ENTROPY METHODS AND NONLINEAR DIFFUSIONS

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Abstract. Entropy methods have been largely studied during recent years. In these lectures, the emphasis will be put on how to relate optimal rates for large time behavior of solutions to diffusion equations in the whole euclidean space with optimal constants in functional inequalities. The following topics will be covered:

(1) Linear Fokker-Planck equation, generalized entropies and gradient flow structure for various notions of distances
(2) Porous media and fast diffusion equations: global decay rates towards Barenblatt profiles can be related to Gagliardo-Nirenberg interpolation inequalities in a certain range of parameters. Outside of this range, a linearized version of the inequalities still holds, the so-called Hardy-Poincaré inequalities, which govern the asymptotic rates of decay.
(3) Best matching Barenblatt profiles provide refined asymptotic expansions of the solutions. Such an approach has rich consequences for underlying functional inequalities and provide refined versions of the Gagliardo-Nirenberg inequalities.
(4) Relative entropy methods also apply to systems like the parabolic-elliptic Keller-Segel model. Various inequalities related to the optimal logarithmic Hardy-Littlewood-Sobolev inequality will be considered in view of a detailed description of the asymptotic behavior of the solutions.

The page of this course is available at

`http://www.ceremade.dauphine.fr/∼dolbeaul/Teaching/files/SNS-2012/index.pdf`
(login: SNS , password: CRM)

Some papers can be found at

`http://www.ceremade.dauphine.fr/∼dolbeaul/Preprints/list/index.php`
(login: Jean , password: Dolbeault)

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Course 1. Fokker-Planck equation, Bakry-Emery method and notions of gradient flow

(1) Heat equation: time-dependent rescalings, L^1 and L^2 estimates based on Gaussian Poincaré and logarithmic Sobolev inequalities
(2) The Bakry-Emery method for the Ornstein-Uhlenbeck flow
(3) φ-entropies: Convex Sobolev inequality, Csiszár-Kullback inequality, Holley-Stroock perturbation lemma, tensorization and sub-additivity properties
(4) Notions of gradient flow associated to generalized Poincaré inequalities

Some reading material. For a brief historical overview of *Entropy methods in partial differential equations*, see

`http://www.ceremade.dauphine.fr/~dolbeaul/Teaching/files/SNS-2012/Historical.pdf`

Some notes on entropy methods for Fokker-Planck equations can be found at

`http://www.ceremade.dauphine.fr/~dolbeaul/Teaching/files/SNS-2012/Linear.pdf`

A proof of Beckner's generalized Poincaré inequalities by the entropy – entropy production method of Bakry & Emery - case of the Ornstein-Uhlenbeck equation is available at

`http://www.ceremade.dauphine.fr/~dolbeaul/Teaching/files/SNS-2012/DNS.pdf`

A proof of the Csiszár-Kullback inequality is available at

`http://www.ceremade.dauphine.fr/~dolbeaul/Teaching/files/SNS-2012/Csiszar-Kullback.pdf`

For a proof of the Holley-Stroock lemma, see

`http://www.ceremade.dauphine.fr/~dolbeaul/Teaching/files/SNS-2012/Perturbations.pdf`

Tensorization and sub-additivity properties in the language of PDEs and convex analysis is available at

`http://www.ceremade.dauphine.fr/~dolbeaul/Teaching/files/SNS-2012/Tensorization.pdf`

For notions of gradient flow associated to generalized Poincaré inequalities, refer to Section 2 of

`http://www.ceremade.dauphine.fr/~dolbeaul/Preprints/Fichiers/DNS-14.pdf`

Some references.

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[2] A. Arnold, J. A. Carrillo, L. Desvillettes, J. Dolbeault, A. Jüngel, C. Lederman, P. A. Markowich, G. Toscani, and C. Villani, *Entropies and equilibria of many-particle systems: an essay on recent research*, Monatsh. Math., 142 (2004), pp. 35–43.
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[6] D. Chafaï, *Entropies, convexity, and functional inequalities: on Φ-entropies and Φ-Sobolev inequalities*, J. Math. Kyoto Univ., 44 (2004), pp. 325–363.
[7] L. Gross, *Logarithmic Sobolev inequalities*, Amer. J. Math., 97 (1975), pp. 1061–1083.
[8] Jean Dolbeault, Bruno Nazaret, and Giuseppe Savaré, *From Poincaré to logarithmic Sobolev inequalities: a gradient flow approach*, tech. rep., Preprint Ceremade no. 1102, 2011.
[9] F. B. Weissler, *Logarithmic Sobolev inequalities for the heat-diffusion semigroup*, Trans. Amer. Math. Soc., 237 (1978), pp. 255–269.

Course 2. The fast diffusion equation (1)

(1) Entropy – entropy production and Gagliardo-Nirenberg inequalities
(2) Extension of the range for entropy – entropy production inequalities, linearization and Hardy-Poincaré inequalities

[1] M. Bonforte, J. Dolbeault, G. Grillo, and J. L. Vázquez, *Sharp rates of decay of solutions to the nonlinear fast diffusion equation via functional inequalities*, Proceedings of the National Academy of Sciences, 107 (2010), pp. 16459–16464.
[2] M. Del Pino and J. Dolbeault, *Best constants for Gagliardo-Nirenberg inequalities and applications to nonlinear diffusions*, J. Math. Pures Appl. (9), 81 (2002), pp. 847–875.

Course 3. The fast diffusion equation (2)

(1) Improving asymptotic rates of convergence and best matching Barenblatt solutions
(2) From asymptotic improvements to global improvements: the Bakry-Emery approach and improved interpolation inequalities

[1] J. Dolbeault and G. Toscani, *Fast diffusion equations: Matching large time asymptotics by relative entropy methods.*, Kinetic and Related Models, 4 (2011), pp. 701–716.
[2] Jean Dolbeault and Giuseppe Toscani, *Improved interpolation inequalities, relative entropy and fast diffusion equations*, tech. rep., Preprint Ceremade no. 1104, 2011.

Course 4

(1) The Keller-Segel model in mathematical biology
(2) Diffusion limits and the 6th Hilbert problem
(3) Hypocoercivity

[1] A. Blanchet, J. Dolbeault, and B. Perthame, *Two-dimensional Keller-Segel model: optimal critical mass and qualitative properties of the solutions*, Electron. J. Differential Equations, 44, 32 pages (2006).
[2] J. Campos and J. Dolbeault, *A functional framework for the Keller-Segel system: logarithmic Hardy-Littlewood-Sobolev and related spectral gap inequalities*. Preprint, 2012.
[3] J. Dolbeault, C. Mouhot, and C. Schmeiser, *Hypocoercivity for kinetic equations with linear relaxation terms*, Comptes Rendus Mathématique, 347 (2009), pp. 511 – 516.
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