quad \text{if} \quad M > 8\pi$$

Blow-up and singular solutions: some results

1. Formal asymptotic expansions in $\mathbb{R}^2$
(Herrero, Velázquez 1997), (Chavanis, Sire 2002-2005), (Campos, PhD thesis, 2012)
(Dejak, Lushnikov, Ovchinnikov, Sigal 2012), (Dejak, Egli, Lushnikov, Sigal 2013)
2. Results in bounded domains: (Kavallaris, Souplet 2009)
3. A first rigorous result in $\mathbb{R}^2$ (radial case)
(Raphaël, Schweyer 2012-2013) stable chemotactic blow-up, universality of the bubble
4. Other results in $\mathbb{R}^2$: (Montaru 2012-2013)
5. Measure valued solutions: (Herrero, Velázquez 1997), (Luckhaus, Sugiyama, Velázquez 2012), (Seki, Sugiyama, Velázquez 2013)
(Haškovec, Schmeiser 2009) the particle system, Wasserstein's distance and free energy
(Bedrossian, Masmoudi 2012) spectral gap and free energy

more results

1. (W. Jäger, S. Luckhaus), (A. Blanchet, JD, B. Perthame)
2. a review of related models: (D. Horstmann D, 2003: "From 1970 until present...") Crowd modeling, social sciences
3. (L. Corrias et al.), (V. Calvez et al.) when other terms are taken into account. Limits: (P. Biler, L. Brandolese)
4. The 8π case: (A. Blanchet, J.A. Carrillo, N. Masmoudi), (E.A. Carlen, J. A. Carrillo, and M. Loss), (E.A. Carlen and A. Figalli),
5. Complex blow-up patterns (Y. Seki, Y. Sugiyama, J.J.L. Velázquez)
6. exploration of the blow-up by formal methods: (J.J.L. Velázquez, M.A. Herrero), (J.J.L. Velázquez et al.)... (S. Luckhaus, Y. Sugiyama, J.J.L. Velázquez 2012)
7. models with nonlinear diffusion terms: (Y. Sugiyama), (A. Blanchet and P. Laurençot),
8. models with prevention of overcrowding: (C. Schmeiser et al.)
9. models with more than one species: (E.E Espejo, K. Vilches, C. Carlos 2013), (F. Dickstein 2013)
10. and many more !... e.g. in bounded domains...

more recent results

1. Large mass and blow-up for the evolution problem (J. Bedrossian, 2015)
2. Rates (A. Montaru. 2015)
3. Regularity of the solutions (J. Bedrossian, N. Masmoudi, 2014), (P. Biler, J. Zienkiewicz, 2015), (Y. Sugiyama, 2015)
4. Gradient flows, construction of the solutions (A. Blanchet, J. A. Carrillo, D. Kinderlehrer, M. Kowalczyk, P. Laurençot, S. Lisini, 2015)
5. More on blow-up (L. Chen, H. Siedentop, 2017), (T.-E. Ghoul, N. Masmoudi, 2018)

The super-critical range: life after blow-up

Regularization

Regularize the Poisson kernel

$$(-\Delta)_\varepsilon^{-1} * \rho\,(x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \log(|x-y|+\varepsilon)\,\rho(y)\,dy$$

[F. Poupaud, Diagonal defect measures, adhesion dynamics and Euler equations, Meth. Appl. Anal. **9** (2002), pp. 533–561]

Proposition (JD, C. Schmeiser 2009)

For every $\varepsilon > 0$, the regularized problem has a global solution satisfying

$$\|\rho^\varepsilon(\cdot,t)\|_{L^1(\mathbb{R}^2)} = \|\rho_I\|_{L^1(\mathbb{R}^2)} := M$$

$$\|\rho^\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R}^2)} \le c\left(1+\frac{1}{\varepsilon^2}\right)$$

with an ε-independent constant c

The nonlinear term

$$m^\varepsilon(t,x) := \int_{\mathbb{R}^2} \mathcal{K}^\varepsilon(x-y)\,\rho^\varepsilon(t,x)\,\rho^\varepsilon(t,y)dy \quad \text{with } \mathcal{K}^\varepsilon(x) = \frac{x^{\otimes 2}}{|x|(|x|+\varepsilon)}$$

Lemma (Poupaud)

The families $\{\rho^\varepsilon(t)\}_{\varepsilon>0}$ and $\{m^\varepsilon(t)\}_{\varepsilon>0}$ are tightly bounded locally uniformly in t, and $\{\rho^\varepsilon(t)\}_{\varepsilon>0}$ is tightly equicontinuous in t

Tight boundedness and equicontinuity of $\rho^\varepsilon(t) \Longrightarrow$ compactness

$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \varphi(x,y)\,\rho^\varepsilon(t,x)\,\rho^\varepsilon(t,y)\,dx\,dy \to \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \varphi(x,y)\,\rho(t,x)\,\rho(t,y)\,dx\,dy$

$\int_{t_1}^{t_2}\int_{\mathbb{R}^2} \varphi(t,x)\,m^\varepsilon(t,x)\,dx\,dt \to \int_{t_1}^{t_2}\int_{\mathbb{R}^2} \varphi(t,x)\,m(t,x)\,dx\,dt$

for all $\varphi \in C_b([t_1,t_2]\times\mathbb{R}^2)$

Defect measure

$$\nu(t,x) = m(t,x) - \int_{\mathbb{R}^2} \mathcal{K}(x-y)\,\rho(t,x)\,\rho(t,y)\,dy\,, \quad \mathcal{K}(x) = \frac{x^{\otimes 2}}{|x|^2}$$

Atomic support

The limit is characterized by the pair (ρ, ν), the atomic support of ρ is an at most countable set

Lemma (Poupaud 2002)

ν is symmetric, nonnegative, and satisfies

$$\mathrm{tr}(\nu(t,x)) \leq \sum_{a \in S_{at}(\rho(t))} (\rho(t)(\{a\}))^2 \delta(x-a)$$

$\mathcal{M}$: spaces of Radon measures
$\mathcal{M}_1^+$: subset of nonnegative bounded measures

$$\mathcal{DM}^+(I;\mathbb{R}^2) = \Big\{(\rho,\nu) : \ \rho(t) \in \mathcal{M}_1^+(\mathbb{R}^2) \ \forall t \in I, \ \nu \in \mathcal{M}(I \times \mathbb{R}^2)^{2\times 2}$$
$$\rho \text{ is tightly continuous with respect to } t$$
$$\nu \text{ is a nonnegative, symmetric, matrix valued measure}$$
$$\mathrm{tr}(\nu(t,x)) \leq \sum_{a \in S_{at}(\rho(t))} (\rho(t)(\{a\}))^2 \delta(x-a)\Big\}$$

Limiting problem

$$\int_0^T \int_{\mathbb{R}^2} \varphi(t,x)\, j[\rho,\nu](t,x)\, dx\, dt$$
$$= -\frac{1}{4\pi} \int_0^T \int_{\mathbb{R}^4} (\varphi(t,x) - \varphi(t,y))\, K(x-y)\, \rho(t,x)\, \rho(t,y)\, dx\, dy\, dt$$
$$- \frac{1}{4\pi} \int_0^T \int_{\mathbb{R}^2} \nu(t,x) \nabla\varphi(t,x)\, dx\, dt$$

for $\varphi \in C_b^1((0,T) \times \mathbb{R}^2)$

Theorem (JD, C. Schmeiser 2009)

For every $T > 0$, ρ^ε converges tightly and uniformly in time to $\rho(t)$ and there exists $\nu(t)$ such that $(\rho,\nu) \in \mathcal{DM}^+((0,T);\mathbb{R}^2)$ is a generalized solution of

$$\partial_t \rho + \nabla \cdot (j[\rho,\nu] - \nabla\rho) = 0$$

$\rho(t=0) = \rho_I$ holds in the sense of tight continuity

Strong formulation (formal) : an *ansatz*

- $\rho = \overline{\rho} + \hat{\rho}$, $\hat{\rho}(t,x) = \sum_{n\in N} M_n(t)\,\delta_n(t,x)$, $\delta_n(t,x) = \delta(x - x_n(t))$
- $(\rho,\nu) \in \mathcal{DM}^+((0,T);\mathbb{R}^2)$

$$\Longrightarrow\ \nu(t,x) = \sum_{n\in N} \nu_n(t)\,\delta_n(t,x),\ \ \mathrm{tr}(\nu_n) \le M_n^2$$

$$j[\rho,\nu] = \overline{\rho}\,\nabla S_0[\overline{\rho}+\hat{\rho}] + \sum_n M_n\,\delta_n\,\nabla S_0\left[\overline{\rho} + \sum_{m\neq n} M_m\,\delta_m\right] + \frac{1}{4\pi}\sum_n M_n\,\nu_n\,\nabla\delta_n$$

$$\begin{aligned}\partial_t\overline{\rho} + \nabla\cdot(\overline{\rho}\,\nabla S_0[\overline{\rho}] - \nabla\overline{\rho}) + \nabla\overline{\rho}\cdot\nabla S_0[\hat{\rho}]&\\ + \sum_n \delta_n(\dot{M}_n - \overline{\rho}\,M_n)&\\ - \sum_n M_n\,\nabla\delta_n\left(\dot{x}_n - \nabla S_0\left[\overline{\rho} + \textstyle\sum_{m\neq n} M_m\,\delta_m\right]\right)&\\ + \sum_n\left(\tfrac{1}{4\pi}\nu_n : \nabla^2\delta_n - M_n\,\Delta\delta_n\right) &= 0\end{aligned}$$

$$\nu_n = 4\pi M_n \,\mathrm{id}$$

As a consequence of $\mathrm{tr}(\nu_n) = 8\pi M_n \le M_n^2$, point masses have to be at least 8π (there is only a finite number of them)

$$\partial_t \overline{\rho} + \nabla \cdot (\overline{\rho}\, \nabla S_0[\overline{\rho}] - \nabla \overline{\rho}) - \frac{1}{2\pi} \nabla \overline{\rho} \cdot \sum_n M_n \, \frac{x - x_n}{|x - x_n|^2} = 0$$

$$\dot{M}_n = \overline{\rho}(x = x_n)\, M_n$$

$$\dot{x}_n = \nabla S_0[\overline{\rho}](x = x_n) - \frac{1}{2\pi} \sum_{m \neq n} M_m \frac{x_n - x_m}{|x_n - x_m|^2}$$

Note that $\frac{d}{dt}\left(\int_{\mathbb{R}^2} \overline{\rho}\, dx + \sum_n M_n\right) = 0$

... Comparison with Velázquez' results

Long time behaviour

Assume again

$$\nu(t,x) = 4\pi \,\mathrm{id} \sum_{a \in S_{at}(\rho(t))} \rho(t)(\{a\})\, \delta(x-a)$$

and

$$\int_{\mathbb{R}^2} |x|^2 \rho_I \, dx < \infty$$

With $\hat{M} = \sum_{a \in S_{at}(\rho(t))} \rho(t)(\{a\})$ and $\bar{M} = M - \hat{M}$

$$\frac{d}{dt}\int_{\mathbb{R}^2} |x|^2 \rho \, dx = 4M - \frac{1}{2\pi}\int_{\mathbb{R}^4} (1-\chi_D)\rho \otimes \rho \, dy\, dx - \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{tr}(\nu)\, dx$$

$$= \bar{M}\left(4 - \frac{M}{2\pi} - \frac{\hat{M}}{2\pi}\right) - \frac{1}{2\pi} \sum_{a \neq b,\ a,b \in S_{at}(\rho(t))} \rho(t)(\{a\})\, \rho(t)(\{b\})$$

... compatible with Wasserstein's framework

(Haškovec, Schmeiser 2009)

Local density profiles

For fixed t and $a \in S_{at}(\rho(t))$, let $\varepsilon\xi = x - a$ and $\varepsilon^2 \rho^\varepsilon = R^\varepsilon$

$$\varepsilon^2 \partial_t R^\varepsilon + \nabla_\xi \cdot (R^\varepsilon \nabla_\xi S_1[R^\varepsilon] - \nabla_\xi R^\varepsilon) = 0$$

R^ε is uniformly bounded, implying compactness of $\nabla_\xi S_1[R^\varepsilon]$. The L^∞-weak* limit R of R^ε (take subsequences, formal) satisfies

$$\nabla_\xi \cdot (R \nabla_\xi S_1[R] - \nabla_\xi R) = 0$$

Observe that

$$\int_{\mathbb{R}^2} R(\xi) d\xi = \frac{1}{8\pi} \int_{\mathbb{R}^4} \frac{|\xi - \eta|}{|\xi - \eta| + 1} R(\xi) R(\eta) d\eta d\xi \leq \frac{1}{8\pi} \left(\int_{\mathbb{R}^2} R(\xi) d\xi \right)^2$$

This shows that either R vanishes or its mass is not smaller than 8π

Free energy (1/2)

$$F_\varepsilon[\rho] := \int_{\mathbb{R}^2} \left(\rho \log \rho - \frac{1}{2} \rho S_\varepsilon[\rho] \right) dx$$
$$= \int_{\mathbb{R}^2} \rho \log \rho \, dx + \frac{1}{4\pi} \int_{\mathbb{R}^4} \log(|x-y| + \varepsilon) \rho(x) \rho(y) dy \, dx$$

and

$$\frac{d}{dt} F_\varepsilon[\rho^\varepsilon] = - \int_{\mathbb{R}^2} \rho^\varepsilon |\nabla (\log \rho^\varepsilon - S_\varepsilon[\rho^\varepsilon])|^2 dx$$

With an arbitrary $a \in \mathbb{R}^2$ and $R(\xi) = \varepsilon^2 \rho(a + \varepsilon \xi)$ we have

$$F_\varepsilon[\rho] = \left(2M - \frac{M^2}{4\pi} \right) \log \frac{1}{\varepsilon} + F_1[R]$$

Free energy (2/2)

Lemma

Let $R \in L^1_+(\mathbb{R}^2)$ *be radial,* $\int_{\mathbb{R}^2} \log(1+|x|)\, R(x)\, dx < \infty$*,* $M = \int_{\mathbb{R}^2} R\, dx$

$$\frac{1}{4\pi} \int_{\mathbb{R}^2} \log(1+|x-y|)\, R(y)\, dy \geq \frac{M}{4\pi} \log|x| \quad \forall\, x \in \mathbb{R}^2$$

$L^1_{+,M} := \{R \in L^1_+(\mathbb{R}^2) : \int_{\mathbb{R}^2} R\, d\xi = M\}$,
$J_M := \inf_{R \in L^1_{+,M}} F_1[R] \geq -\infty$

Theorem

$J_M = -\infty$ *for* $M > 8\pi$*, and* $J_M > -\infty$ *for* $M \leq 8\pi$*. If* $M > 8\pi$*, there exists a radial nonincreasing minimizer*

Keller-Segel model: the subcritical range

Existence and free energy

$M = \int_{\mathbb{R}^2} n_0 \, dx \le 8\pi$: global existence (W. Jäger, S. Luckhaus 1992), (JD, B. Perthame 2004), (A. Blanchet, JD, B. Perthame 2006)

If u solves

$$\frac{\partial u}{\partial t} = \nabla \cdot [u \, (\nabla (\log u) - \nabla v)]$$

the free energy

$$F[u] := \int_{\mathbb{R}^2} u \log u \, dx - \frac{1}{2} \int_{\mathbb{R}^2} u \, v \, dx$$

satisfies

$$\frac{d}{dt} F[u(t,\cdot)] = - \int_{\mathbb{R}^2} u \left| \nabla (\log u) - \nabla v \right|^2 \, dx$$

(log HLS) inequality (E. Carlen, M. Loss 1992):
F is bounded from below if $M \le 8\pi$

... $M = 8\pi$ the critical case (A. Blanchet, J.A. Carrillo, N. Masmoudi 2008), (A. Blanchet et al.)

The existence setting for the subcritical regime

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u\, \nabla v) & x \in \mathbb{R}^2\,,\ t>0 \\ -\Delta v = u & x \in \mathbb{R}^2\,,\ t>0 \\ u(\cdot, t=0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

Initial conditions

$$n_0 \in L^1_+(\mathbb{R}^2, (1+|x|^2)\, dx)\ , \quad n_0 \log n_0 \in L^1(\mathbb{R}^2, dx)\ , \quad M := \int_{\mathbb{R}^2} n_0(x)\, dx < 8\,\pi$$

Global existence and mass conservation: $M = \int_{\mathbb{R}^2} u(x,t)\, dx\ \forall\, t \geq 0$

$v = -\frac{1}{2\pi}\, \log |\cdot| * u$

Time-dependent rescaling

$$u(x,t)=\frac{1}{R^2(t)}\,n\left(\frac{x}{R(t)},\tau(t)\right)\quad\text{and}\quad v(x,t)=c\left(\frac{x}{R(t)},\tau(t)\right)$$

with $R(t)=\sqrt{1+2t}$ and $\tau(t)=\log R(t)$

$$\begin{cases}\dfrac{\partial n}{\partial t}=\Delta n-\nabla\cdot\left(n\left(\nabla c-x\right)\right) & x\in\mathbb{R}^2\,,\ t>0\\ c=-\dfrac{1}{2\pi}\,\log|\cdot|*n & x\in\mathbb{R}^2\,,\ t>0\\ n(\cdot,t=0)=n_0\ge 0 & x\in\mathbb{R}^2\end{cases}$$

(A. Blanchet, JD, B. Perthame) Convergence in self-similar variables

$$\lim_{t\to\infty}\|n(\cdot,\cdot+t)-n_\infty\|_{L^1(\mathbb{R}^2)}=0\quad\text{and}\quad\lim_{t\to\infty}\|\nabla c(\cdot,\cdot+t)-\nabla c_\infty\|_{L^2(\mathbb{R}^2)}=0$$

means *intermediate asymptotics* in original variables:

$$\|u(x,t)-\tfrac{1}{R^2(t)}\,n_\infty\left(\tfrac{x}{R(t)},\tau(t)\right)\|_{L^1(\mathbb{R}^2)}\searrow 0$$

The stationary solution in self-similar variables

$$n_\infty = M \, \frac{e^{\,c_\infty - |x|^2/2}}{\int_{\mathbb{R}^2} e^{c_\infty - |x|^2/2}\, dx} = -\Delta c_\infty \,, \qquad c_\infty = -\frac{1}{2\pi} \log|\cdot| * n_\infty$$

- Radial symmetry (Y. Naito)
- Uniqueness (P. Biler, G. Karch, P. Laurençot, T. Nadzieja)
- As $|x| \to +\infty$, n_∞ is dominated by $e^{-(1-\varepsilon)|x|^2/2}$ for any $\varepsilon \in (0,1)$ (A. Blanchet, JD, B. Perthame)
- Bifurcation diagram of $\|n_\infty\|_{L^\infty(\mathbb{R}^2)}$ as a function of M

$$\lim_{M \to 0_+} \|n_\infty\|_{L^\infty(\mathbb{R}^2)} = 0$$

(D.D. Joseph, T.S. Lundgren) (JD, R. Stańczy)
(The bifurcation diagram will be shown later)

The stationary solution when mass varies

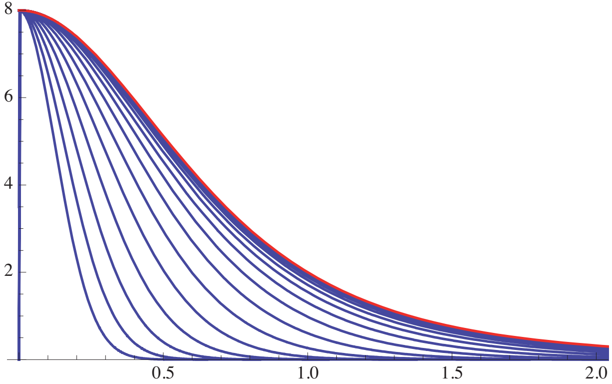


Figure: Representation of the solution appropriately scaled so that the 8π case appears as a limit (in red)

The free energy in self-similar variables

$$\frac{\partial n}{\partial t} = \nabla\Big[n\,(\log n - x + \nabla c)\Big]$$

$$F[n] := \int_{\mathbb{R}^2} n \log n \; dx + \int_{\mathbb{R}^2} \frac{1}{2}\,|x|^2\, n \; dx - \frac{1}{2}\int_{\mathbb{R}^2} n\, c \; dx$$

satisfies

$$\frac{d}{dt} F[n(t,\cdot)] = -\int_{\mathbb{R}^2} n\,|\nabla\,(\log n) + x - \nabla c|^2 \; dx$$

A last remark on 8π and scalings: $n^\lambda(x) = \lambda^2\, n(\lambda\, x)$

$$F[n^\lambda] = F[n] + \int_{\mathbb{R}^2} n \log(\lambda^2)\; dx + \int_{\mathbb{R}^2} \tfrac{\lambda^{-2}-1}{2}\,|x|^2\, n\; dx + \frac{1}{4\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} n(x)\, n(y)\; \log\frac{1}{\lambda}\; dx\, dy$$

$$F[n^\lambda] - F[n] = \underbrace{\left(2M - \frac{M^2}{4\pi}\right)}_{>0 \text{ if } M<8\pi} \log\lambda + \frac{\lambda^{-2}-1}{2}\int_{\mathbb{R}^2} |x|^2\, n\; dx$$

Keller-Segel with subcritical mass in self-similar variables

$$\begin{cases} \dfrac{\partial n}{\partial t} = \Delta n - \nabla \cdot (n\,(\nabla c - x)) & x \in \mathbb{R}^2\,,\ t > 0 \\ c = -\dfrac{1}{2\pi}\,\log|\cdot| * n & x \in \mathbb{R}^2\,,\ t > 0 \\ n(\cdot, t=0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

$$\lim_{t\to\infty} \|n(\cdot,\cdot+t) - n_\infty\|_{L^1(\mathbb{R}^2)} = 0 \quad \text{and} \quad \lim_{t\to\infty} \|\nabla c(\cdot,\cdot+t) - \nabla c_\infty\|_{L^2(\mathbb{R}^2)} = 0$$

$$n_\infty = M\,\frac{e^{\,c_\infty - |x|^2/2}}{\int_{\mathbb{R}^2} e^{c_\infty - |x|^2/2}\,dx} = -\Delta c_\infty\,, \qquad c_\infty = -\frac{1}{2\pi}\log|\cdot| * n_\infty$$

First result: small mass case

Theorem (A. Blanchet, JD, M. Escobedo, J. Fernández)

There exists a positive constant M^* *such that, for any initial data* $n_0 \in L^2(n_\infty^{-1}\,dx)$ *of mass* $M < M^*$ *satisfying the above assumptions, there is a unique solution* $n \in C^0(\mathbb{R}^+, L^1(\mathbb{R}^2)) \cap L^\infty((\tau,\infty)\times\mathbb{R}^2)$ *for any* $\tau > 0$

Moreover, there are two positive constants, C *and* δ, *such that*

$$\int_{\mathbb{R}^2} |n(t,x) - n_\infty(x)|^2 \,\frac{dx}{n_\infty} \le C\,e^{-\,\delta\,t} \quad \forall\, t > 0$$

As a function of M, δ *is such that* $\lim_{M\to 0_+} \delta(M) = 1$

Four steps proof

The condition $M \leq 8\,\pi$ is necessary and sufficient for the global existence of the solutions, but there are two extra smallness conditions in our proof:

- Uniform estimate: the *method of the trap*
- L^p and H^1 estimates in the self-similar variables
- *Spectral gap* of a linearized operator $\mathcal{L}$
- Duhamel formula and nonlinear estimates

Linearization

We can introduce two functions f and g such that

$$n = n_\infty \, (1 + f) \quad \text{and} \quad c = c_\infty (1 + g)$$

and rewrite the Keller-Segel model as

$$\frac{\partial f}{\partial t} = \mathcal{L} \, f + \frac{1}{n_\infty} \nabla (f \, n_\infty \nabla (c_\infty \, g))$$

where the linearized operator is

$$\mathcal{L} \, f = \frac{1}{n_\infty} \nabla \cdot \big(n_\infty \nabla (f - c_\infty \, g) \big)$$

and

$$-\Delta (c_\infty \, g) = n_\infty \, f$$

Keller-Segel model: functional framework and sharp asymptotics

- bifurcation diagrams
- spectrum of the linearized operator
- symmetrization
- nonlinear estimates
- rates of convergence for subcritical masses

... some preliminaries are needed

A parametrization of the solutions and the linearized operator

(J. Campos, JD)

$$-\Delta c = M \frac{e^{-\frac{1}{2}|x|^2+c}}{\int_{\mathbb{R}^2} e^{-\frac{1}{2}|x|^2+c}\,dx}$$

Solve

$$-\varphi'' - \frac{1}{r}\varphi' = e^{-\frac{1}{2}r^2+\varphi}\,, \quad r>0$$

with initial conditions $\varphi(0)=a$, $\varphi'(0)=0$ and get with $r=|x|$

$$M(a) := 2\pi \int_{\mathbb{R}^2} e^{-\frac{1}{2}r^2+\varphi_a}\,dx$$

$$n_a(x) = M(a)\,\frac{e^{-\frac{1}{2}r^2+\varphi_a(r)}}{2\pi\int_{\mathbb{R}^2} r\,e^{-\frac{1}{2}r^2+\varphi_a}\,dx} = e^{-\frac{1}{2}r^2+\varphi_a(r)}$$

Mass

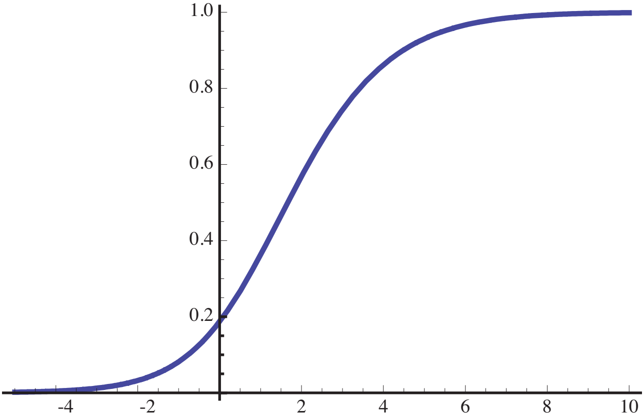


Figure: *The mass can be computed as* $M(a) = 2\pi \int_0^\infty n_a(r)\, r\, dr$*. Plot of* $a \mapsto M(a)/8\pi$

Bifurcation diagram

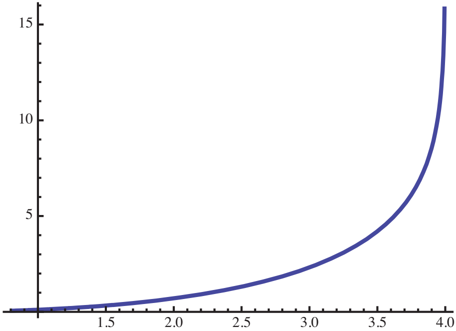


Figure: *The bifurcation diagram can be parametrized by* $a \mapsto \left(\frac{1}{2\pi} M(a), \|c_a\|_\infty\right)$ *with* $\|c_a\|_\infty = c_a(0) = a - b(a)$ *(cf. Keller-Segel system in a ball with no flux boundary conditions)*

Spectrum of $\mathcal{L}$ (lowest eigenvalues only)

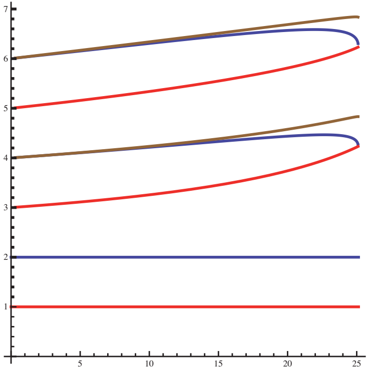


Figure: The lowest eigenvalues of $-\mathcal{L} = (-\Delta)^{-1}(n_a\, f)$ (shown as a function of the mass) are 0, 1 and 2, thus establishing that the spectral gap of $-\mathcal{L}$ is 1

(A. Blanchet, JD, M. Escobedo, J. Fernández), (J. Campos, JD), (V. Calvez, J.A. Carrillo), (J. Bedrossian, N. Masmoudi)

Spectral analysis in the functional framework determined by the relative entropy method

Simple eigenfunctions

Kernel Let $f_0 = \frac{\partial}{\partial M} c_\infty$ be the solution of

$$-\Delta\, f_0 = n_\infty\, f_0$$

and observe that $g_0 = f_0 / c_\infty$ is such that

$$\frac{1}{n_\infty} \nabla \cdot \big(n_\infty \nabla (f_0 - c_\infty\, g_0) \big) =: \mathcal{L}\, f_0 = 0$$

Lowest non-zero eigenvalues $f_1 := \frac{1}{n_\infty} \frac{\partial n_\infty}{\partial x_1}$ associated with $g_1 = \frac{1}{c_\infty} \frac{\partial c_\infty}{\partial x_1}$ is an eigenfunction of $\mathcal{L}$, such that $-\mathcal{L}\, f_1 = f_1$

With $D := x \cdot \nabla$, let $f_2 = 1 + \frac{1}{2}\, D \log n_\infty = 1 + \frac{1}{2\, n_\infty}\, D\, n_\infty$. Then

$$-\Delta\, (D\, c_\infty) + 2\, \Delta c_\infty = D\, n_\infty = 2\, (f_2 - 1)\, n_\infty$$

and so $g_2 := \frac{1}{c_\infty}\, (-\Delta)^{-1} (n_\infty\, f_2)$ is such that $-\mathcal{L}\, f_2 = 2\, f_2$

Functional setting...

Lemma (A. Blanchet, JD, B. Perthame)

Sub-critical HLS inequality (A. Blanchet, JD, B. Perthame)

$$F[n] := \int_{\mathbb{R}^2} n \log\left(\frac{n}{n_\infty}\right) dx - \frac{1}{2}\int_{\mathbb{R}^2} (n - n_\infty)\,(c - c_\infty)\, dx \geq 0$$

achieves its minimum for $n = n_\infty$

$$\mathsf{Q}_1[f] = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2}\, F[n_\infty(1 + \varepsilon\, f)] \geq 0$$

if $\int_{\mathbb{R}^2} f\, n_\infty\, dx = 0$. Notice that f_0 generates the kernel of Q_1

Lemma (J. Campos, JD)

Poincaré type inequality. For any $f \in H^1(\mathbb{R}^2, n_\infty\, dx)$ *such that* $\int_{\mathbb{R}^2} f\, n_\infty\, dx = 0$*, we have*

$$\int_{\mathbb{R}^2} |\nabla(-\Delta)^{-1}(f\, n_\infty)|^2\, n_\infty\, dx = \int_{\mathbb{R}^2} |\nabla(g\, c_\infty)|^2\, n_\infty\, dx \leq \int_{\mathbb{R}^2} |f|^2\, n_\infty\, dx$$

… and eigenvalues

With g such that $-\Delta(g\,c_\infty) = f\,n_\infty$, Q_1 determines a scalar product

$$\langle f_1, f_2 \rangle := \int_{\mathbb{R}^2} f_1\, f_2\, n_\infty\, dx - \int_{\mathbb{R}^2} f_1\, n_\infty\, (g_2\, c_\infty)\, dx$$

on the orthogonal space to f_0 in $L^2(n_\infty\, dx)$

$$\mathsf{Q}_2[f] := \int_{\mathbb{R}^2} |\nabla(f - g\,c_\infty)|^2\, n_\infty\, dx \quad \text{with} \quad g = -\frac{1}{c_\infty}\,\frac{1}{2\pi}\,\log|\cdot| * (f\,n_\infty)$$

is a positive quadratic form, whose polar operator is the self-adjoint operator $\mathcal{L}$

$$\langle f, \mathcal{L}\, f \rangle = \mathsf{Q}_2[f] \quad \forall\, f \in \mathcal{D}(\mathsf{L}_2)$$

Lemma (J. Campos, JD)

$\mathcal{L}$ has pure discrete spectrum and its lowest eigenvalue is 1

Linearized Keller-Segel theory

$$\mathcal{L}\, f = \frac{1}{n_\infty} \nabla \cdot \big(n_\infty \nabla (f - c_\infty\, g) \big)$$

Corollary (J. Campos, JD)

$$\langle f, f \rangle \le \langle \mathcal{L}\, f, f \rangle$$

The linearized problem takes the form

$$\frac{\partial f}{\partial t} = \mathcal{L}\, f$$

where $\mathcal{L}$ is a self-adjoint operator on the orthogonal of f_0 equipped with $\langle \cdot, \cdot \rangle$. A solution of

$$\frac{d}{dt} \langle f, f \rangle = -\, 2 \, \langle \mathcal{L}\, f, f \rangle$$

has therefore exponential decay

More on functional inequalities

A subcritical logarithmic HLS inequality

Recall that

Lemma (A. Blanchet, JD, B. Perthame)

Sub-critical HLS inequality (A. Blanchet, JD, B. Perthame)

$$F[n] := \int_{\mathbb{R}^2} n \log\left(\frac{n}{n_\infty}\right) dx - \frac{1}{2}\int_{\mathbb{R}^2} (n - n_\infty)\,(c - c_\infty)\, dx \geq 0$$

achieves its minimum for $n = n_\infty$

Lemma (J. Campos, JD)

Poincaré type inequality For any $f \in H^1(\mathbb{R}^2, n_\infty\, dx)$ *such that* $\int_{\mathbb{R}^2} f\, n_\infty\, dx = 0$*, we have*

$$\int_{\mathbb{R}^2} |\nabla(-\Delta)^{-1}(f\, n_\infty)|^2\, n_\infty\, dx = \int_{\mathbb{R}^2} |\nabla(g\, c_\infty)|^2\, n_\infty\, dx \leq \int_{\mathbb{R}^2} |f|^2\, n_\infty\, dx$$

... Legendre duality

An Onofri type inequality

Theorem (J. Campos, JD)

For any $M \in (0, 8\pi)$, *if* $n_\infty = M \, \frac{e^{c_\infty - \frac{1}{2}\,|x|^2}}{\int_{\mathbb{R}^2} e^{c_\infty - \frac{1}{2}\,|x|^2}\,dx}$ *with* $c_\infty = (-\Delta)^{-1}\, n_\infty$, $d\mu_M = \frac{1}{M}\, n_\infty\, dx$, *we have the inequality*

$$\log\left(\int_{\mathbb{R}^2} e^{\varphi}\, d\mu_M\right) - \int_{\mathbb{R}^2} \varphi\, d\mu_M \le \frac{1}{2\,M} \int_{\mathbb{R}^2} |\nabla\varphi|^2\, dx \quad \forall\, \varphi \in \mathcal{D}_0^{1,2}(\mathbb{R}^2)$$

Corollary (J. Campos, JD)

The following Poincaré inequality holds

$$\int_{\mathbb{R}^2} \left|\psi - \overline{\psi}\right|^2 n_\infty\, dx \le \int_{\mathbb{R}^2} |\nabla\psi|^2\, dx \quad \text{where} \quad \overline{\psi} = \int_{\mathbb{R}^2} \psi\, d\mu_M$$

An improved interpolation inequality (coercivity estimate)

Lemma (J. Campos, JD)

For any $f \in L^2(\mathbb{R}^2, n_\infty\,dx)$ such that $\int_{\mathbb{R}^2} f\, f_0\, n_\infty\,dx = 0$ holds, we have

$$-\frac{1}{2\pi}\iint_{\mathbb{R}^2\times\mathbb{R}^2} f(x)\,n_\infty(x)\,\log|x-y|\,f(y)\,n_\infty(y)\,dx\,dy \le (1-\varepsilon)\int_{\mathbb{R}^2} |f|^2\,n_\infty\,dx$$

for some $\varepsilon > 0$, where $g\,c_\infty = G_2 * (f\,n_\infty)$ and, if $\int_{\mathbb{R}^2} f\,n_\infty\,dx = 0$ holds,

$$\int_{\mathbb{R}^2} |\nabla(g\,c_\infty)|^2\,dx \le (1-\varepsilon)\int_{\mathbb{R}^2} |f|^2\,n_\infty\,dx$$

Equivalence of the norms

As a consequence

$$\langle f, f \rangle := \int_{\mathbb{R}^2} |f|^2 \, n_\infty \, dx - \int_{\mathbb{R}^2} f \, n_\infty \, (g \, c_\infty) \, dx$$

is equivalent to

$$\int_{\mathbb{R}^2} |f|^2 \, n_\infty \, dx$$

under the condition that $\int_{\mathbb{R}^2} f \, f_0 \, n_\infty \, dx = 0$

A similar result is true in the critical case:
(J. Bedrossian, N. Masmoudi), (P. Raphaël, R. Schweyer)

A spectral gap estimate

Theorem (J. Campos, JD)

For any function $f \in \mathcal{D}(\mathsf{L}_2)$, *we have*

$$\langle f, f \rangle = \mathsf{Q}_1[f] \le \mathsf{Q}_2[f] = \langle f, \mathcal{L}\, f \rangle \,.$$

The nonlinear Keller-Segel model, a functional analysis approach

Exponential convergence for any mass $M \leq 8\pi$

If $n_{0,*}(\sigma)$ stands for the symmetrized function associated to n_0, assume that for any $s \geq 0$

$$(H) \quad \exists\, \varepsilon \in (0, 8\,\pi - M) \quad \text{such that} \quad \int_0^s n_{0,*}(\sigma)\, d\sigma \leq \int_{B\left(0,\sqrt{s/\pi}\right)} n_{\infty, M+\varepsilon}(x)\, dx$$

Theorem (J. Campos, JD)

Under the above assumption, if $n_0 \in L^2_+(n_\infty^{-1}\, dx)$ *and* $M := \int_{\mathbb{R}^2} n_0\, dx < 8\,\pi$, *then any solution with initial datum* n_0 *is such that*

$$\int_{\mathbb{R}^2} |n(t,x) - n_\infty(x)|^2 \frac{dx}{n_\infty(x)} \leq C\, e^{-\,2\,t} \quad \forall\, t \geq 0$$

for some positive constant C, *where* n_∞ *is the unique stationary solution with mass* M

Sketch of the proof

- (J. Campos, JD) Uniform convergence of $n(t,\cdot)$ to n_∞ can be established for any $M \in (0, 8\pi)$ by an adaptation of the symmetrization techniques of (J.I. Díaz, T. Nagai, J.M. Rakotoson)
- Uniform estimates (with no rates) easily result
- Estimates based on Duhammel's formula inspired by (M. Escobedo, E. Zuazua) allow to prove uniform convergence
- Spectral estimates can be incorporated to the relative entropy approach
- Exponential convergence of the relative entropy follows

Step 1: symmetrization (1/2)

To any measurable function $u : \mathbb{R}^2 \mapsto [0,+\infty)$,
we associate the distribution function defined by $\mu(t,\tau) := |\{u > \tau\}|$
and its decreasing rearrangement given by

$$u_* : [0,+\infty) \;\to\; [0,+\infty] \,, \quad s \;\mapsto\; u_*(s) = \inf\{\tau \geq 0 \;:\; \mu(t,\tau) \leq s\} \,.$$

1. For every measurable function $F : \mathbb{R}^+ \mapsto \mathbb{R}^+$, we have

$$\int_{\mathbb{R}^2} F(u)\, dx = \int_{\mathbb{R}^+} F(u_*)\, ds$$

2. If $u \in W^{1,q}(0,T;L^p(\mathbb{R}^N))$ is a nonnegative function, with $1 \leq p < \infty$ and $1 \leq q \leq \infty$, then $u_* \in W^{1,q}(0,T;L^p(0,\infty))$ and the formula

$$\int_0^{\mu(t,\tau)} \frac{\partial u_*}{\partial t}(t,\sigma)\, d\sigma = \int_{\{u(t,\cdot)>\tau\}} \frac{\partial u}{\partial t}(t,x)\, dx$$

holds for almost every $t \in (0,T)$ (J.I. Díaz, T. Nagai, J.M. Rakotoson)

Step 1: symmetrization (2/2)

Lemma

If n is a solution, then the function

$$k(t,s) := \int_0^s n_*(t,\sigma)\,d\sigma$$

satisfies $k \in L^\infty\left([0,+\infty)\times(0,+\infty)\right) \cap H^1\left([0,+\infty); W^{1,p}_{\mathrm{loc}}(0,+\infty)\right) \cap L^2\left([0,+\infty); W^{2,p}_{\mathrm{loc}}(0,+\infty)\right)$ *and*

$$\begin{cases} \frac{\partial k}{\partial t} - 4\,\pi\,s\,\frac{\partial^2 k}{\partial s^2} - (k + 2\,s)\,\frac{\partial k}{\partial s} \le 0 & a.e.\ in\ (0,+\infty)\times(0,+\infty) \\ k(t,0) = 0\,, \quad k(t,+\infty) = \int_{\mathbb{R}^2} n_0\,dx & for\ t \in (0,+\infty) \\ k(0,s) = \int_0^s (n_0)_*\,d\sigma & for\ s \ge 0 \end{cases}$$

Step 2: Uniform estimates

Proposition (J.I. Díaz, T. Nagai, J.M. Rakotoson)

Let f, g be two continuous functions on $Q = \mathbb{R}^+ \times (0,+\infty)$ such that ...

$$\begin{cases} \frac{\partial f}{\partial t} - 4\,\pi\,s\,\frac{\partial^2 f}{\partial s^2} - (f+2\,s)\,\frac{\partial f}{\partial s} \le \frac{\partial g}{\partial t} - 4\,\pi\,s\,\frac{\partial^2 g}{\partial s^2} - (g+2\,s)\,\frac{\partial g}{\partial s} \;\; a.e.\;\; in\; Q \\ f(t,0) = 0 = g(t,0) \quad and \quad f(t,+\infty) \le g(t,+\infty) \;\; for\; any\; t \in (0,+\infty) \\ f(0,s) \le g(0,s) \;\; for\; s \ge 0\,,\; and\; g(t,s) \ge 0 \;\; in\; Q \end{cases}$$

then $f \le g$ on Q

Corollary

Assume that $n_0 \in L^2_+(n_\infty^{-1}\,dx)$ satisfies (H) and $M := \int_{\mathbb{R}^2} n_0\,dx < 8\,\pi$. Then there exist positive constants $C_1 = C_1(M,p)$ and $C_2 = C_2(M,p)$ such that

$$\|n\|_{L^p(\mathbb{R}^2)} \le C_1 \quad and \quad \|\nabla c\|_{L^\infty(\mathbb{R}^2)} \le C_2$$

Step 3: Estimates based on Duhammel's formula

Consider the kernel associated to the Fokker-Planck equation

$$K(t,x,y) := \frac{1}{2\pi\,(1-e^{-2\,t})}\,e^{-\frac{1}{2}\,\frac{|x-e^{-t}y|^2}{1-e^{-2\,t}}} \quad x\in\mathbb{R}^2\,, \quad y\in\mathbb{R}^2\,, \quad t>0$$

If n is a solution, then

$$n(t,x) = \int_{\mathbb{R}^2} K(t,x,y)\,n_0(y)\;dy + \int_0^t \int_{\mathbb{R}^2} \nabla_x K(t-s,x,y)\cdot n(s,y)\,\nabla c(s,y)\;dy\;ds$$

Corollary

Assume that n is a solution. Then

$$\lim_{t\to\infty} \|n(t,\cdot)-n_\infty\|_p = 0 \quad \textit{and} \quad \lim_{t\to\infty} \|\nabla c(t,\cdot)-\nabla c_\infty\|_q = 0$$

for any $p\in[1,\infty]$ and any $q\in[2,\infty]$

Step 4: Spectral estimates can be incorporated

With $\mathsf{Q}_1[f] = \langle f, f \rangle$ and $\mathsf{Q}_2[f] = \langle f, \mathcal{L}\, f \rangle$

1. For any function f in the orthogonal of the kernel of $\mathcal{L}$, we have

$$\mathsf{Q}_1[f] \leq \mathsf{Q}_2[f]$$

2. For any radial function $f \in \mathcal{D}(\mathsf{L}_2)$, we have

$$2\,\mathsf{Q}_1[f] \leq \mathsf{Q}_2[f]$$

Cf. (V. Calvez, J.A. Carrillo) in the radial case

Step 5: Exponential convergence of the relative entropy

$$\frac{\partial f}{\partial t} = \mathcal{L}\, f - \frac{1}{n_\infty}\, \nabla\left[n_\infty\, f\, \nabla(g\, c_\infty)\right]$$

$$\frac{d}{dt}\,\mathsf{Q}_1[f(t,\cdot)] = -\,2\,\mathsf{Q}_2[f(t,\cdot)] + \int_{\mathbb{R}^2} \nabla(f - g\, c_\infty)\, f\, n_\infty \cdot \nabla(g\, c_\infty)\; dx$$

$$\frac{d}{dt}\,\mathsf{Q}_1[f(t,\cdot)] \le -\,2\,\mathsf{Q}_2[f(t,\cdot)] + \delta(t,\varepsilon)\,\sqrt{\mathsf{Q}_1[f(t,\cdot)]\,\mathsf{Q}_2[f(t,\cdot)]}$$

$$\mathsf{Q}_1[f(t,\cdot)] \le \mathcal{Q} \quad \forall\, t \ge 0$$

$$\frac{d}{dt}\,\mathsf{Q}_1[f(t,\cdot)] \le -\,\mathsf{Q}_1[f(t,\cdot)]\left[2 - \,\delta(t,\varepsilon)\left(\left(\mathsf{Q}_1[f(t,\cdot)]\right)^{\frac{1-\varepsilon}{2-\varepsilon}} + \mathsf{Q}_1[f(t,\cdot)]\right)^{\frac{1}{2+\varepsilon}}\right)\right]$$

As a consequence, we finally get that

$$\limsup_{t\to\infty} \color{red}{e^{2\,t}}\, \mathsf{Q}_1[f(t,\cdot)] < \infty$$

Some key ideas

1. Lyapunov / Entropy functionals and functional inequalities
2. Linearization and best constants
3. Functional framework for linearized operators can be deduced from the entropy functional

- (G. Fernández, S. Mischler, 2013)

- weak notion of solution (based on free energy estimates)
- uniqueness, smoothing
- linearized and nonlinear stability in rescaled variables and exponential convergence under weaker assumptions - sharp rates in $L^{4/3}(\mathbb{R}^2)$

Extensions, consequences

- parabolic-parabolic models
  (JD, G. Jankowiak, P. Markowich)
  (G. Jankowiak)
- improved functional inequalities
  (JD, G. Jankowiak)

Parabolic-parabolic models

Parabolic-parabolic models for crowd motion

(JD, G. Jankowiak, P. Markowich) A model for crowd motion

$$\partial_t \rho = \Delta \rho - \nabla \cdot \big(\rho \, (1-\rho) \, \nabla D \big)$$

$$\partial_t D = \kappa \, \Delta D - \delta \, D + g(\rho)$$

on a bounded domain Ω with no-flux boundary conditions

$$\big(\nabla \rho - \rho \, (1-\rho) \, \nabla D \big) \cdot \nu = 0 \quad \text{on} \quad \partial \Omega$$

Model (I): $g(\rho) = \rho \, (1-\rho)$ or Model (II): $g(\rho) = \rho$
Any stationary solution solves

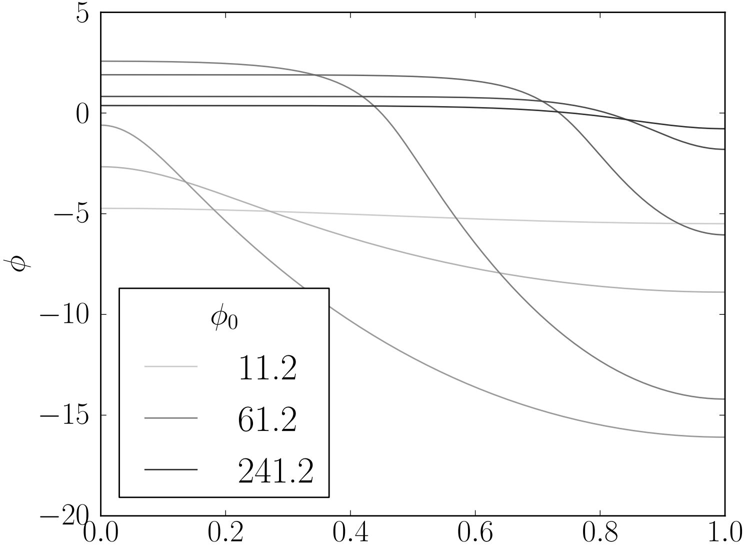
$$\nabla \rho - \rho \, (1-\rho) \, \nabla D = 0 \quad \text{on} \quad \Omega \quad \Longleftrightarrow \quad \rho = \frac{1}{1+e^{-\varphi}}$$

where $\varphi = D - \varphi_0$ and $\int_\Omega \frac{1}{1+e^{\varphi_0 - D}} \, dx = M$

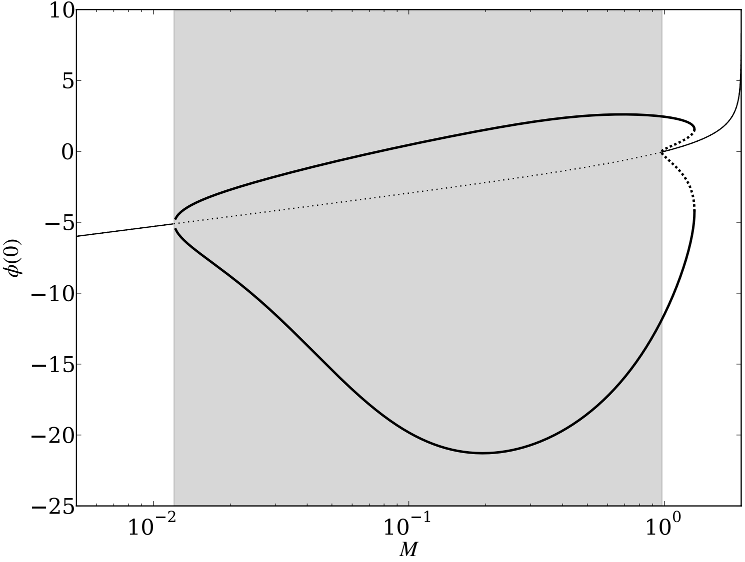
$$- \kappa \, \Delta \varphi + \delta \, (\varphi + \varphi_0) - f(\varphi) = 0 \quad \text{on} \quad \Omega$$

with homogeneous Neumann boundary conditions

Model (I), $d = 1$, $\delta = 10^{-3}$



Model (I), $\kappa = 5 \times 10^{-4}$, $\delta = 10^{-3}$



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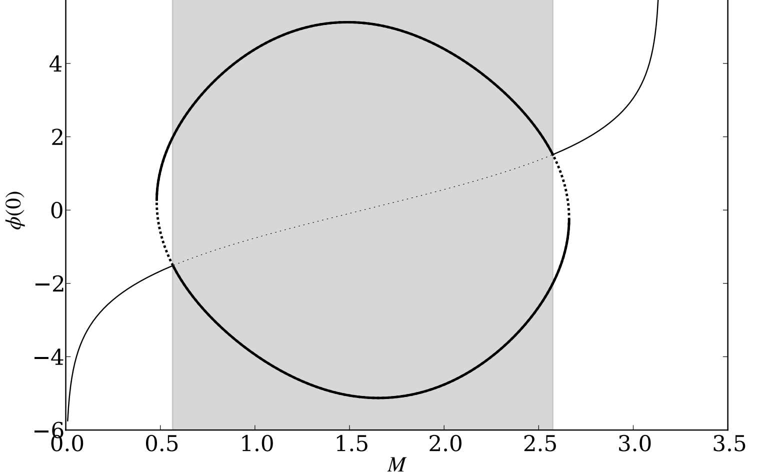

Parabolic-parabolic Keller-Segel model

(G. Jankowiak) Analysis of the stability of self-similar solutions, including for masses larger than 8π

Logarithmic Hardy-Littlewood-Sobolev and Onofri inequalities: duality, flows

Critical case: the logarithmic HLS inequality

The classical logarithmic Hardy-Littlewood-Sobolev (logHLS) in $\mathbb{R}^2$

$$\int_{\mathbb{R}^2} n\,\log\left(\frac{n}{M}\right)\,dx+\frac{2}{M}\int_{\mathbb{R}^2\times\mathbb{R}^2} n(x)\,n(y)\,\log|x-y|\,dx\,dy+M\,\left(1+\log\pi\right)\geq 0$$

Equality is achieved by

$$\mu(x):=\frac{1}{\pi\,(1+|x|^2)^2}\quad\forall\,x\in\mathbb{R}^2$$

Notice that $-\Delta\log\mu=8\,\pi\,\mu$ can be inverted as

$$(-\Delta)^{-1}\mu=\frac{1}{8\,\pi}\,\log\left(\pi\,\mu\right)$$

With $M=8\,\pi$ and $n_\infty=8\,\pi\,\mu$ (logHLS) can be rewritten as

$$\int_{\mathbb{R}^2} n\,\log\left(\frac{n}{n_\infty}\right)\,dx\geq\frac{1}{2}\int_{\mathbb{R}^2}(n-n_\infty)\,(-\Delta)^{-1}(n-n_\infty)\,dx$$

Subritical case: the logarithmic HLS inequality

The minimum of

$$\int_{\mathbb{R}^2} n \log\left(\frac{n}{M}\right) dx + \frac{2}{M}\int_{\mathbb{R}^2\times\mathbb{R}^2} n(x)\,n(y)\,\log|x-y|\,dx\,dy + \frac{1}{2}\int_{\mathbb{R}^2}|x|^2\,n\,dx$$

is achieved by the stationary solution n_∞ of the Keller-Segel model and can again be written as

$$\int_{\mathbb{R}^2} n \log\left(\frac{n}{n_\infty}\right) dx \geq \frac{1}{2}\int_{\mathbb{R}^2}(n-n_\infty)\,(-\Delta)^{-1}(n-n_\infty)\,dx$$

Critical case: Legendre duality

Onofri's inequality

$$F_1[u] := \log\left(\int_{\mathbb{R}^d} e^u \, d\mu\right) \le \frac{1}{16\,\pi} \int_{\mathbb{R}^2} |\nabla u|^2 \, dx + \int_{\mathbb{R}^2} u\,\mu \, dx =: F_2[u]$$

By duality: $F_i^*[v] = \sup\left(\int_{\mathbb{R}^d} v\,u\,d\mu - F_i[u]\right)$ we can relate Onofri's inequality with (logHLS)

For any $v \in \mathcal{L}^1_+(\mathbb{R}^2)$ with $\int_{\mathbb{R}^2} v\,dx = 1$, such that $v \log v$ and $(1 + \log|x|^2)\,v \in \mathcal{L}^1(\mathbb{R}^2)$, we have

$$F_1^*[v] - F_2^*[v] = \int_{\mathbb{R}^2} v \,\log\left(\frac{v}{\mu}\right) dx - 4\,\pi \int_{\mathbb{R}^2} (v-\mu)\,(-\Delta)^{-1}(v-\mu)\,dx \ge 0$$

(E. Carlen, M. Loss 1992 & V. Calvez, L. Corrias 2008)
The same property holds in the subcritical case

The two-dimensional case: (logHLS) and flows

(E. Carlen, J. Carrillo, M. Loss 2010)

$$\mathsf{H}_2[v] := \int_{\mathbb{R}^2} (v-\mu)\,(-\Delta)^{-1}(v-\mu)\;dx - \frac{1}{4\,\pi}\int_{\mathbb{R}^2} v\,\log\left(\frac{v}{\mu}\right)\,dx$$

is related to Gagliardo-Nirenberg inequalities if $v_t = \Delta\sqrt{v}$

- Alternatively, assume that v is a positive solution of

$$\frac{\partial v}{\partial t} = \Delta \log\left(\frac{v}{\mu}\right) \quad t>0\,, \quad x\in\mathbb{R}^2$$

Proposition (JD 2011)

If v is a solution with nonnegative initial datum v_0 in $\mathcal{L}^1(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2} v_0\,dx = 1$, $v_0\,\log v_0 \in \mathcal{L}^1(\mathbb{R}^2)$ and $v_0\,\log\mu \in \mathcal{L}^1(\mathbb{R}^2)$, then

$$\frac{d}{dt}\mathsf{H}_2[v(t,\cdot)] = \frac{1}{16\,\pi}\int_{\mathbb{R}^2} |\nabla u|^2\,dx - \int_{\mathbb{R}^d}\left(e^{\frac{u}{2}}-1\right)u\,d\mu \ge F_2[u] - F_1[u]$$

with $\log(v/\mu) = u/2$

Hierarchies of inequalities, improved inequalities

Theorem (JD, Jankowiak 2013)

If $d \geq 3$*, with* $q = \frac{d+2}{d-2}$

$$\mathsf{S}_d \, \|u^q\|^2_{\frac{2\,d}{d+2}} - \int_{\mathbb{R}^d} u^q \, (-\Delta)^{-1} u^q \, dx$$
$$\leq \mathsf{S}_d \, \|u\|_{2^*}^{\frac{8}{d-2}} \left[\mathsf{S}_d \, \|\nabla u\|_2^2 - \|u\|_{2^*}^2 \right]$$
$$\forall \, u \in \mathrm{H}^1(\mathbb{R}^d)$$

and, when $d = 2$*, for any function* $f \in \mathcal{D}(\mathbb{R}^2)$

$$\left(\int_{\mathbb{R}^d} e^{\,f} \, d\mu \right)^2 - 4\pi \int_{\mathbb{R}^d} e^f \, \mu \, (-\Delta)^{-1} \, e^f \, \mu \, dx$$
$$\leq \left(\int_{\mathbb{R}^d} e^{\,f} \, d\mu \right)^2 \left[\frac{1}{16\,\pi} \, \|\nabla f\|_2^2 + \int_{\mathbb{R}^d} f \, d\mu - \log \left(\int_{\mathbb{R}^d} e^{\,f} \, d\mu \right) \right]$$

Thank you for your attention !