Fast diffusion, mean field drifts and reverse HLS inequalities

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Outline

From drift-diffusion with non-linear diffusions and mean field drift to reverse HLS inequalities

- introduction to fast diffusion equations, large time and linearization
- a mean-field drift-diffusion model: flocking
- Keller-Segel: large time asymptotics
- drift-diffusion equations and reverse HLS inequalities

$$\frac{\partial u}{\partial t} = \Delta u^m + \nabla \cdot \Big(u \left(\nabla V + \nabla W * u \right) \Big)$$

Pure nonlinear (fast) diffusion equations

$$\frac{\partial u}{\partial t} = \Delta u^m$$

▷ intermediate asymptotics by entropy methods and Gagliardo-Nirenberg interpolation inequalities
Best constants correspond to best rates

▷ optimal rates of decay versus (improved) asymptotic rates, linearization, weighted Poincaré inequalities
Weights arise by "quadratization" of the entropy

▷ Caffarelli-Kohn-Nirenberg inequalities and symmetry breaking
The linearization explains why the Bakry-Emery or "carré du champ" method gives optimal results

A simple mean field model

A simple version of the Cucker-Smale model for bird flocking (simplified version, homogeneous in space)

$$\frac{\partial f}{\partial t} = D\,\Delta_v f + \nabla_v \cdot \left(\nabla_v \varphi(v)\, f - \mathbf{u}_f\, f\right)$$

where $\mathbf{u}_f = \int_{\mathbb{R}^d} v\, f\, dv$ is the average velocity

▷ linear diffusion, simplest mean field term

Even if we add a non-linear mean-field coupling, it is still possible to characterize the stationary solutions as critical points of an entropy and relate their stability properties with a rate of convergence

The parabolic elliptic Keller-Segel model

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u \, \nabla v) & x \in \mathbb{R}^2 \,, \; t > 0 \\ -\Delta v = u & x \in \mathbb{R}^2 \,, \; t > 0 \\ u(\cdot, t = 0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

The drift can be computed as a convolution with the Green function

$$v(x) = -\frac{1}{2\pi} \log |\cdot| * u$$

▷ linear diffusion, attractive Poisson coupling

This systems is critical and there is a critical mass which decides between blow-up and a diffusion dominated regime with intermediate asymptotics. Linearization in an appropriate functional setting allows to characterize the optimal convergence rates in self-similar variables

The reverse Hardy-Littlewood-Sobolev inequalities

It is about the stationary solutions of

$$\frac{\partial u}{\partial t} = \Delta u^m + \nabla \cdot \big(u \, \nabla W * u\big)$$

with $m < 1$ and $W(x) = |x|^\lambda$ with $\lambda > 0$, which are the minimizers of a free energy.

$\triangleright$ The corresponding inequality is a reverse Hardy-Littlewood-Sobolev inequality. The equation combines a nonlinear diffusion with $m < 1$ and a strong force of coupling: $\lambda > 0$. *So far a concentration phenomenon cannot be discarded in certain regimes of the parameters. Not much is known on the evolution problem*

Some keywords and background references

Keywords...

- Nernst-Planck, Debye-Hückel and Smoluchowski-Poisson systems
- Keller-Segel model (and its numerous variants)
- Viczek and Cucker-Smale models

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▷ F. Hoffmann. Keller-Segel-Type Models and Kinetic Equations for Interacting Particles: Long-Time Asymptotic Analysis. PhD Thesis, University of Cambridge, 2017