SYMMETRY AND NONLINEAR DIFFUSION FLOWS

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ABSTRACT. These notes are the lecture notes of the course

Symmetry and nonlinear diffusion flows
https://www.ceremade.dauphine.fr/~dolbeaul/Teaching/files/UChile-2017/LectureNotes.pdf
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The material has been collected from various papers and publications. It is an updated version of a course taught in June 2017 at the Universidad Autónoma de Madrid. The goal is to provide additional details on the proofs and references, without covering the most general cases. Topics covered are:

- a summary of known results on φ-entropies and related functional inequalities based on simple linear diffusion equations, including results based on the Bakry-Emery method,
- interpolation inequalities on compact manifolds, with the sphere as main example, with an emphasis on the use of the fast diffusion flow in order to cover the whole range of parameters up to the critical exponent,
- Rényi entropy powers compared to relative entropy methods on the Euclidean space as a new tool for capturing optimal constants in the large time regime, with applications to Gagliardo-Nirenberg inequalities,
- Considerations on branches of solutions and bifurcations in semilinear elliptic equations: known rigidity results can be reinterpreted as stationary points of flows based on nonlinear diffusions,
- Symmetry and symmetry breaking results in Caffarelli-Kohn-Nirenberg inequalities: how to introduce nonlinear flows in presence of weights for proving symmetry reults,
- Further considerations on large time asymptotics, linearization and optimal constants.

CONTENTS

Warning: The material of these notes has been collected from various papers and publications, and some notations may still be inconsistants. Simplified cases have been selected for clarity and the reader has to know that in the reference list, more general results can be found. Also be aware that not all details are given, so that some of the proofs are still given only at formal level. It has to be expected that there are still some typos in many of the computations !
The text provided here may evolve, so please consider the latest version available on the website.

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1. A REVIEW OF RESULTS ON φ-ENTROPIES

This section is intended to be part of a paper with X. Li, on *φ-entropies: convexity, coercivity and hypocoercivity for Fokker-Planck and kinetic Fokker-Planck equations*. Most of the corresponding material can also be found in [40, Section 2].

Let us consider a convex function φ on $\mathbb{R}^+$. By definition, the *φ-entropy* of a nonnegative function $w \in \mathrm{L}^1(\mathbb{R}^d, d\gamma)$ is the functional

$$\mathscr{E}[w] := \int_{\mathbb{R}^d} \varphi(w)\, d\gamma\,,$$

where φ is a nonnegative convex continuous function on $\mathbb{R}^+$ such that $\varphi(1) = 0$ and $1/\varphi''$ is concave on $(0, +\infty)$, *i.e.*,

$$\varphi'' \geq 0\,, \quad \varphi \geq \varphi(1) = 0 \quad \text{and} \quad (1/\varphi'')'' \leq 0\,. \tag{1}$$

Notice that the last condition means $2\,(\varphi''')^3 \leq \varphi''\,\varphi^{(iv)}$ a.e. A classical example of a such a function φ is given by

$$\varphi_p(w) := \tfrac{1}{p-1}\left(w^p - 1 - p\,(w-1)\right) \quad p \in (1,2]\,,$$

where, in the case $p = 2$, $\varphi_2(w) = (w-1)^2$ and the limit case as $p \to 1_+$ is given by the standard Gibbs entropy

$$\varphi_1(w) := w \log w - (w-1)\,.$$

Many results corresponding to the case $p = 2$ can be obtained, *e.g.*, by spectral methods. The case $p = 1$ is important in probability theory and statistical physics. Our goal is to emphasize that they share properties which can be put in a common framework. Throughout this paper we shall assume that $d\gamma$ is a nonnegative bounded measure, which is absolutely continuous with respect to Lebesgue's measure and write

$$d\gamma = e^{-\psi}\, dx$$

where ψ is a *potential* such that $e^{-\psi}$ is in $\mathrm{L}^1(\mathbb{R}^d, dx)$. Up to the addition of a constant to ψ, we can assume without loss of generality that $d\gamma$ is a probability measure. A review of the main results on φ-entropies is given in Section 1.

Without entering in the technical details, let us illustrate the use of the φ-entropy in the case of diffusion equations. A typical application of the φ-entropy is the control of the rate of relaxation of the solution to the *Ornstein-Uhlenbeck equation*

$$\frac{\partial w}{\partial t} = \mathsf{L}\, w := \Delta w - \nabla\psi \cdot \nabla w\,, \tag{2}$$

which is also known as the backward Kolmogorov equation. If we solve the equation with a nonnegative initial datum w_0 such that $\int_{\mathbb{R}^d} w_0\, d\gamma = 1$, then the solution satisfies $\int_{\mathbb{R}^d} w(t,\cdot)\, d\gamma = 1$ for any $t > 0$ and $\lim_{t\to+\infty} w(t,\cdot) = 1$. The Ornstein-Uhlenbeck operator L defined on $\mathrm{L}^2(\mathbb{R}^d, d\gamma)$ is self-adjoint and such that

$$-\int_{\mathbb{R}^d} (\mathsf{L}\, w_1)\, w_2\, d\gamma = \int_{\mathbb{R}^d} \nabla w_1 \cdot \nabla w_2\, d\gamma \quad \forall\, w_1,\, w_2 \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)\,.$$

As a consequence, it is straightforward to observe that for any solution w with initial datum such that $\mathscr{E}[w_0]$ is finite, then

$$\frac{d}{dt}\mathscr{E}[w] = -\int_{\mathbb{R}^d} \varphi''(w)\, |\nabla_x w|^2\, d\gamma =: -\mathscr{I}[w]\,,$$

where $\mathscr{I}[w]$ denotes the *φ-Fisher information* functional. If for some $\lambda > 0$ we can establish the *entropy – entropy production* inequality

$$\mathscr{I}[w] \geq \lambda\, \mathscr{E}[w] \quad \forall\, w \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)\,, \tag{3}$$

then we deduce that

$$\mathscr{E}[w(t,\cdot)] \leq \mathscr{E}[w_0]\, e^{-\lambda\, t} \quad \forall\, t \geq 0\,,$$

which controls the convergence of w to 1 as $t \to +\infty$, for instance in $\mathrm{L}^p(\mathbb{R}^d, d\gamma)$ by a generalized *Csiszár-Kullback inequality* if $\varphi = \varphi_p$, $1 \leq p \leq 2$. The entropy – entropy production inequality is the Poincaré inequality associated with $d\gamma$ if $\varphi = \varphi_2$, and the logarithmic Sobolev inequality if $\varphi = \varphi_1$.

We recall that the study of (2) is equivalent to the study of the Fokker-Planck equation

$$\frac{\partial u}{\partial t} = \Delta u + \nabla_x \cdot (u\, \nabla_x \psi)\,. \tag{4}$$

A nonnegative solution with initial datum $u_0 \in \mathrm{L}^1(\mathbb{R}^d, dx)$ and $\int_{\mathbb{R}^d} u_0\,dx = M > 0$ has constant mass $M = \int_{\mathbb{R}^d} u(t,\cdot)\,dx$ for any $t > 0$, and converges towards the unique stationary solution

$$u_\star = M\,e^{-\psi}$$

if we assume that the potential ψ is normalized as above so that $\int_{\mathbb{R}^d} e^{-\psi}\,dx = 1$. Without loss of generality, we shall also assume that $M = 1$. Then one observes that $w = u/u_\star$ solves (2), which allows to control the rate of convergence of u to $u_\star$.

1.1. **Generalized Csiszár-Kullback-Pinsker inequality.** By assumption (1), we know that $\mathscr{E}$ is nonnegative and achieves its minimum at $w \equiv 1$. It results from the strict convexity of φ that $\mathscr{E}[w]$ controls a norm of $(w-1)$ under a generic assumption compatible with the expression of φ_p. The classical result of [59, 23, 55] has been extended in[54, 64, 15, 24]. Here is a statement, with a short proof taken from Section 1.4 of [10] for completeness.

Proposition 1. *Let $p \in [1,2]$, $w \in \mathrm{L}^1 \cap \mathrm{L}^p(\mathbb{R}^d, d\gamma)$ be a nonnegative function, and assume that $\varphi \in C^2(0,+\infty)$ is a nonnegative strictly convex function such that $\varphi(1) = \varphi'(1) = 0$. If $A := \inf_{s\in(0,\infty)} s^{2-p}\,\varphi''(s) > 0$, then*

$$\mathscr{E}[w] \ge 2^{-\frac{2}{p}}\,A\,\min\left\{1, \|w\|^{p-2}_{\mathrm{L}^p(\mathbb{R}^d,d\gamma)}\right\}\,\|w-1\|^2_{\mathrm{L}^p(\mathbb{R}^d,d\gamma)}\,.$$

When $\varphi = \varphi_p$, we find that $A = p$. This inequality has many variants and extensions: it is not limited to $\mathbb{R}^d$ but also holds on bounded domains or manifolds and the relative φ-entropy $\int_{\mathbb{R}^d}\big(\varphi(w_1) - \varphi(w_2) - \varphi'(w_1)\,(w_2 - w_1)\big)\,d\gamma$ can also be used to measure $\|w_2 - w_1\|^2_{\mathrm{L}^p(\mathbb{R}^d,d\gamma)}$.

Proof. Up to the addition of a small constant, we can assume that $w > 0$ and argue by density. A Taylor expansion at order two shows that

$$\mathscr{E}[w] = \frac{1}{2}\int_{\mathbb{R}^d}\varphi''(\xi)\,|w-1|^2\,d\gamma \ge \frac{A}{2}\int_{\mathbb{R}^d}\xi^{p-2}\,|w-1|^2\,d\gamma$$

where ξ lies between 1 and w. With $\alpha = p\,(2-p)/2$ and $h > 0$, for any measurable set $\mathscr{A} \subset \mathbb{R}^d$, we get

$$\int_{\mathscr{A}}|w-1|^p\,h^{-\alpha}\,h^{\alpha}\,d\gamma \le \left(\int_{\mathscr{A}}|w-1|^2\,h^{p-2}\,d\gamma\right)^{p/2}\left(\int_{\mathscr{A}}h^p\,d\gamma\right)^{(2-p)/2}$$

by Hölder's inequality. We apply this formula to two different sets.
On $\mathscr{A} = \{x \in \mathbb{R}^d : w(x) > 1\}$, use $\xi^{p-2} > w^{p-2}$ and take $h = w$:

$$\int_{\{w>1\}}|w-1|^2\,\xi^{p-2}\,d\gamma \ge \left(\int_{\{w>1\}}|w-1|^p\,d\gamma\right)^{2/p}\,\|w\|^{p-2}_{\mathrm{L}^p(\mathbb{R}^d,d\gamma)}\,.$$

On $\mathscr{A} = \{x \in \mathbb{R}^d : w(x) \le 1\}$, use $\xi^{p-2} \ge 1$ and take $h = 1$:

$$\int_{\{w\le1\}}|w-1|^2\,\xi^{p-2}\,d\gamma \ge \left(\int_{\{w\le1\}}|w-1|^p\,d\gamma\right)^{2/p}\,.$$

Adding these two estimates and using with $r = 2/p \ge 1$ the elementary inequality $(a+b)^r \le 2^{r-1}(a^r + b^r)$ for any a, $b \ge 0$ concludes the proof. □

1.2. **Convexity, tensorization and sub-additivity.** Let us turn our attention to (3). To start with, we observe that the functional $w \mapsto \mathscr{I}[w] = \int_{\mathbb{R}^d}\varphi''(w)\,|\nabla w|^2\,d\gamma$ is convex if and only if $1/\varphi''$ is concave. Now let us consider two probability measures $d\gamma_1$ and $d\gamma_2$ defined respectively on $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$, such that Inequality (3) holds with $\gamma = \gamma_i$, and $i = 1, 2$:

$$\int_{\mathbb{R}^{d_i}}\varphi''(w)\,|\nabla w|^2\,d\gamma_i =: \mathscr{I}_{\gamma_i}[w] \ge \lambda_i\,\mathscr{E}_{\gamma_i}[w] \quad \forall\, w \in \mathrm{H}^1(\mathbb{R}^{d_i}, d\gamma_i)\,, \tag{5}$$

Here we denote by $\mathscr{E}_\gamma$ the φ-entropy for functions which are not normalized, that is,

$$\mathscr{E}_\gamma[w] := \int_{\mathbb{R}^d}\varphi(w)\,d\gamma - \varphi\left(\int_{\mathbb{R}^d}w\,d\gamma\right).$$

Assuming that $d\gamma$ is a probability measure, by Jensen's inequality we know that $w \mapsto \mathscr{E}_\gamma[w]$ is nonnegative because φ is convex. As we shall see below, $w \mapsto \mathscr{E}_\gamma[w]$ is also convex, which is the key ingredient for *tensorization*. The question at stake is to know if Inequality (3) holds on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ for the measure $d\gamma = d\gamma_1 \otimes \gamma_2$. Most of the results of Section 1.2 have been stated in [22] or are considered as classical.

Theorem 2. *Assume that φ satisfies* (1). *If $d\gamma_1$ and $d\gamma_2$ are two probability measures on $\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}$ satisfying* (5) *with positive constants λ_1 and λ_2, then $d\gamma_1\otimes\gamma_2$ is such that the following inequality holds:*

$$\mathscr{I}_{\gamma_1\otimes\gamma_2}[w]=\int_{\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}}\varphi''(w)\,|\nabla w|^2\,d\gamma_1\,d\gamma_2\ge\min\{\lambda_1,\lambda_2\}\,\mathscr{E}_{\gamma_1\otimes\gamma_2}[w]\quad\forall\,w\in\mathrm{H}^1(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2},d\gamma)\,.$$

It is straightforward to notice that the Fisher information is additive

$$\mathscr{I}\gamma_1\otimes\gamma_2[w]=\int_{\mathbb{R}^{d_2}}\mathscr{I}\gamma_1[w]\,d\gamma_2+\int_{\mathbb{R}^{d_1}}\mathscr{I}\gamma_2[w]\,d\gamma_1\,,$$

so that the proof of Theorem 2 can be reduced to the proof of a *sub-additivity* property of the φ-entropies that goes as follows.

Proposition 3. *Assume that φ satisfies* (1) *and consider two probability measures $d\gamma_1$ and $d\gamma_2$ on $\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}$. Then for any $w\in\mathrm{L}^1(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2},d\gamma_1\otimes\gamma_2)$, we have*

$$\mathscr{E}_{\gamma_1\otimes\gamma_2}[w]\le\int_{\mathbb{R}^{d_2}}\mathscr{E}_{\gamma_1}[w]\,d\gamma_2+\int_{\mathbb{R}^{d_1}}\mathscr{E}_{\gamma_2}[w]\,d\gamma_1\quad\forall\,w\in\mathrm{L}^1(d\gamma_1\otimes\gamma_2)\,.$$

This last result relies on convexity properties that we are now going to study. As a preliminary step, we establish an inequality of Jensen type.

Lemma 4. *Let $w\in\mathrm{L}^1(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2},d\gamma_1\otimes\gamma_2)$ be a function of two variables $(x_1,x_2)\in\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}$. If $\mathscr{F}_{\gamma_1}$ is a convex functional on $\mathrm{L}^1(d\gamma_1)$ such that*

$$\frac{d}{dt}\int_{\mathbb{R}^{d_2}}\mathscr{F}_{\gamma_1}\left[t\,w+(1-t)\textstyle\int_{\mathbb{R}^{d_2}}w\,d\gamma_2\right]d\gamma_2{}_{\,|t=0}=0\,,\tag{6}$$

then the following inequality holds:

$$\int_{\mathbb{R}^{d_2}}\mathscr{F}_{\gamma_1}[w]\,d\gamma_2\ge\mathscr{F}_{\gamma_1}\left[\int_{\mathbb{R}^{d_2}}w\,d\gamma_2\right].$$

Proof. Let $w_t=t\,w+(1-t)\int_{\mathbb{R}^{d_2}}w\,d\gamma_2$. By convexity of $\mathscr{F}_{\gamma_1}$,

$$\mathscr{F}_{\gamma_1}[w_t]\le t\,\mathscr{F}_{\gamma_1}[w]+(1-t)\,\mathscr{F}_{\gamma_1}\left[\int_{\mathbb{R}^{d_2}}w\,d\gamma_2\right].$$

Hence it follows that

$$\mathscr{F}_{\gamma_1}[w_t]-\mathscr{F}_{\gamma_1}\left[\int_{\mathbb{R}^{d_2}}w\,d\gamma_2\right]\le t\left(\mathscr{F}_{\gamma_1}[w]-\mathscr{F}_{\gamma_1}\left[\int_{\mathbb{R}^{d_2}}w\,d\gamma_2\right]\right),$$

from which we deduce that

$$0=\frac{d}{dt}\mathscr{F}_{\gamma_1}[w_t]_{|t=0}\le\mathscr{F}_{\gamma_1}[w]-\mathscr{F}_{\gamma_1}\left[\int_{\mathbb{R}^{d_2}}w\,d\gamma_2\right].$$

Conclusion holds after integrating with respect to γ_2. □

The second observation is the proof of the convexity of $w\mapsto\mathscr{E}_\gamma[w]$. The following result is taken from [56].

Lemma 5. *If φ satisfies* (1), *then $\mathscr{E}_\gamma$ is convex.*

Proof. We give a two steps proof of this result, for completeness.
1) Define $x_t=t\,y+(1-t)\,x$, $t\in(0,1)$. Since $1/\varphi''$ is concave,

$$\frac{1}{\varphi''(x_t)}\ge\frac{t}{\varphi''(y)}+\frac{1-t}{\varphi''(x)}\,.\tag{7}$$

The function φ is convex, hence $\varphi''(x)>0$ and $\varphi''(y)>0$ and so

$$\frac{1}{\varphi''(x_t)}\ge\frac{t}{\varphi''(y)}\quad\text{and}\quad\frac{1}{\varphi''(x_t)}\ge\frac{1-t}{\varphi''(x)}\,.$$

This means

$$\varphi''(y)\ge t\,\varphi''(x_t)\quad\text{and}\quad\varphi''(x)\ge(1-t)\,\varphi''(x_t)\,.$$

We can also rewrite (7) as

$$\varphi''(x)\,\varphi''(y)\ge\left(t\,\varphi''(x)+(1-t)\,\varphi''(y)\right)\varphi''(x_t)\,.$$

Consider the function

$$F_t(x,y):=t\,\varphi(y)+(1-t)\,\varphi(x)-\varphi(x_t)$$

and observe that

$$\mathrm{Hess}(F_t) = \begin{pmatrix} (1-t)\,\varphi''(x) - (1-t)^2\,\varphi''(x_t) & -\,t\,(1-t)\,\varphi''(x_t) \\ -\,t\,(1-t)\,\varphi''(x_t) & t\,\varphi''(y) - t^2\,\varphi''(x_t) \end{pmatrix}$$

is nonnegative since both diagonal terms are nonnegative and the determinant is nonnegative. The matrix $\mathrm{Hess}(F_t)$ is therefore nonnegative and F_t is convex.
2) We observe that

$$t\,\mathscr{E}_\gamma[w_1] + (1-t)\,\mathscr{E}_\gamma[w_0] - \mathscr{E}_\gamma[t\,w_1 + (1-t)\,w_0] = \int_{\mathbb{R}^d} F_t(w_1, w_0)\,d\gamma - F_t\left(\int_{\mathbb{R}^d} w_1\,d\gamma, \int_{\mathbb{R}^d} w_0\,d\gamma\right)$$

is nonnegative by Jensen's inequality, which proves the result. □

Proof of Proposition 3. We claim that $\mathscr{F}_{\gamma_1} = \mathscr{E}_{\gamma_1}$ satisfies (6). Indeed, let us consider $w_t = t\,w + (1-t)\,w_0$ with $w_0 := \int_{\mathbb{R}^{d_2}} w\,d\gamma_2$. A simple computation shows that

$$\frac{d}{dt}\mathscr{F}_{\gamma_1}[w_t] = \int_{\mathbb{R}^{d_1}} \varphi'(w_t)\,(w - w_0)\,d\gamma_1 - \varphi'\left(\int_{\mathbb{R}^{d_1}} w_t\,d\gamma_1\right)\int_{\mathbb{R}^{d_1}} (w - w_0)\,d\gamma_1\,,$$

and, as a consequence at $t = 0$,

$$\frac{d}{dt}\mathscr{F}_{\gamma_1}[w_t]_{|t=0} = \int_{\mathbb{R}^{d_1}} \varphi'(w_0)\,(w - w_0)\,d\gamma_1 - \varphi'\left(\int_{\mathbb{R}^{d_1}} w_0\,d\gamma_1\right)\int_{\mathbb{R}^{d_1}} (w - w_0)\,d\gamma_1\,.$$

Since w_0 does not depend on x_2, an integration with respect to γ_2 concludes the proof of (6). From Lemma 4, we get

$$\int_{\mathbb{R}^{d_2}} \mathscr{E}_{\gamma_1}[w]\,d\gamma_2 \geq \mathscr{E}_{\gamma_1}\left[\int_{\mathbb{R}^{d_2}} w\,d\gamma_2\right].$$

By definition of $\mathscr{E}_{\gamma_1}$, this means

$$\int_{\mathbb{R}^{d_2}}\left[\int_{\mathbb{R}^{d_1}} \varphi(w)\,d\gamma_1 - \varphi\left(\int_{\mathbb{R}^{d_1}} w\,d\gamma_1\right)\right]d\gamma_2 \geq \int_{\mathbb{R}^{d_1}} \varphi\left(\int_{\mathbb{R}^{d_2}} w\,d\gamma_2\right)d\gamma_1 - \varphi\left(\iint_{\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}} w\,d\gamma_1\otimes\gamma_2\right),$$

from which we deduce

$$\int_{\mathbb{R}^{d_2}}\left[\int_{\mathbb{R}^{d_1}} \varphi(w)\,d\gamma_1 - \varphi\left(\int_{\mathbb{R}^{d_1}} w\,d\gamma_1\right)\right]d\gamma_2 + \int_{\mathbb{R}^{d_1}}\left[\int_{\mathbb{R}^{d_2}} \varphi(w)\,d\gamma_2 - \varphi\left(\int_{\mathbb{R}^{d_2}} w\,d\gamma_2\right)\right]d\gamma_1$$
$$\geq \iint_{\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}} \varphi(w)\,d\gamma_1\otimes\gamma_2 - \varphi\left(\iint_{\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}} w\,d\gamma_1\otimes\gamma_2\right).$$

This ends the proof of Proposition 3. □

Proof of Theorem 2. The proof is an easy consequence of Proposition 3 and of the observation that

$$\begin{aligned}\min\{\lambda_1,\lambda_2\}\,\mathscr{E}_{\gamma_1\otimes\gamma_2}[w] &\leq \lambda_1\int_{\mathbb{R}^{d_2}} \mathscr{E}_{\gamma_1}[w]\,d\gamma_2 + \lambda_2\int_{\mathbb{R}^{d_1}} \mathscr{E}_{\gamma_2}[w]\,d\gamma_1 \\ &\leq \iint_{\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}} \varphi''(w)\left[\,|\nabla_{x_1} w|^2 + |\nabla_{x_2} w|^2\,\right]d\gamma_1\otimes\gamma_2 \\ &\leq \iint_{\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}} \varphi''(w)\,|\nabla w|^2\,d\gamma_1\otimes\gamma_2 = \mathscr{I}_{\gamma_1\otimes\gamma_2}[w]\,.\end{aligned}$$

□

As a concluding remark, we observe that tensorization is not limited to probability measures on $\mathbb{R}^d$. The main interest of such an approach when dealing with $\mathbb{R}^d$ is that it is enough to establish the inequality when $d = 1$. In the case $d = 1$, sharp criteria can be found in [9] (also see [8]). There are many related issues that can be traced back to the work of Muckenhoupt, *e.g.*, [58] and Hardy (see [50]).

1.3. **Entropy – entropy production inequalities: perturbation results.** Perturbing the measure in the case of a Poincaré inequality is essentially trivial. In the case of the logarithmic Sobolev inequality, this has been done by Holley and Stroock in [51]. More general entropy functionals have been considered in [64], which cover all φ-entropies. Also see [1, 22]. Let us establish a result in this spirit.

Assume that for some probability measure $d\gamma$ and for some $\lambda > 0$, Inequality (3) holds, that is,

$$\lambda\left[\int_{\mathbb{R}^d} \varphi(w)\,d\gamma - \varphi(\overline{w})\right] \leq \int_{\mathbb{R}^d} \varphi''(w)|\nabla w|^2\,d\gamma \quad \forall\, w \in \mathrm{H}^1(d\gamma)\,. \tag{8}$$

Here we denote by $\overline{w}$ the average of w with respect to $d\gamma$: $\overline{w} := \int_{\mathbb{R}^d} w\,d\gamma$. Assume that $d\mu$ is a measure which is absolutely continuous with respect to $d\gamma$ and such that

$$e^{-b}\,d\gamma \le d\mu \le e^{-a}\,d\gamma$$

for some constants a, $b \in \mathbb{R}$. The statement below generalizes the one of Lemma 5.2 of [13].

Lemma 6. *Under the above assumption, if φ is a C^2 function such that $\varphi'' > 0$, then*

$$e^{a-b}\,\lambda \int_{\mathbb{R}^d} \left[\varphi(w) - \varphi(\widetilde{w}) - \varphi'(\widetilde{w})(w - \widetilde{w})\right] d\mu \le \int_{\mathbb{R}^d} \varphi''(w)\,|\nabla w|^2\,d\mu \quad \forall\, w \in \mathrm{H}^1(d\mu)\,,$$

where $\widetilde{w} := \int_{\mathbb{R}^d} w\,d\mu \,/\, \int_{\mathbb{R}^d} d\mu$.

Proof. We start by observing that

$$e^b \int_{\mathbb{R}^d} \varphi''(w)|\nabla w|^2\,d\mu \ge \int_{\mathbb{R}^d} \varphi''(w)|\nabla w|^2\,d\gamma = \mathscr{I}_\gamma[w]$$
$$\ge \lambda\,\mathscr{E}_\gamma[w] = \lambda\left[\int_{\mathbb{R}^d} \varphi(w)\,d\gamma - \varphi(\overline{w})\right] = \lambda \int_{\mathbb{R}^d} \big(\varphi(w) - \varphi(\overline{w}) - \varphi'(\overline{w})\,(w - \overline{w})\big)\,d\gamma\,.$$

By convexity of φ, we know that $\varphi(w) - \varphi(\overline{w}) - \varphi'(\overline{w})\,(w - \overline{w}) \ge 0$, so that

$$\lambda\,\mathscr{E}_\gamma[w] \ge \lambda\,e^a \int_{\mathbb{R}^d} \big(\varphi(w) - \varphi(\overline{w}) - \varphi'(\overline{w})\,(w - \overline{w})\big)\,d\mu = \lambda\,e^a \int_{\mathbb{R}^d} \big(\varphi(w) - \varphi(\overline{w}) - \varphi'(\overline{w})\,(\widetilde{w} - \overline{w})\big)\,d\mu\,.$$

By convexity of φ again, $\varphi(\overline{w}) + \varphi'(\overline{w})\,(\widetilde{w} - \overline{w}) \le \varphi(\widetilde{w})$, which shows that

$$\lambda\,\mathscr{E}_\gamma[w] \ge \lambda\,e^a \int_{\mathbb{R}^d} \big(\varphi(w) - \varphi(\widetilde{w})\big)\,d\mu = e^a\,\lambda \int_{\mathbb{R}^d} \left[\varphi(w) - \varphi(\widetilde{w}) - \varphi'(\widetilde{w})(w - \widetilde{w})\right] d\mu$$

and completes the proof. □

1.4. **Entropy – entropy production inequalities and linear flows.** Let us consider the counterpart of the *Ornstein-Uhlenbeck equation* (2) on a smooth convex bounded domain Ω

$$\frac{\partial w}{\partial t} = \mathsf{L}\,w := \Delta w - \nabla\psi\cdot\nabla w\,, \tag{9}$$

supplemented with homogenous Neumann boundary conditions

$$\nabla w\cdot\nu = 0 \quad \text{on} \quad \partial\Omega\,,$$

where ν denotes a unit outward pointing normal vector orthogonal to $\partial\Omega$. Let us consider the measure $d\gamma = \left(\int_\Omega e^{-\psi}\,dx\right)^{-1} e^{-\psi}\,dx$. If w solves (9) with a nonnegative initial datum w_0 such that $\int_\Omega w_0\,d\gamma = 1$, then mass is conserved so that $\int_\Omega w(t,\cdot)\,d\gamma = 1$ for any $t \ge 0$ and converges to 1 as $t \to +\infty$. The next question is how to measure the rate of convergence using the φ-entropy. For simplicity, let us assume that $\varphi = \varphi_p$ for some $p \in [1,2]$. An answer is given by the formal computation on $\mathbb{R}^d$, adapted to the bounded domain Ω. Because of the boundary condition, it is straightforward to check that

$$\frac{d}{dt}\int_\Omega \frac{w^p - 1}{p-1}\,d\gamma = -\frac{4}{p}\int_\Omega |\nabla w^{p/2}|^2\,d\gamma$$

if $p > 1$ and a similar results holds when $p = 1$. Hence, if for some $\lambda > 0$ we can prove that

$$\int_\Omega \frac{w^p - 1}{p-1}\,d\gamma \le \frac{4}{p\,\lambda}\int_\Omega |\nabla w^{p/2}|^2\,d\gamma \quad \text{for any } w \text{ such that} \quad \int_\Omega w\,d\gamma = 1\,, \tag{10}$$

then we can conclude that $\int_\Omega \frac{w^p-1}{p-1}\,d\gamma$ decays like $e^{-\lambda\,t}$. The main idea of the Bakry-Emery method, or *carré du champ* method, as it is exposed in [5] is that (10) can be established using the flow itself, by computing $\frac{d}{dt}\int_\Omega |\nabla z|^2\,d\gamma$ with $z := w^{p/2}$. Let us sketch the main steps of the proof.

As a preliminary observation, we notice that L is self-adjoint in $\mathrm{L}^2(\Omega, d\gamma)$ in the sense that

$$\int_\Omega w_1\,(\mathsf{L}\,w_2)\,d\gamma = -\int_\Omega \nabla w_1\cdot\nabla w_2\,d\gamma = \int_\Omega (\mathsf{L}\,w_1)\,w_2\,d\gamma$$

and also that

$$[\nabla, \mathsf{L}] = -\operatorname{Hess}\psi\,.$$

Using $w = z^{2/p}$ we deduce from (9) that

$$\frac{\partial z}{\partial t} = \mathsf{L}\,z + \frac{2-p}{p}\,\frac{|\nabla z|^2}{z}\,. \tag{11}$$

We adopt the convention that $a\cdot b=\sum_{i=1}^d a_i\,b_i$ if $a=(a_i)_{1\le i\le d}$ and $b=(b_i)_{1\le i\le d}$ are two vectors with values in $\mathbb{R}^d$. If $m=(m_{i,j})_{1\le i,j\le d}$ and $n=(n_{i,j})_{1\le i,j\le d}$ are two matrices, then $m:n=\sum_{i,j=1}^d m_{i,j}\,n_{i,j}$. Also $a\otimes b$ denotes the matrix $(a_i\,b_j)_{1\le i,j\le d}$. We shall use indifferently $|a|^2=a\cdot a$ and $|m|^2=m:m$ for vectors and matrices. With these notations, let us use (11) to compute

$$\begin{aligned}
\frac12\,\frac{d}{dt}\int_\Omega|\nabla z|^2\,d\gamma &= \int_\Omega\nabla z\cdot\nabla\left(\mathsf L\,z+\frac{2-p}{p}\,\frac{|\nabla z|^2}{z}\right)d\gamma\\
&= \int_\Omega\nabla z\cdot\big(\mathsf L\,\nabla z-\operatorname{Hess}\psi\,\nabla z\big)\,d\gamma+\frac{2-p}{p}\int_\Omega\nabla z\cdot\left(2\operatorname{Hess}z\,\frac{\nabla z}{z}-\frac{|\nabla z|^2}{z}\,\nabla z\right)d\gamma\\
&= -\int_\Omega\left\|\operatorname{Hess}z\right\|^2d\gamma-\int_\Omega\operatorname{Hess}\psi:\nabla z\otimes\nabla z\,d\gamma+\int_{\partial\Omega}\operatorname{Hess}z:\nabla z\otimes\nu\,e^{-\psi}\,d\sigma\\
&\quad+2\,\frac{2-p}{p}\int_\Omega\operatorname{Hess}z:\frac{\nabla z\otimes\nabla z}{z}\,d\gamma-\frac{2-p}{p}\int_\Omega\left\|\frac{\nabla z\otimes\nabla z}{z}\right\|^2d\gamma\\
&= -\frac2p\,(p-1)\int_\Omega\left\|\operatorname{Hess}z\right\|^2d\gamma-\int_\Omega\operatorname{Hess}\psi:\nabla z\otimes\nabla z\,d\gamma\\
&\quad-\frac{2-p}{p}\int_\Omega\left\|\operatorname{Hess}z-\frac{\nabla z\otimes\nabla z}{z}\right\|^2d\gamma+\int_{\partial\Omega}\operatorname{Hess}z:\nabla z\otimes\nu\,e^{-\psi}\,d\sigma\,.
\end{aligned}$$

Here $d\sigma$ denotes the surface measure induced by Lebesgue's measure on $\partial\Omega$. We learn from Grisvard's lemma, see for instance Lemma 5.2 in [46] or [48], that $\int_{\partial\Omega}\operatorname{Hess}z:\nabla z\otimes\nu\,e^{-\psi}\,d\sigma$ is nonpositive as soon as Ω is convex and $\nabla z\cdot\nu=0$ on $\partial\Omega$. As soon as we know that either

$$\operatorname{Hess}\psi\ge\lambda_\star\,\mathrm{Id}$$

for some $\lambda_\star>0$, or the inequality

$$\frac2p\,(p-1)\int_\Omega|\nabla X|^2\,d\gamma+\int_\Omega\operatorname{Hess}\psi:X\otimes X\,d\gamma\ge\lambda(p)\int_\Omega|X|^2\,d\gamma\quad\forall\,X\in\mathrm H^1(\Omega,d\gamma)^d$$

holds for some $\lambda(p)>0$, which is a weaker assumption for any $p>1$, then we obtain that

$$\frac{d}{dt}\int_\Omega|\nabla z|^2\,d\gamma\le-2\,\lambda(p)\int_\Omega|\nabla z|^2\,d\gamma\,.$$

Of course, $\lambda(p)\ge\lambda_\star$. By convention, we take $\lambda(1)=\lambda_\star$.

Proposition 7. *Assume that $p\in[1,2]$, $\varphi=\varphi_p$ and, with the above notations, $\lambda(p)>0$. If Ω is a smooth convex bounded domain in $\mathbb{R}^d$, then (10) holds with $\lambda=2\,\lambda(p)$.*

Proof. It is straightforward. In view of the above computations, we know that

$$\frac{d}{dt}\left(\frac{4}{p\,\lambda}\int_\Omega|\nabla w^{p/2}|^2\,d\gamma-\int_\Omega\frac{w^p-1}{p-1}\,d\gamma\right)\le0$$

and $\lim_{t\to+\infty}\int_\Omega\frac{w^p-1}{p-1}\,d\gamma=\lim_{t\to+\infty}\int_\Omega|\nabla w^{p/2}|^2\,d\gamma=0$. This is enough to conclude that, for any $t\ge0$,

$$\frac{4}{p\,\lambda}\int_\Omega|\nabla w^{p/2}|^2\,d\gamma-\int_\Omega\frac{w^p-1}{p-1}\,d\gamma\ge0\,.$$

□

We conclude this section with the unbounded case $\Omega=\mathbb{R}^d$. For any $p\in[1,2]$, let us assume that the inequality

$$\frac2p\,(p-1)\int_{\mathbb{R}^d}|\nabla X|^2\,d\gamma+\int_{\mathbb{R}^d}\operatorname{Hess}\psi:X\otimes X\,d\gamma\ge\lambda(p)\int_{\mathbb{R}^d}|X|^2\,d\gamma\quad\forall\,X\in\mathrm H^1(\mathbb{R}^d,d\gamma)^d$$

holds for some $\lambda(p)>0$. For $p>1$, this assumption is a spectral gap condition on a vector valued Schrödinger operator: see for instance [42] for further details. With this assumption in hand, we have the following functional inequality, which interpolates between the logarithmic Sobolev inequality and the Poincaré inequality.

Corollary 8. *Assume that $q\in[1,2)$ and let us consider the probability measure $d\gamma=e^{-\psi}\,dx$. Then with $\lambda=\lambda(2/q)$, we have*

$$\frac{\|f\|^2_{\mathrm L^2(\mathbb{R}^d,d\gamma)}-\|f\|^2_{\mathrm L^q(\mathbb{R}^d,d\gamma)}}{2-q}\le\frac1\lambda\int_{\mathbb{R}^d}|\nabla f|^2\,d\gamma\quad\forall\,f\in\mathrm H^1(\mathbb{R}^d,d\gamma)\,.\tag{12}$$

Proof. By homogeneity, we know from Proposition 7 that

$$\int_\Omega \frac{w^p-\overline{w}^p}{p-1}\,d\gamma \le \frac{2}{p\,\lambda(p)}\int_\Omega |\nabla w^{p/2}|^2\,d\gamma$$

for all w such that $f=w^{p/2}$. Here we take $p=2/q$. The conclusion holds by approximating $\mathbb{R}^d$ by a growing sequence of bounded convex domains. □

An equivalent form of (12) is

$$\mathscr{I}[w]\ge\lambda\,\mathscr{E}[w]\quad\forall\,w\in\mathrm{H}^1(\mathbb{R}^d,d\gamma)\text{ such that }\int_{\mathbb{R}^d} w\,d\gamma=1 \tag{13}$$

with the notation $\varphi=\varphi_p$ and $p=2/q\in[1,2]$.

Remark 1. *The optimality of the constant $\lambda=1$ in (12) is easy to obtain when $\psi=\frac12|x|^2$. With $q=1$, (12) is the Gaussian Poincaré inequality*

$$\|f-\bar f\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)}\le\int_\Omega|\nabla f|^2\,d\gamma\quad\forall\,f\in\mathrm{H}^1(\mathbb{R}^d,d\gamma)\quad\textit{with}\quad \bar f=\int_{\mathbb{R}^d} f\,d\gamma\,,$$

with equality if $f=f_1$, $f_1(x)=x_1$. By taking the limit as $q\to2_-$ in (12), we recover Gross' logarithmic Sobolev inequality

$$\int_{\mathbb{R}^d} f^2\log\left(\frac{f^2}{\|f\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)}}\right)d\gamma\le2\int_\Omega|\nabla f|^2\,d\gamma\quad\forall\,f\in\mathrm{H}^1(\mathbb{R}^d,d\gamma)\,.$$

Uniformly with respect to $q\in(1,2)$, the equality case in (12) with $\lambda=1$ is achieved by considering $1+\varepsilon f_1$ as a test function in the limit as $\varepsilon\to0$.

From the point of view of the evolution equation, it is easy to see that the equality in (10) is achieved asymptotically as $t\to+\infty$ by taking $w=u/u_\star$ where u is the solution of (4) given by

$$u(t,x)=u_\star\,(x-x_\star(t))$$

with $x_\star(t)=x_0\,e^{-t}$ for any fixed $x_0\in\mathbb{R}^d$.

1.5. **Improved entropy – entropy production inequalities.** In the proof of Proposition 7, the term

$$\int_\Omega\|\mathrm{Hess}\,z-\nabla z\otimes\nabla z/z\|^2\,d\gamma$$

has been dropped. In some cases, one can recombine the other terms differently and obtain an improved inequality if $q\in(1,2)$. See [3] (and also [2] for a spectral point of view or [30] in the case of the sphere). The boundary term $\int_{\partial\Omega}\mathrm{Hess}\,z:\nabla z\otimes\nu\,e^{-\psi}\,d\sigma$ may also be of importance, as it is suggested in nonlinear problems by [39].

Let us give an example of an improvement, based on [3], in the special case $\psi(x)=|x|^2/2$. Using $\mathrm{Hess}\,\psi=\mathrm{Id}$, after approximating $\mathbb{R}^d$ by bounded domains, we obtain that

$$\frac12\frac{d}{dt}\int_{\mathbb{R}^d}|\nabla z|^2\,d\gamma+\int_{\mathbb{R}^d}|\nabla z|^2\,d\gamma\le-\int_{\mathbb{R}^d}\left\|\mathrm{Hess}\,z-\frac{2-p}{p}\,\frac{\nabla z\otimes\nabla z}{z}\right\|^2d\gamma-\frac2p\,\kappa_p\int_{\mathbb{R}^d}\frac{|\nabla z|^4}{z^2}\,d\gamma$$

with $\kappa_p=(p-1)\,(2-p)/p$. A simple Cauchy-Schwarz inequality shows that

$$\left(\int_{\mathbb{R}^d}|\nabla z|^2\,d\gamma\right)^2\le\int_{\mathbb{R}^d}\frac{|\nabla z|^4}{z^2}\,d\gamma\int_{\mathbb{R}^d}z^2\,d\gamma\,.$$

With the previous notations, we have $\int_{\mathbb{R}^d}z^2\,d\gamma=\int_{\mathbb{R}^d}w^p\,d\gamma=1+(p-1)\,\mathscr{E}[w]$ and $\int_{\mathbb{R}^d}|\nabla z|^2\,d\gamma=\frac p4\,\mathscr{I}[w]$ so that

$$\frac12\frac{d}{dt}\int_{\mathbb{R}^d}|\nabla z|^2\,d\gamma+\int_{\mathbb{R}^d}|\nabla z|^2\,d\gamma\le-\frac2p\,\kappa_p\,\frac{\left(\int_{\mathbb{R}^d}|\nabla z|^2\,d\gamma\right)^2}{\int_{\mathbb{R}^d}|z|^2\,d\gamma}$$

can be rewritten as

$$\frac{d}{dt}\mathscr{I}[w]+2\,\mathscr{I}[w]\le-\kappa_p\,\frac{\mathscr{I}[w]^2}{1+(p-1)\,\mathscr{E}[w]}\,. \tag{14}$$

We recall that we consider here the case $\varphi=\varphi_p$, $p\in(1,2)$, so that κ_p is positive and we can take advantage of (14) to obtain an improved version of Corollary 8. The following result follows the scheme of Theorem 2 in [3].

Proposition 9. *Assume that $q\in(1,2)$ and let us consider the Gaussian probability measure $d\gamma=(2\pi)^{-d/2}\,e^{-|x|^2/2}\,dx$. Then there exists a strictly convex function F on $\mathbb{R}^+$ such that $F(0)=0$ and $F'(0)=1$, for which*

$$\frac{1}{q}\,F\left(q\,\frac{\|f\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)}-1}{2-q}\right)\le\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)}$$

for any $f\in\mathrm{H}^1(\mathbb{R}^d,d\gamma)$ such that $\|f\|_{\mathrm{L}^q(\mathbb{R}^d,d\gamma)}=1$.

Proof. The proofs follows the strategy of [3]. Let $\mathsf{e}(t):=\frac{1}{p-1}\left(\int_{\mathbb{R}^d}f^2\,d\gamma-1\right)$ where $f=w^{p/2}$. We deduce from (14) that

$$\mathsf{e}''+2\,\mathsf{e}'\ge\frac{\kappa_p\,|\mathsf{e}'|^2}{1+(p-1)\,\mathsf{e}}\ge\frac{\kappa_p\,|\mathsf{e}'|^2}{1+\mathsf{e}}\,.$$

The function $F(s):=\frac{1}{1-\kappa_p}\left[1+s-(1+s)^{\kappa_p}\right]$ solves $F'=1+\kappa_p\,\frac{F}{1+s}$ and we can check that (14) is equivalent to

$$\frac{d}{dt}\left(\frac{\mathsf{e}'+2\,F\big(\mathsf{e}\big)}{\big(1+\mathsf{e}\big)^{\kappa_p}}\right)\ge0\,.$$

Since $\lim_{t\to+\infty}\big(\mathsf{e}'(t)+2\,F\big(\mathsf{e}(t)\big)\big)=0$, we have shown that $\mathsf{e}'+2\,F\big(\mathsf{e}\big)\le0$. □

From the point of view of entropy – production of entropy inequalities, we have obtained that

$$\mathscr{I}[w]\ge2\,F\,(\mathscr{E}[w])$$

where F is a strictly convex function such that $F(0)=0$ and $F'(0)=1$. Using the homogeneity and substituting $f/\|f\|_{\mathrm{L}^q(\mathbb{R}^d,d\gamma)}$ to f, similar estimates have been used in [3] to prove that

$$\tfrac{2}{(2-q)^2}\left[\|f\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)}-\|f\|^{2(2-q)}_{\mathrm{L}^q(\mathbb{R}^d,d\gamma)}\,\|f\|^{2(q-1)}_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)}\right]\le\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)}\;\forall\,f\in\mathrm{H}^1(\mathbb{R}^d,d\gamma)\,.$$

2. The carré du champ method on the sphere

For a presentation of this topic, we refer to [30, 36] for computations based on the ultraspherical operator and to [33, 29] for computtations on general manifolds.

3. Symmetry and symmetry breaking results in critical Caffarelli-Kohn-Nirenberg inequalities

This section is reproduced without changes from the paper on the *Symmetry of optimizers of the Caffarelli-Kohn-Nirenberg inequalities* by J. Dolbeault, M.J. Esteban and M. Loss, to appear in the Proceedings of ICMP 2015, [35]. A more exhaustive review of the symmetry and symmetry breaking issues can be found in [37], with an emphasis on the bifurcation point of view.

3.1. **Introduction.**

Symmetries of optimizers in variational problems is a central theme in the calculus of variations. Sophisticated methods like rearrangement inequalities, reflection methods and moving plane methods belong now to the standard repertoire of any analyst. There are, however, examples where these methods cannot be applied. Variational problems that depend on parameters very often cannot be treated by such methods, simply because, depending on the parameters, the optimizers are symmetric and sometimes not. Famous examples are the minimizers of the Ginzburg-Landau functional in superconductivity, where, depending on the strength of the quartic interaction the minimizers form a single, symmetric vortex or a vortex lattice. Clearly such problems cannot be treated by general methods. For certain parameters they ought to work while in others they cannot. Thus, rather special techniques, tailored to the problems at hand, have to be developed to prove symmetry in the desired regions.

One class of such examples is given by the Caffarelli-Kohn-Nirenberg inequalities [16]. In these notes, we shall specifically consider the case of the inequality

$$\text{(CKN)}\qquad\int_{\mathbb{R}^d}\frac{|\nabla w|^2}{|x|^{2a}}\,dx\ge\mathsf{C}^d_{a,b}\left(\int_{\mathbb{R}^d}\frac{|w|^p}{|x|^{bp}}\,dx\right)^{2/p}$$

with $a\le b\le a+1$ if $d\ge3$, $a<b\le a+1$ if $d=2$, and $a<a_c$ where

$$a_c:=\frac{d-2}{2}\,,\quad p=\frac{2\,d}{d-2+2\,(b-a)}\,.$$

The function w is in a suitable function space which contains, for instance, all smooth functions with compact support. The constant $\mathsf{C}^d_{a,b}$ is, by definition, the best possible constant. Rotating the function w does not change the value of the various expressions in (CKN), i.e., the inequality is rotationally invariant. The special case where $a \geq 0$ has been treated by various authors (see the references in [34]). Rearrangement inequalities can be used to reduce the problem to the set of radial functions, for which the optimality issue can then be solved explicitly.

For the case where $a<0$ the problem is much more subtle. Nevertheless, Catrina and Wang [21], proved that the optimizers, i.e., the functions that yield equality in (CKN), exist in the open strip $a<b<a+1$. This result establishes the existence of non-negative solutions $w \in \mathrm{L}^p(\mathbb{R}^d;|x|^{-b\,p}\,dx)$ of the equation

$$-\operatorname{div}\big(|x|^{-2a}\,\nabla w\big)=|x|^{-b\,p}\,w^{p-1}\,. \tag{15}$$

Moreover, in the same paper Catrina and Wang also showed that, in some region in the (a,b) plane, the rotational symmetry of the optimizers is broken. A more detailed analysis by Felli and Schneider [45] shows that the region where the optimizers have a broken symmetry contains the set $\mathscr{R}_{\rm FS}:=\{(a,b)\,:\,a<0\,,\;b<b_{\rm FS}(a)\}$ where

$$b_{\rm FS}(a):=\frac{d\,(a_c-a)}{2\sqrt{(a_c-a)^2+d-1}}+a-a_c\,.$$

We call this region $\mathscr{R}_{\rm FS}$ the *Felli-Schneider region.*

In [45] more is shown. The optimizers in the *radial class* can be determined explicitly which allows to compute the second variation operator about these solutions. The lowest eigenvalue of this operator is strictly negative for $(a,b)\in\mathscr{R}_{\rm FS}$, equals zero on the curve $b=b_{\rm FS}(a)$ and is strictly positive in the open complement of the Felli-Schneider region: there, the radial optimizers are stable. Needless to say that positivity of the second variation does not imply the radial symmetry of the (global) optimizers for the (CKN) inequality. Thus, it is a natural question whether or not the optimizers possess rotational symmetry in the complement of $\mathscr{R}_{\rm FS}$. Let $2^*:=\frac{2\,d}{d-2}$ if $d\geq 3$ and $2^*:=\infty$ if $d=2$. The following theorem is proved in [34]:

Theorem 10. *Let $d\geq 2$, $p\in(2,2^*)$, $a<0$ and b in the complement of the Felli-Schneider region and such that $p=\frac{2\,d}{d-2+2\,(b-a)}$. Then any non-negative solution $w\in\mathrm{L}^p(\mathbb{R}^d;|x|^{-b\,p}\,dx)$ of (15) must be of the form*

$$\big(A+B\,|x|^{2\alpha}\big)^{-\frac{n-2}{2}}$$

where A, B are positive constants,

$$\alpha=\frac{(1+a-b)\,(a_c-a)}{a_c-a+b} \tag{16}$$

and

$$n=\frac{2\,p}{p-2}\,. \tag{17}$$

In particular this holds for the optimizers of (CKN).

There are some interesting consequences. Using the change of variables

$$w(r,\omega)=r^{a-a_c}\,\phi\big(\log r,\omega\big)\,,$$

equation (15) can be cast in the form

$$-\,\partial_z^2\phi-\Delta_\omega\,\phi+\Lambda\,\phi=\phi^{p-1}\,. \tag{18}$$

Here, $\frac{x}{|x|}=\omega\in\mathbb{S}^{d-1}$, $r=|x|$, $z=\log r$, Δ_ω is the Laplace-Beltrami operator on the sphere $\mathbb{S}^{d-1}$ and

$$\Lambda=(a-a_c)^2\,.$$

Thus, ϕ is a function on the cylinder $\mathbb{R}\times\mathbb{S}^{d-1}$. Moreover, as noticed in [21], (CKN) is transformed into

$$\|\partial_z\phi\|^2_{\mathrm{L}^2(\mathbb{R}\times\mathbb{S}^{d-1})}+\|\nabla_\omega\phi\|^2_{\mathrm{L}^2(\mathbb{R}\times\mathbb{S}^{d-1})}+\Lambda\,\|\phi\|^2_{\mathrm{L}^2(\mathbb{R}\times\mathbb{S}^{d-1})}\geq\mathsf{C}^d_{a,b}\,\|\phi\|^2_{\mathrm{L}^p(\mathbb{R}\times\mathbb{S}^{d-1})}\,. \tag{19}$$

Corollary 11. *Let $d\geq 2$, $p\in(2,2^*)$. Any non-negative solution $\phi\in\mathrm{L}^p(\mathbb{R}\times\mathbb{S}^{d-1};dz\,d\omega)$ of (18) is, up to translations, of the form*

$$\phi_\Lambda(z)=\Big(\tfrac{2}{p\,\Lambda}\cosh^2\big(\tfrac{p-2}{2}\sqrt{\Lambda}\,z\big)\Big)^{-\frac{1}{p-2}}\,,$$

if and only if

$$\Lambda\leq 4\,\frac{d-1}{p^2-4}\,.$$

In this range, equality in (19) *is achieved if and only if $\phi(z)=\phi_\Lambda(z+z_0)$ for some $z_0\in\mathbb{R}$.*

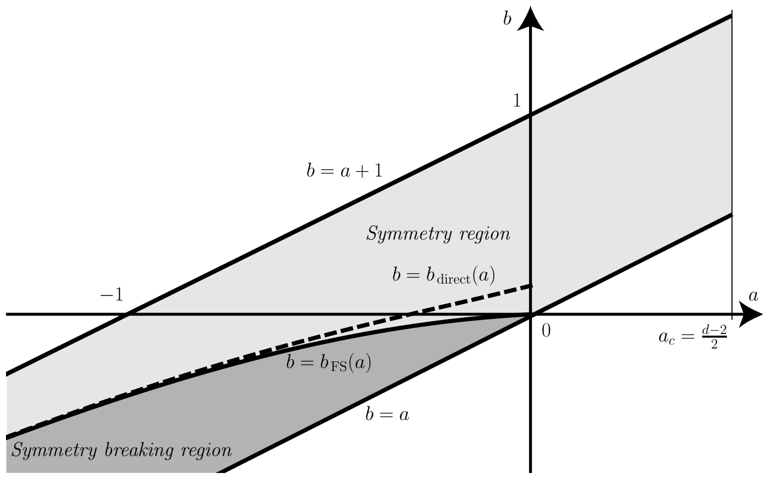

FIGURE 1. *This figure is taken from* [34]. The *Felli-Schneider region*, or symmetry breaking region, appears in dark grey and is defined by $a<0$, $a<b<b_{\rm FS}(a)$. We prove that symmetry holds in the light grey region defined by $b_{\rm FS}(a)\le b<a+1$ when $a<0$. Symmetry also holds for any $b\in[a,a+1)$ if $a\in[0,a_c)$. The curve $a\mapsto b_{\rm direct}(a)$ corresponds to the dashed curve. The above plot is done with $d=3$.

To put this result in perspective we compare it with a result in [12].

Theorem 12. *Let $d\ge2$, $p\in(2,2^*)$. On the sphere $\mathbb S^d$ consider the equation*

$$-\Delta u+\lambda\,u=u^{p-1}$$

with $\lambda>0$. Here Δ represents the Laplace-Beltrami operator on $\mathbb S^d$. Then the constant function $u\equiv\lambda^{1/(p-2)}$ is the only non-negative solution if and only if

$$\lambda\le\frac{d}{p-2}\,.$$

Thus, Corollary 11 can be viewed as an extension of the above mentioned *rigidity* result to the non-compact case of a cylinder. As a special case, this also allows to identify the equality case in the interpolation inequality (19) on the cylinder.

In the next sections some ideas about the proof are given: we start by the simple case of the standard Sobolev inequality in Section 3.2, explain in Section 3.3 how to recast (CKN) as a Sobolev type inequality in an artificial *dimension* n, where n is not necessarily an integer, and conclude by explaining how the main estimates can be produced using a *fast diffusion* flow.

3.2. **Heuristics for the proof of Theorem 10.** In order to avoid long computations it is best to explain the ideas in a 'simple' example. For any $d\ge3$, the Sobolev inequality

$$\int_{\mathbb R^d}|\nabla u|^2\,dx\ge\mathsf C_d\left(\int_{\mathbb R^d}|u|^p\,dx\right)^{2/p}\,,\quad\text{with}\quad p=2^*=\frac{2\,d}{d-2}\tag{20}$$

is extremely well understood [63, 4, 57]. Once more $\mathsf C_d$ denotes the sharp constant. Note that this inequality appears as a special case of (CKN) if one sets $a=b=0$, in which case $\mathsf C_d=\mathsf C^d_{0,0}$. There is equality in (20) if and only if u is a translate of the Aubin-Talenti function

$$\left(c_\star\,\lambda+\frac{|x|^2}{\lambda}\right)^{-(d-2)/2}\,,$$

where $c_\star$ and λ are positive constants. There have been some proofs using flow methods to understand this inequality [19, 17]. The flow used for the case at hand is a porous medium / fast diffusion flow. It is given by

$$\frac{\partial v}{\partial t}=\Delta\,v^{1-\frac1d}\tag{21}$$

and has the self-similar solutions

$$v_\star(x,t)=\left(c_\star\,t+\frac{|x|^2}{t}\right)^{-d}\,.$$

This function has slow decay in the x variable. The obvious similarity of the expressions of the Aubin-Talenti and self-similar functions suggests a reformulation of the Sobolev functional by setting

$$v = u^{\frac{2d}{d-2}} .$$

Let us define a *pressure variable* p by

$$v = \mathsf{p}^{-d} .$$

A short computation shows

Lemma 13. *The Sobolev inequality, written in terms of v and p, is given by*

$$a_c^2 \int_{\mathbb{R}^d} v\,|\nabla \mathsf{p}|^2\,dx \geq \mathsf{C}_d \left(\int_{\mathbb{R}^d} v\,dx\right)^{\frac{d-2}{d}} . \tag{22}$$

Assume now that v satisfies the fast diffusion equation (21). This implies that p evolves by the equation

$$\frac{\partial \mathsf{p}}{\partial t} = \frac{d-1}{d}\left(\mathsf{p}\,\Delta\,\mathsf{p} - d\,|\nabla \mathsf{p}|^2\right) .$$

The right side of (22) does not change if v evolves via (21). For the left side we have

Lemma 14. *Assume that v evolves via (21). Then*

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^d} v\,|\nabla \mathsf{p}|^2\,dx &= -2\int_{\mathbb{R}^d}\left[\tfrac{1}{2}\,\Delta\,|\nabla \mathsf{p}|^2 - \nabla \mathsf{p}\cdot\nabla\Delta\,\mathsf{p} - \tfrac{1}{d}\,(\Delta\,\mathsf{p})^2\right]\mathsf{p}^{1-d}\,dx \\ &= -2\int_{\mathbb{R}^d} \mathrm{Tr}\left[\mathrm{H}_{\mathsf{p}} - \tfrac{1}{d}\,(\mathrm{Tr}\,\mathrm{H}_{\mathsf{p}})\,\mathrm{Id}\right]^2\mathsf{p}^{1-d}\,dx \end{aligned}$$

where $\mathrm{H}_{\mathsf{p}} = (\nabla\otimes\nabla)\,\mathsf{p}$ *denotes the Hessian matrix of* p. *Moreover,*

$$\mathrm{H}_{\mathsf{p}} - \tfrac{1}{d}\,(\mathrm{Tr}\,\mathrm{H}_{\mathsf{p}})\,\mathrm{Id} = 0$$

if and only if $\mathsf{p}(x) = a + b\cdot x + c\,|x|^2$ *for some* $(a,b,c)\in\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}$.

The proof is a somewhat longish but straightforward computation. Note, that it is precisely the particular choice of v and p that renders the time derivative in such a simple form.

To summarize, while the right side of the Sobolev inequality stays fixed the left side diminishes under the flow. *The idea is to use the fast diffusion flow to drive the functional towards its optimal value.* Actually we use the fact that if v is optimal in (22), or if it is a critical point, the functional has to be stationary under the action of the flow, which allows to identify p, hence v. To exploit this idea for the (CKN) inequality we have to rewrite it in the form of a Sobolev type inequality.

3.3. **A modified Sobolev inequality.** The first step in the proof is to rewrite the problem in a form that resembles the Sobolev inequality. If we write

$$w(r,\omega) = u(s,\omega) \quad \text{with} \quad s = r^\alpha ,$$

the inequality (CKN) takes the form

$$\int_{\mathbb{R}_+\times\mathbb{S}^{d-1}}\left[\alpha^2\left(\frac{\partial u}{\partial s}\right)^2 + \frac{|\nabla_\omega u|^2}{s^2}\right] s^{n-1}\,ds\,d\omega \geq \mathsf{C}_{a,b}^d\,\alpha^{1-\frac{2}{p}}\left(\int_{\mathbb{R}_+\times\mathbb{S}^{d-1}}|u|^p\,s^{n-1}\,ds\,d\omega\right)^{\frac{2}{p}}$$

where $d\omega$ denotes the uniform measure on the sphere $\mathbb{S}^{d-1}$, ∇_ω denotes the gradient on $\mathbb{S}^{d-1}$ and where α and n are given by (16) and (17). We shall abbreviate

$$\mathsf{D}\,u := \left(\alpha\,\tfrac{\partial u}{\partial s}, \tfrac{1}{s}\,\nabla_\omega u\right), \quad |\mathsf{D}\,u|^2 = \alpha^2\left(\tfrac{\partial u}{\partial s}\right)^2 + \tfrac{|\nabla_\omega u|^2}{s^2} .$$

Our inequality is therefore equivalent to a Sobolev type inequality and takes the form

$$\int_{\mathbb{R}^d}|\mathsf{D}\,u|^2\,d\mu \geq \mathsf{C}_{a,b}^d\,\alpha^{1-\frac{2}{p}}\left(\int_{\mathbb{R}^d}|u|^p\,d\mu\right)^{\frac{2}{p}} , \quad \text{with} \quad p = \frac{2\,n}{n-2} . \tag{23}$$

This inequality generalizes (20). Here the measure $d\mu$ is defined on $\mathbb{R}^+\times\mathbb{S}^{d-1}$ by

$$d\mu = s^{n-1}\,ds\,d\omega .$$

As in Section 3.2, we may consider $v = u^p$ and define a pressure variable p such that $v = \mathsf{p}^{-n}$, so that $u = \mathsf{p}^{-(n-2)/2}$. With these notations, (23) can be rewritten as

$$\tfrac{1}{4}\,(n-2)^2 \int_{\mathbb{R}^d} v\,|\mathsf{D}\mathsf{p}|^2\,d\mu \ge \mathsf{C}^d_{a,b}\,\alpha^{1-\frac{2}{p}} \left(\int_{\mathbb{R}^d} v\,d\mu\right)^{\frac{2}{p}}. \tag{24}$$

With straightforward abuses of notations, we shall write $\int_{\mathbb{R}^d} f\,d\mu = \int_{\mathbb{R}^d} f\,d\mu$ and identify $\mathrm{L}^p(\mathbb{R}_+\times\mathbb{S}^{d-1};d\mu)$ with $\mathrm{L}^p(\mathbb{R}^d;|x|^{n-d}\,dx)$ or simply $\mathrm{L}^p(\mathbb{R}^d;d\mu)$.

One should note that n is, in general, not an integer and the above inequality reduces to Sobolev's inequality only if $n=d$. Of particular significance is that the curve

$$b = b_{\mathrm{FS}}(a)\,,$$

when represented in the new variables α and n, is given by the equation $\alpha = \alpha_{\mathrm{FS}}$ with

$$\alpha_{\mathrm{FS}} := \sqrt{\frac{d-1}{n-1}}\,.$$

Thus, for $\alpha > \alpha_{\mathrm{FS}}$ the minimizers are not radial. The equation (15) transforms into the equation

$$-\,\mathscr{L}\,u = u^{p-1}\,, \tag{25}$$

where $\mathscr{L}$ is the Laplacian associated with the quadratic form given by the left side of (23), i.e., $\mathscr{L} = -\,\mathsf{D}^*\cdot\mathsf{D}$. Theorem 10 can be reformulated as

Theorem 15. *Let $d\ge 2$, $p\in(2,2^*)$, $n=\frac{2p}{p-2}>d$ and $\alpha\le\alpha_{\mathrm{FS}}$. Then any non-negative solution $u\in\mathrm{L}^p(\mathbb{R}^d;d\mu)$ of* (25) *must be of the form*

$$\big(A+B\,|x|^2\big)^{-\frac{n-2}{2}} \tag{26}$$

where A, B are positive constants, and n is given by (17)*. As a special case, equality in* (24) *is achieved if and only if u is given by* (26)*.*

The upshot of this work can be summarized in the following fashion: *Any optimizer in the radial class that is not unstable under small perturbations is in fact a global minimizer for the* (CKN) *inequality.*

3.4. **The flow.** We consider the fast diffusion flow

$$\frac{\partial v}{\partial t} = \mathscr{L}\,v^{1-\frac{1}{n}}\,. \tag{27}$$

It is easily seen that the flow (27) has the self-similar solutions

$$v_\star(t;s,\omega) = t^{-n}\left(c_\star + \frac{s^2}{2\,(n-1)\,\alpha^2\,t^2}\right)^{-n}.$$

The basic idea is now quite simple. We consider a non-negative solution $u\in\mathrm{L}^p(\mathbb{R}^d;d\mu)$ of (25) and set $v=u^p$. We also consider the pressure variable p such that $v=\mathsf{p}^{-n}$. The first thing to note is that the right side of (24) does not change if we evolve v and hence u under the flow (27). Further, if we differentiate the left side of (24) along the flow we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^d} v\,|\mathsf{D}\mathsf{p}|^2\,d\mu = -2\int_{\mathbb{R}^d}\left[\tfrac12\,\mathscr{L}\,|\mathsf{D}\mathsf{p}|^2 - \mathsf{D}\mathsf{p}\cdot\mathsf{D}\mathscr{L}\,\mathsf{p} - \tfrac1n\,(\mathscr{L}\,\mathsf{p})^2\right]d\mu\,.$$

On the other hand simple computations show that

$$\tfrac14\,(n-2)^2\,\frac{d}{dt}\left(\int_{\mathbb{R}^d} v\,|\mathsf{D}\mathsf{p}|^2\,d\mu\right)\Big|_{t=0} = -2\int_{\mathbb{R}^d}(\mathscr{L}\,u)\,u^{1-p}\left(\mathscr{L}\,u^{p(n-1)/n}\right)d\mu \tag{28}$$

when expressed in terms of u. Now we take $v=u^p$, where u is the solution to (25), as initial datum for (15). With this choice, the right side in (28) is actually zero. Indeed, by multiplying both sides of (25) by $u^{1-p}\left(\mathscr{L}\,u^{p(n-1)/n}\right)$ one obtains

$$\int_{\mathbb{R}^d}(\mathscr{L}\,u)\,u^{1-p}\left(\mathscr{L}\,u^{p(n-1)/n}\right)d\mu = \int_{\mathbb{R}^d} u^{p-1}\,u^{1-p}\left(\mathscr{L}\,u^{p(n-1)/n}\right)d\mu = 0\,.$$

The interesting point, and the heart of the argument, is that

$$0 = \int_{\mathbb{R}^d}\left[\tfrac12\,\mathscr{L}\,|\mathsf{D}\mathsf{p}|^2 - \mathsf{D}\mathsf{p}\cdot\mathsf{D}\mathscr{L}\,\mathsf{p} - \tfrac1n\,(\mathscr{L}\,\mathsf{p})^2\right]d\mu$$

can be written as a sum of non-negative terms precisely when $\alpha \le \alpha_{\rm FS}$, and the vanishing of these terms shows that u must be of the form $(A+B\,s^2)^{-(n-2)/2}$. In this way one obtains a classification of the non-negative solutions of (25) provided they are in $\mathrm{L}^p(\mathbb{R}^d;d\mu)$. To simplify notations, we shall omit the index ω, so that from now on ∇ and Δ respectively refer to the gradient and to the Laplace-Beltrami operator on $\mathbb{S}^{d-1}$. With the notation $'=\partial_s$, our identity can be reworked as follows.

Lemma 16. *Assume that $d\ge 3$, $n>d$ and let $\mathsf p$ be a positive function in $C^3(\mathbb{S}^{d-1})$. Then*

$$\tfrac12\,\mathscr L\,|\mathsf D\mathsf p|^2-\mathsf D\mathsf p\cdot\mathsf D\mathscr L\,\mathsf p-\tfrac1n\,(\mathscr L\,\mathsf p)^2 = \alpha^4\,\tfrac{n-1}{n}\left[\mathsf p''-\frac{\mathsf p'}{r}-\frac{\Delta\,\mathsf p}{\alpha^2\,(n-1)\,r^2}\right]^2+\frac{2\,\alpha^2}{r^2}\left|\nabla\mathsf p'-\frac{\nabla\mathsf p}{r}\right|^2 + \frac{1}{r^4}\left[\tfrac12\,\Delta\,|\nabla\mathsf p|^2-\nabla\mathsf p\cdot\nabla\Delta\,\mathsf p-\tfrac1{n-1}\,(\Delta\,\mathsf p)^2-(n-2)\,\alpha^2\,|\nabla\mathsf p|^2\right].$$

The only term in Lemma 16 that does not have a sign is the last one. When integrated against $\mathsf p^{1-n}$ over $\mathbb{S}^{d-1}$, however, this term can be written as a sum of squares. The following lemma holds for $d\ge 3$. For the case $d=2$ we refer the reader to [34].

Lemma 17. *Assume that $d\ge 3$ and that $\mathsf p$ is a positive function in $C^3(\mathbb{S}^{d-1})$. Then*

$$\begin{aligned}
&\int_{\mathbb{S}^{d-1}}\left[\tfrac12\,\Delta\,|\nabla\mathsf p|^2-\nabla\mathsf p\cdot\nabla\Delta\,\mathsf p-\tfrac1{n-1}\,(\Delta\,\mathsf p)^2-(n-2)\,\alpha^2\,|\nabla\mathsf p|^2\right]\mathsf p^{1-n}\,d\omega\\
&=\ \tfrac{(n-2)\,(d-1)}{(n-1)\,(d-2)}\int_{\mathbb{S}^{d-1}}\left\|\mathsf L\,\mathsf p-\tfrac{3\,(n-1)\,(n-d)}{2\,(n-2)\,(d+1)}\,\mathsf M\,\mathsf p\right\|^2\mathsf p^{1-n}\,d\omega\\
&\quad+\tfrac{n-d}{2\,(d+1)}\left[\tfrac{n+3}{2}+\tfrac{3\,(n-1)\,(n+1)\,(d-2)}{2\,(n-2)\,(d+1)}\right]\int_{\mathbb{S}^{d-1}}\frac{|\nabla\mathsf p|^4}{\mathsf p^2}\,\mathsf p^{1-n}\,d\omega\\
&\quad+(n-2)\left[\alpha_{\rm FS}^2-\alpha^2\right]\int_{\mathbb{S}^{d-1}}|\nabla\mathsf p|^2\,\mathsf p^{1-n}\,d\omega
\end{aligned}$$

where $\mathsf L\,\mathsf p:=(\nabla\otimes\nabla)\,\mathsf p-\frac1{d-1}\,(\Delta\,\mathsf p)\,g$ and $\mathsf M\,\mathsf p:=\frac{\nabla\mathsf p\otimes\nabla\mathsf p}{\mathsf p}-\frac1{d-1}\,\frac{|\nabla\mathsf p|^2}{\mathsf p}\,g$. Here g is the standard metric on $\mathbb{S}^{d-1}$ and $\mathsf L\,\mathsf p$ denotes the trace free Hessian of $\mathsf p$.

The key device used for the proof of this lemma is the Bochner-Lichnerowicz-Weitzenböck formula. If $\mathscr M$ is a compact Riemannian manifold, then for any smooth function $f:\mathscr M\to\mathbb R$ we have

$$\tfrac12\,\Delta\,|\nabla f|^2=\|\mathrm H_f\|^2+\nabla\,\Delta f\cdot\nabla f+\mathrm{Ric}(\nabla f,\nabla f)$$

where $\|\mathrm H_f\|^2$ is the trace of the square of the Hessian of f and $\mathrm{Ric}(\nabla f,\nabla f)$ is the Ricci curvature tensor contracted against $\nabla f\otimes\nabla f$. If $\mathscr M=\mathbb{S}^{d-1}$, then $\mathrm{Ric}(\nabla f,\nabla f)=(d-2)\,|\nabla f|^2$. The main point in Lemma 17 is that, provided $\alpha\le\alpha_{\rm FS}$, all terms are non-negative.

It is quite easy to see that the vanishing of these terms entails that $\mathsf p$ can only depend on the variable $s=|x|$ and must be of the form (26).

While the formal computations are straightforward there is the perennial issue of the boundary terms that occur in all the integration by parts. This is due to the fact that one is dealing with solutions of (25) and it is not at all clear that the boundary terms vanish. This requires a detailed regularity analysis of the solutions of (25). The task is non-trivial because the exponent p is critical for the scaling in the s variable. The reader may consult [34] for details.

The computations outlined above can be carried over to the case where $\mathbb{S}^{d-1}$ is replaced by a compact Riemannian manifold $\mathscr M$ of dimension $d-1$. The results are then expressed in terms of the Ricci curvature of the manifold. Again the reader may consult [34] for details.

4. Symmetry and symmetry breaking results in subcritical Caffarelli-Kohn-Nirenberg inequalities

An abridged version of this section titled *Symmetry by flow* has been published as an Oberwolfach report (# MFO-1628, J. Dolbeault, joint work with M.J. Esteban, M. Loss and M. Muratori).

With the norms defined by

$$\|w\|_{q,\gamma}:=\left(\int_{\mathbb R^d}|w|^q\,|x|^{-\gamma}\,dx\right)^{1/q},\quad \|w\|_q:=\|w\|_{q,0}\,,$$

let us consider the family of *Caffarelli-Kohn-Nirenberg interpolation inequalities* given by

$$\|w\|_{2p,\gamma} \le \mathsf{C}_{\beta,\gamma,p}\,\|\nabla w\|_{2,\beta}^{\vartheta}\,\|w\|_{p+1,\gamma}^{1-\vartheta} \tag{29}$$

in a suitable functional space $\mathrm{H}^p_{\beta,\gamma}(\mathbb{R}^d)$ obtained by completion of smooth functions with support in $\mathbb{R}^d\setminus\{0\}$, w.r.t. the norm given by $\|w\|^2 := (p_\star - p)\,\|w\|^2_{p+1,\gamma} + \|\nabla w\|^2_{2,\beta}$. Here $\mathsf{C}_{\beta,\gamma,p}$ denotes the optimal constant, the parameters β, γ and p are subject to the restrictions

$$d \ge 2\,,\quad \gamma - 2 < \beta < \frac{d-2}{d}\,\gamma\,,\quad \gamma \in (-\infty, d)\,,\quad p \in \big(1, p_\star\big] \quad \text{with } p_\star := \frac{d-\gamma}{d-\beta-2} \tag{30}$$

and the exponent ϑ is determined by the scaling invariance, *i.e.*,

$$\vartheta = \frac{(d-\gamma)\,(p-1)}{p\big(d+\beta+2-2\gamma-p\,(d-\beta-2)\big)}\,.$$

These inequalities have been introduced, among other inequalities, by L. Caffarelli, R. Kohn and L. Nirenberg in [16].

Equality in (29) is achieved by Aubin-Talenti type functions

$$w_\star(x) = \Big(1+|x|^{2+\beta-\gamma}\Big)^{-1/(p-1)} \qquad \forall\,x\in\mathbb{R}^d$$

if we know that *symmetry* holds, that is, if we know that the equality is achieved among radial functions. It is indeed not very difficult to check that $w_\star$ is the unique radial critical point, up to the transformations associated with the invariances of the equation. Of course, the set of functions generated by the dilations and the multiplication by an arbitrary constant is then optimal whenever $w_\star$ is optimal. On the contrary, there is *symmetry breaking* if this is not the case, because the equality case is then achieved only by a non-radial extremal function.

Deciding whether symmetry or symmetry breaking holds is a central problem in physics. It is well known that symmetric energy functionals might have states of lowest energy that may or may not have these symmetries. In the case under study, (29) involves radial weights and is therefore invariant under rotation. When, in the language of physics, symmetry is *broken*, this means that the symmetry group of the minimizer is smaller than the symmetry group of the functional. Needless to say, for computing the optimal value of the functional it is of great advantage that an optimizer be symmetric. The optimal constant $\mathsf{C}_{\beta,\gamma,p}$ can then be explicitly computed in terms of the Γ function. Otherwise, this is a difficult question which has only numerical solutions and involve a delicate energy minimization as shown in [27, 28]. In other contexts the breaking of symmetry leads to various interesting phenomena and this is why it is important to decide what symmetry types, if any, an optimizer has.

Depending on the parameters, to decide whether a minimizer has the full symmetry or not can be difficult. To show that symmetry is broken one can minimize the functional in the *class of symmetric functions* and then check whether the value of the functional can be lowered by perturbing the minimizer away from the symmetric situation. This is the method that has been used to establish that *symmetry breaking* occurs in (29) if

$$\gamma < 0 \quad \text{and} \quad \beta_{\mathrm{FS}}(\gamma) < \beta < \frac{d-2}{d}\,\gamma \tag{31}$$

where

$$\beta_{\mathrm{FS}}(\gamma) := d-2-\sqrt{(\gamma-d)^2-4\,(d-1)}\,.$$

In the critical case $p = p_\star$, the method was implemented by F. Catrina and Z.-Q. Wang in [21], and the sharp result has been obtained by V. Felli and M. Schneider in [45]. The same condition was recently obtained in the subcritical case $p < p_\star$, in [14]. Here by *critical* we simply mean that $\|w\|_{2p,\gamma}$ scales like $\|\nabla w\|_{2,\beta}$. One has to observe that proving symmetry breaking by establishing the linear instability is a *local* method, which is based on a painful but rather straightforward linearization around the special function $w_\star$.

A real difficulty occurs when the minimizer in the symmetric class is stable, *i.e.*, all local perturbations that break the symmetry increase the energy: in our case, non-radial perturbations. It is obvious that, in general, one cannot conclude that the minimizer is symmetric because the minimizer in the symmetric class and the actual minimizer might not be close in any reasonable notion of distance. In general it is very difficult to decide, assuming stability, wether the minimizer is symmetric or not. This is a *global* problem and not amenable to linear methods.

There are no general techniques available for understanding the symmetry of minimizers. The question is quite obvious when the weights and the nonlinearity do not cooperate to decrease the energy under symmetrization and in most of the cases it turns out that moving planes and related comparison techniques also fail. Let us recast our purpose in a larger perspective. In analysis, the focus has to be and has always been on relevant and non-trivial examples, such as finding the sharp constant in Sobolev's inequality [4, 63], the Hardy-Littlewood-Sobolev inequality [57] or the logarithmic Sobolev inequality [49], to mention classical examples. In the context of elliptic PDEs, various techniques have been developed to tackle the question of the symmetry. The *symmetrization methods* and *moving plane techniques* can be applied, in the case of Caffarelli-Kohn-Nirenberg inequalities, to prove that symmetry holds if $p = p_\star$ and $\beta > 0$. Still using symmetrization methods, a better range has been achieved in [11], that can still be enlarged by direct energy estimates as in [32]. Various perturbation techniques have also been implemented, as in [41], to extend the region of the parameters for which symmetry is known, but the method is then, at least in [41] and related papers, not constructive. In any case, none of these methods has been proved so far to cover the whole range of linear stability of the symmetric minimizers. To establish the optimal symmetry range in (29), and thus determine the sharp constant in the Caffarelli-Kohn-Nirenberg inequalities, a new method had to be designed. What has been proved in [34] in the critical case $p = p_\star$, and extended in [38] to the sub-critical case $1 < p < p_\star$, is that the symmetry breaking range given in (31) is optimal: symmetry holds in the complementary region of the admissible parameters.

At first sight, the strategy used in [34, 38] might look somewhat strange: we directly prove the uniqueness of the critical points, up to the invariances of the equation, and since the problem has radial critical points, these are the only ones. Actually there is a good reason for that, which arises from the monotonicity of an appropriate functional in the functional space under the action of a nonlinear flow. The stationarity under the flow characterizes all critical points. This also explains why we are able to extend a local property (the linear stability of radial solutions) to a global stability result (the uniqueness, up to the invariances, of the critical point).

Our method, which is of Bakry-Emery type, provides us with the optimal range of symmetry. This is a remarkable fact that can be explained as follows. In the framework of nonlinear flows, the optimality in the entropy-entropy production inequality is achieved by a linearization which also corresponds to large-time asymptotics. As a consequence the best constant in the inequality is equal to the optimal constant which arises from the Bakry-Emery computation (see [6, 52]), which is also attained in the large-time asymptotics.

In [34, 38], we analyze the symmetry properties not only of the extremal functions of (29), but also of all positive solutions in $\mathrm{H}^p_{\beta,\gamma}(\mathbb{R}^d)$ of the corresponding Euler-Lagrange equations, *i.e.*, up to a multiplication by a constant and a dilation, of

$$-\operatorname{div}\left(|x|^{-\beta}\,\nabla w\right) = |x|^{-\gamma}\left(w^{2p-1} - w^p\right) \quad \text{in} \quad \mathbb{R}^d\setminus\{0\}\,. \tag{32}$$

Theorem 18. *Under Condition* (30) *assume that*

$$\text{either} \quad \beta \le \beta_{\mathrm{FS}}(\gamma) \quad \forall\,\gamma<0\,, \quad \text{or} \quad \gamma \ge 0\,. \tag{33}$$

Assume that $d \ge 2$. Then all positive solutions in $\mathrm{H}^p_{\beta,\gamma}(\mathbb{R}^d)$ to (32) *are radially symmetric and, up to a scaling and a multiplication by a constant, equal to $w_\star$.*

In Fig. 2, the grey area corresponds to the cone determined by

$$d-2+\frac{\gamma-d}{p} \le \beta < \frac{d-2}{d}\,\gamma \quad \text{and} \quad \gamma \in (-\infty, d)$$

in (30). The light grey area is the region of symmetry, while the dark grey area is the region of symmetry breaking. The threshold is determined by the hyperbola

$$(d-\gamma)^2 - (\beta-d+2)^2 - 4\,(d-1) = 0$$

or, equivalently $\beta = \beta_{\mathrm{FS}}(\gamma)$.

Our result is a *rigidity result*: we prove that all positive solutions are, up to the trivial invariances corresponding to the natural symmetries, equal to the known solution $w_\star$. The method is based on the use of a test function, which is adapted to the solution of our equation. Let us put this method in perspective, by illustrating it with two examples of a similar approach.

- The *Pohozaev method* amounts to test an elliptic equation (typically involving a supercritical exponent) with a solution u against (local) dilations, *i.e.*, against $x\cdot\nabla u$, and the standard conclusion is that there is no non-trivial solution. The result can be reinterpreted as a uniqueness result of the solution

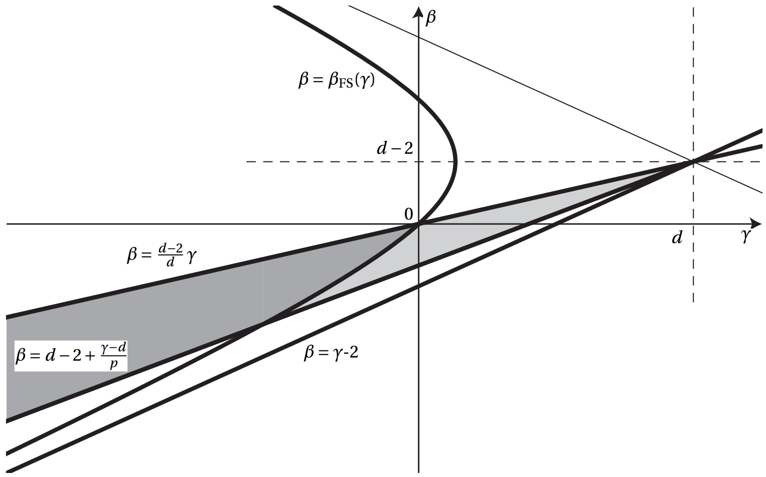

FIGURE 2. *This figure is taken from* [38] *and corresponds to* $d = 4$, $p = 1.2$. *Also see* [14]. *Admissible parameters are in the half-cone in grey. Symmetry breaking occurs in the dark grey area, while optimal functions are radially symmetric in the light grey region.*

$u = 0$. Less known is the fact that the Pohozaev method can also be used to establish uniqueness results: see for instance [61, 62, 43].

- By testing with Δu the elliptic PDE $\Delta u + u^q - \lambda\, u = 0$ where Δ denotes the Laplace-Beltrami operator on a compact Riemannian manifold with positive curvature, B. Gidas and J. Spruck in [47] and M.-F. Bidaut-Véron and L. Véron in [12] were able to prove that in a certain range of the parameter $\lambda > 0$, the unique positive solution is the constant function $u \equiv \lambda^{1/(q-1)}$. The method is actually more involved and computations have to be done with a power of u: see [36]. The approach is related with the *carré du champ*, or Bakry-Emery method, for which we refer to [5, 7]. Nonlinear flows naturally appear as shown in [26, 33]. The method uses estimates on the curvature, but some extensions, like [31, 39] (or, as we shall see, [44]), suggest that variants of the method can also be applied in the case of an Euclidean space.

To determine by which function we have to test (32), a detour by evolution equations is useful. Heuristically, the main idea is to use a nonlinear flow in order to choose the direction in which we vary a critical point and get some additional information which characterizes the solution. In bounded domains or manifolds, the above methods aim at proving *rigidity results*, that is, the fact that there is a simple, well identified solution – typically a constant function – and that this solution is unique. When dealing with unbounded domains, no such trivial solutions have to be expected and more elaborated strategies have to be employed. Natural candidates are of course the self-similar solutions as shown in [25, 19, 20] in the case of Gagliardo-Nirenberg inequalities. However, the *carré du champ* strategy usually relies on self-similar variables and ends in painful computations: see for instance [18]. When dealing with weights, the complexity of the computations has made this approach, so far, untractable. A breakthrough came in [60, 44] with the observation that the use of self-similar variables can actually be avoided: it is enough to prove a concavity property of the *Rényi entropy powers* along the nonlinear flow. This observation has guided the approach in [34, 35, 38]. We are now going to sketch the main steps of the proof.

1) The first step is based on a change of variables which amounts to rephrase our problem in a space of higher, *artificial dimension* $n > d$ (here n is a dimension at least from the point of view of the scaling properties), or to be precise to consider a weight $|x|^{n-d}$ which is the same in all norms. With

$$\alpha = 1 + \frac{\beta - \gamma}{2} \quad \text{and} \quad n = 2\,\frac{d-\gamma}{\beta + 2 - \gamma}\,,$$

we claim that Inequality (29) can be rewritten for a function $v(|x|^{\alpha-1}\,x) = w(x)$ as

$$\|v\|_{2p,d-n} \le \mathsf{K}_{\alpha,n,p}\, \|\mathsf{D}_\alpha\, v\|_{2,d-n}^{\vartheta}\, \|v\|_{p+1,d-n}^{1-\vartheta} \quad \forall\, v \in \mathrm{H}^p_{d-n,d-n}(\mathbb{R}^d)\,, \tag{34}$$

with the notations

$$s = |x|\,, \quad \omega = \frac{x}{s}\,, \quad \mathsf{D}_\alpha\, v = \left(\alpha\,\frac{\partial v}{\partial s}, \frac{1}{s}\,\nabla_\omega v\right).$$

The optimal constant $\mathsf{K}_{\alpha,n,p}$ is explicitly computed in terms of $\mathsf{C}_{\beta,\gamma,p}$ and the condition (30) is equivalent to

$$d\ge 2\,,\quad \alpha>0\,,\quad n>d\quad\text{and}\quad p\in\big(1,p_\star\big]\,.$$

By our change of variables, $w_\star$ is changed into

$$v_\star(x):=\big(1+|x|^2\big)^{-1/(p-1)}\quad\forall\,x\in\mathbb{R}^d\,.$$

The symmetry condition (33) now reads

$$\alpha\le\alpha_{\rm FS}\quad\text{with}\quad\alpha_{\rm FS}:=\sqrt{\frac{d-1}{n-1}}\,. \tag{35}$$

2) In a second step, let us consider the derivative of a generalized *Rényi entropy power* functional, which is defined by

$$\mathcal{G}[u]:=\left(\int_{\mathbb{R}^d}u^m\,d\mu\right)^{\sigma-1}\int_{\mathbb{R}^d}u\,|\mathsf{D}_\alpha\mathsf{P}|^2\,d\mu$$

where $\sigma=\frac{2}{d}\,\frac{1}{1-m}-1$. Here P is the *pressure* variable

$$\mathsf{P}:=\frac{m}{1-m}\,u^{m-1}$$

and the exponents m and p in (34) are related by

$$p=\frac{1}{2\,m-1}\quad\Longleftrightarrow\quad m=\frac{p+1}{2\,p}\,.$$

Notice that $\sigma=1$ if $p=p_\star$ as was considered in [34, 35]. The measure $d\mu=s^{n-1}\,ds\,d\omega=s^{n-d}\,dx$, with $s=|x|$, takes into account the weight. A minimization of $\mathcal{G}$ under a mass constraint $\int_{\mathbb{R}^d}u\,d\mu=M>0$, given, is equivalent to the computation of the optimal constant in (34) and if symmetry holds, then (34) is equivalent to $\mathcal{G}[u]\ge\mathcal{G}[v_\star^{2p}]$ under the condition that $\int_{\mathbb{R}^d}u\,d\mu=M=\int_{\mathbb{R}^d}v_\star^{2p}\,d\mu$. Indeed, up to a numerical constant which is irrelevant, $(\mathcal{G}[u])^{\vartheta/2}$ and M are proportional to, respectively, $\|\nabla w\|_{2,\beta}^{\vartheta}\,\|w\|_{p+1,\gamma}^{1-\vartheta}$ and $\|w\|_{2p,\gamma}^{2p}$. We will actually not try to prove Theorem 18 directly, but consider the action of a nonlinear flow on $\mathcal{G}$. Let us introduce the diffusion operator $\mathcal{L}_\alpha=-\,\mathsf{D}_\alpha^*\,\mathsf{D}_\alpha$, which is given in spherical coordinates by

$$\mathcal{L}_\alpha\,u=\alpha^2\left(u''+\frac{n-1}{s}\,u'\right)+\frac{1}{s^2}\,\Delta_\omega\,u$$

where Δ_ω denotes the Laplace-Betrami operator acting on the $(d-1)$-dimensional sphere $\mathbb{S}^{d-1}$ of the angular variables, and $'$ denotes here the derivative with respect to s. With these notations, we consider the fast diffusion equation

$$\frac{\partial u}{\partial t}=\mathcal{L}_\alpha\,u^m \tag{36}$$

in the subcritical range $1-1/n<m<1$ and in the critical case $m=1-1/n$. The key computation is the proof that

$$\begin{aligned}-\frac{d}{dt}\mathcal{G}[u(t,\cdot)]\left(\int_{\mathbb{R}^d}u^m\,d\mu\right)^{1-\sigma}\quad\ge\quad&(1-m)\,(\sigma-1)\int_{\mathbb{R}^d}u^m\left|\mathcal{L}_\alpha\,\mathsf{P}-\frac{\int_{\mathbb{R}^d}u\,|\mathsf{D}_\alpha\mathsf{P}|^2\,d\mu}{\int_{\mathbb{R}^d}u^m\,d\mu}\right|^2\,d\mu\\&+2\int_{\mathbb{R}^d}\left(\alpha^4\left(1-\tfrac{1}{n}\right)\left|\mathsf{P}''-\tfrac{\mathsf{P}'}{s}-\tfrac{\Delta_\omega\mathsf{P}}{\alpha^2\,(n-1)\,s^2}\right|^2+\tfrac{2\,\alpha^2}{s^2}\left|\nabla_\omega\mathsf{P}'-\tfrac{\nabla_\omega\mathsf{P}}{s}\right|^2\right)u^m\,d\mu\\&+2\int_{\mathbb{R}^d}\left((n-2)\left(\alpha_{\rm FS}^2-\alpha^2\right)|\nabla_\omega\mathsf{P}|^2+c(n,m,d)\,\tfrac{|\nabla_\omega\mathsf{P}|^4}{\mathsf{P}^2}\right)u^m\,d\mu=:\mathcal{H}[u]\end{aligned}$$

for some numerical constant $c(n,m,d)>0$. Hence if $\alpha\le\alpha_{\rm FS}$, the r.h.s. $\mathcal{H}[u]$ vanishes if and only if P is an affine function of $|x|^2$, which proves the symmetry result.

3) This method has a hidden difficulty. In order to justify the above computation, many integrations by parts have to be performed, which require a sufficient decay of the function u and of its derivatives as $|x|\to+\infty$ and also, because of the weight, good properties as $x\to 0$. So far, such properties are not known for a general solution of (36). However, we may consider a positive solution to (32) and, up to the above changes of variables, take the corresponding function u as an initial datum for (36). On the one hand, since u is a critical point of $\mathcal{G}$ under mass constraint, we know that $\frac{d}{dt}\mathcal{G}[u(t,\cdot)]=0$ at $t=0$. On the other hand, because u solves an elliptic PDE, it is possible to establish all regularity and decay estimates that are needed to do the

integrations by parts, hence $\mathscr{H}[u]=0$. In that way we conclude that w is equal to $w_\star$ up to a scaling and a multiplication by a constant if (30) and (33) hold.

Applying the flow at $t=0$ to a critical point amounts to write the Euler-Lagrange equation and test it with $\mathscr{L}_\alpha\,u^m$. In other words, what we write is

$$0=\int_{\mathbb{R}^d}\mathrm{d}\mathscr{G}[u]\cdot\mathscr{L}_\alpha\,u^m\,d\mu\ge\mathscr{H}[u]\ge0$$

where the last inequality holds because $\mathscr{H}[u]$ is the integral of a sum of squares (with nonnegative constants in front of each term). If we undo the change of variables, our method amounts to rewrite (32) as

$$\frac{(p-1)^2}{p\,(p+1)}\,w^{1-3p}\,\mathrm{div}\big(|x|^{-\beta}\,w^{2p}\,\nabla w^{1-p}\big)+|\nabla w^{1-p}|^2+|x|^{-\gamma}\,\big(c_1\,w^{1-p}-c_2\big)=0$$

for some constants c_1, c_2 and test it against $|x|^\gamma\,\mathrm{div}\big(|x|^{-\beta}\,\nabla w^{1+p}\big)$. This is of course strictly equivalent to the approach based on the nonlinear flow, but the guidelines provided by the Rényi entropy powers approach and the *carré du champ* method is definitely very useful to order the computations and prove the positivity result on $\mathscr{H}[u]$.

Let us conclude by a few additional remarks.

• The functional which plays the role of a generalized *Rényi entropy power* functional is $\mathscr{F}[u]:=\big(\int_{\mathbb{R}^d}u^m\,d\mu\big)^\sigma$. The reader is invited to check that $\frac{d}{dt}\mathscr{F}[u(t,\cdot)]=\sigma\,(1-m)\,\mathscr{G}[u(t,\cdot)]$ if u evolves according to (36). Hence the positivity of $\mathscr{H}[u]$ actually means that, as a function of t, $\mathscr{F}[u(t,\cdot)]$ is concave.

• The origin of the various terms in $\mathscr{H}[u]$ can be identified, and this is why the method based on Rényi entropy powers is efficient: the first term in the expression of $\mathscr{H}[u]$ arises from a Cauchy-Schwarz estimate, while the last one is due to the diffusion with respect to the angular variable ω on the sphere $\mathbb{S}^{d-1}$. In the framework of entropy methods, reorganizing algorithmically a functional like $\mathscr{H}$ as a sum of squares using integration by parts and various algebraic manipulations is a difficult task, as it is well known, for instance from [53] in a much simpler setting (also see [52] for a reference work, when there are no weights). No such general algorithm is available so far in cases involving weights.

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