Costly Exploration, Stochastic Discoveries, and the Effect on Proven Reserve and the Price Path of Exhaustible Resources

Ivar Ekeland[*], Wolfram Schlenker[†], Peter Tankov[‡], and Brian Wright[§]

May 9, 2021

Abstract

For the past nine decades Hotelling's rule, implied by optimal consumption of a known finite supply of a commodity, has been the theoretical core of the economics of nonrenewable resources. Yet typical markets for a nonrenewable commodity include, besides known "proven" reserves, a finite quantity of resources available for exploration to find additional reserves. Four decades ago, Arrow and Chang proposed a model including identical reserve deposits Poisson distributed across an unexplored resource, explorable at constant cost with infinite speed, but its full implications remain unclear. We completely solve a dynamic stochastic model of this type for socially optimal exploration, consumption and price. We prove that there is a frontier of critical proven reserves, increasing in explored resources. Given reserves above this frontier, exploration is zero and price follows Hotelling's rule, until consumption reduces reserves to the critical frontier. Then exploration occurs instantaneously across the resource until new discoveries raise reserves above the frontier and price drops, or unexplored area is exhausted and price jumps up to the Hotelling path. The expected price path rises at the rate of interest, and is an upper bound on the expected path conditional on positive unexplored resources, consistent with many tests of Hotelling's rule. Starting with resources below an endogenous threshold, every path of price realizations prior to exhaustion lies below this bound.

[*]CEREMADE, Universite Paris-Dauphine
[†]School of International and Public Affairs and The Earth Institute, Columbia University
[‡]ENSAE, Institut Polytechnique de Paris
[§]Department of Agricultural and Resource Economics, University of California at Berkeley

The seminal paper by Hotelling (1931) showed that the price of an exhaustible resource with zero extraction cost will rise at the rate of interest in both the competitive equilibrium and social optimum. There is an active literature arguing that empirical price series of exhaustible resources violate the Hotelling rule, and what model modifications are necessary. When fitting quadratic time trends to the price path of many exhaustible resources, Slade (1982) found that for a large fraction of resources, there is no positive trend. Several papers have suggested reasons why resource prices initially might not rise over time. First, if there is a positive marginal extraction cost, the rent, defined as the difference between the price and the marginal cost will rise at the rate of interest. A declining price might still result in increasing rents if extraction cost are falling at an even faster rate than the price. Applying a cost function approach to historic price series, Halvorsen and Smith (1991) still reject the Hotelling rule. Second, Berck and Roberts (1996) point out that both demand shocks (energy consumption is co-integrated with GDP, which is itself difference stationary) and supply shocks (e.g., new discoveries) lead to permanent price shocks. The price series is difference and not trend stationary and fitting time trends in the price level will give biased test results. While negative price shocks are possible, a declining price path for many decades would suggest a succession of negative shocks in a row, which eventually will appear unlikely. Third, exploratory activity is sometimes allowed to increase reserves. Pindyck (1978) models exploration as potentially yielding unlimited reserves, but given amounts of exploratory activity result in ever smaller discoveries. The result in this deterministic model is a U-shaped price path where the initially cheap technology is used to build up reserves before further exploration becomes too costly and reserves are drawn down again. The deterministic model implies a smooth price-path, which does not fit realized price series as well. Our paper examines the implication of a costly stochastic exploration process that resolves how many reserves are available. We expand on the proposed model setup of (Arrow and Chang 1982) that assumes there is unexplored area, which we standardize without loss of generality to the the interval $[0, 1]$. Arrow and Chang (1982) conclude that "The price history will show fluctuations with little upward trend when X is large; presumably the upward trend is stronger as X approaches zero, but this requires a probabilistic analysis not yet performed." We present such a probabilistic analysis. This unexplored area harbors an unknown amount of reserves, which are assumed to be Poisson distributed. Equivalently, the amount of unexplored area between successive finds of constant size a is exponentially distributed. Costly exploration will reveal how many reserves are "hidden" in this unexplored area. Knowing the exact amount of reserves would allow optimal allocation of the stock over time. On the other hand, delaying costly exploration into the future allows them to be further discounted. The model solves for the optimal time when to engage in costly exploration activity to further reduce this uncertainty. We purposefully assume constant marginal extraction cost to illustrate the economic incentive on when to explore. Convex exploration cost would give an incentive to spread the exploration activity over time and obfuscate the economic value of information obtained from the search process.

Before we present and solve this model, we would like to highlight the importance of new discoveries for exhaustible resources, sometimes also called the cake eating problem.

The Limit to Growth Literature in the 1970s emphasized that keeping current consumption rates would deplete them by 2030 the latest. This prediction has not materialized. The reason was not that we were using less of these resources over time. Quite to the contrary, for most minerals, consumption has increased over time and the cumulative use since 1970 was larger than the proven reserves in 1970. To the contrary, new discoveries resulted in additional reserves, pushing the current reserve level to higher levels than what was available in 1970 as new discoveries exceeded cumulative consumption. In the following we refer to the unexplored area that can yield new discoveries as *resources*, and the proven reserve stock simply as *reserves*.

We provide one specific example: Rystad Energy provides historic field-level estimates of proven reserves for crude oil. This micro-level data set has been used to study the misallocation of oil production around the world (Asker, Collard-Wexler and De Loecker 2019) and the effects of a carbon tax on oil prices and CO_2 emissions (Heal and Schlenker 2020). Figure 1 plots the level of reserves from 1950-2019 as well as annual production. For reserves, we merge each asset, the unit of analysis in the data set that are "producing," "Discovery," or "Under development" with estimates of the size of the reserve to obtain the red line. The blue line shows production per year (note the different scale by a factor of 33). Overlapping lines suggest that 33 years at current production levels are left. The striking feature is twofold: first, cumulative production from 1970 onward (area under the blue curve) far exceeds reserves in 1970, i.e., the Limits to Growth literature prediction that current reserves would run out in the early 2000s was correct. Second, the proven reserves do not decrease over time, suggesting that new discoveries far outpaced production level.[1] New discoveries and the effect on the reserve stock are an important empirical component of some exhaustible resources, and we study the discovery process further in this paper.

Before we dive into the formal technical details of the model, some intuition might be helpful. The optimal exploration policy balances two countervailing effects: On the one hand one would like to delay the exploration process as long as possible to defer these costs to the future. On the other had one would like to start the exploration process as soon as possible to reduce uncertainty about the remaining reserve stock and improve the intertemporal allocation of consumption. When reserves, and hence consumption, are abundant, the cost of not knowing the exact amount of the reserves becomes less severe as marginal utility is low and changing the optimal consumption path would not impact the overall value tremendously. On the other hand, when reserve stocks are very low, it becomes imperative to know how much is left in the ground for the optimal consumption pathway, as changes in the consumption rate have a higher marginal value. We establish below that there are critical reserve and price levels. If reserves exceed the critical level, or price is lower than the critical level, one follows the standard Hotelling path of consumption, where price rises at the rate of interest, reserves are drawn down, and there is no exploration. Once the critical level, which depends on the unexplored area, is reached, exploration starts. Since marginal

[1] The separation into what are "proven reserve" and what are estimates of remaining "undiscovered reserves" is somewhat arbitrary, but the qualitative finding that reserve levels are increasing over time is common across data sources. For example, another data set, BP Statistical Review of World Energy, found that proven oil reserves increased even more, from 682 billion barrels in 1980 to 1.7 trillion barrels in 2019.

exploration costs are zero, exploration happens at an infinite rate in zero time. The explored area until the next discovery is exponentially distributed. Once a new reserve of constant size a is found, there are two possibilities: first, if the new discovery pushes the known reserve level above the critical threshold, which itself might have increased as it is a function of the explored area, exploration stops and one reverts to consuming the reserve until it is drawn down to the critical level again. Second, if the size of the new discovery is lower than the increase in the critical reserve level compared to when exploration was started, the new reserve level remains below the critical level and one will immediately start the next exploration process.

There are several important implications of the model. First, the critical level when exploration starts is increasing in the explored area. As a result, proven reserves will increase over time as long as the unexplored area is not exhausted. The reserve stock is not a viable indicator of scarcity, i.e., the fact that reserves are increasing over time, as is observed for oil, does not imply that scarcity decreases.

Second, while prices rise at the rate of interest in expectation, the distribution of future prices is highly skewed. Imagine a case where there is a lot of unexplored area left. The probability of not finding any further discoveries is very low, as the amount of area until the next discovery is exponentially distributed. While this case is very rare, it would constitute a large surprise with a very large price jump. The other extreme is that almost no area is required to find the new reserve of constant size a. Price would drop, but the drop is bound by having additional reserves in the amount of a. This leads to a highly asymmetric distribution of future prices. This is comparable to a roulette game with 36 numbers, where the payoff is 36 times the bet, i.e., it is fair in expectations. While the expected winning is zero, a player will loose with very high probability, i.e., 35 out of 36 times.

Third, we show that in the limit as the unexplored area approaches zero, finding another reserve will lead to a price drop, while running out of unexplored area will lead to a price increase. Going backward, conditional on not having run out of unexplored area (which eliminates the largest price increases), the observed price path rises at less than the rate of interest. This gives an explanation why forward-looking tests of the Hotelling rule generally due not reject it (Miller and Upton 1985), while backward looking tests do (Halvorsen and Smith 1991).

Lastly, there is also a methodological innovation: Continuous Exploration region, smooth pasting price

1 Model Setup

Consider the familiar cake-eating problem (Hotelling 1931). There is a single, infinite-lived consumer, who discounts future utilities at the constant rate $r > 0$. There is a single, non-renewable, good, available in finite quantity $R_0 > 0$. If the agent consumes quantity c in the infinitesimal time interval $[t,\ t + dt]$, he/she derives utility $u(c)\,dt$. In Hotelling's paper the utility is measured in monetary units (as a social value drawn from consumption) and it is assumed that the rate of time preference r coincides with the market rate of interest. The agent's problem then is to adjust his/her consumption so as to maximise his/her aggregate utility. Mathematically speaking, this translates into the optimization problem:

$$\begin{aligned}
\mathcal{U}(R_0) &= \max_{c(\cdot)} \int_0^\infty u\left(c\left(t\right)\right) e^{-rt} dt \qquad \text{subject to} \\
\frac{dR}{dt} &= -c, \quad c\left(t\right) \geq 0 \\
R\left(0\right) &= R_0.
\end{aligned}$$

Here $\mathcal{U}(R_0)$ is the present value of the future utility the agent draws from the quantity R_0 of the good, $c(t)$ is the rate of consumption at time t, $R(t)$ is the remaining quantity of the good, r is the interest rate and the utility function $u : (0,\ \infty) \to \mathbb{R}$ is concave, increasing and C^2, with $u''\left(c\right) < 0$ for $c > 0$. The optimal solution $c\left(t\right)$ is easily found by the method of Lagrange multipliers:

$$u'\left(c\left(t\right)\right) = \lambda e^{rt}, \tag{1}$$

where the Lagrange multiplier λ is determined from the condition that the lifetime consumption must be equal to the initial quantity of the good:

$$\int_0^\infty c\left(t\right) dt = R_0. \tag{2}$$

By standard arguments, the Lagrange multiplier λ is equal to the derivative of the value function, and since the utility is measured in monetary units, it can also be interpreted as the price of the good at time $t = 0$:

$$\mathcal{U}'(R_0) = \lambda = p_0.$$

Repeating the same argument at time t, the price at time t may be found:

$$p_t = \mathcal{U}'(R_t) = \lambda e^{rt} = p_0 r^{rt}.$$

In the words of Hotelling, "It is a matter of indifference to the owner whether he receives for a unit of his product a price p_0 now or a price $p_0 e^{rt}$ after time t". In this simple Hotelling model of optimal consumption of a finite stock of reserves R, rising price and falling reserves are both signals of increasing scarcity.

We extend this model to include a known stock of a potentially mineral-bearing resource, identified for example by its observable geological characteristics, containing an unknown number of deposits. This dynamic stochastic problem of optimization of social welfare entails choices regarding both the rate of consumption and the timing of exploration.

Exploration of the unit interval of the resource $[0,\ 1]$ proceeds from left to right. There are two state variables, $0 < x < 1$, the space already explored, and $R > 0$, the reserves already discovered (henceforth called the reserves).

We assume that exploration is instantaneous. During exploration, there is zero consumption; during consumption, nothing is explored.

If exploration starts from x, explored resources move immediately to $x' = \inf\{x+h, 1\}$ where $h > 0$ is the first jump time of a Poisson process with parameter λ. If $x' = 1$, unexplored resources are exhausted, no additional deposits are found, and the model transitions to the deterministic Hotelling consumption regime, with initial resources R. If $x' < 1$, then a deposit of $a > 0$ of reserves has been found at x', and the system moves immediately from (x, R) to the new state $(x', R+a)$. If exploration is costless, the whole resource is explored at the outset, allowing optimization of subsequent consumption exploiting knowledge of the total stock of reserves by following the deterministic Hotelling path. Henceforth, we assume a cost $k > 0$ for exploration of the resource: if we start exploring at x and find a deposit at x', then the exploration cost incurred is $k(x'-x)$. We completely solve the consumption and exploration problem for this initial regime.

Our results are as follows. The agent's strategy is completely determined by the critical reserve level as function of unexplored area $R^* : [0,1] \to (0,\infty)$, such that it is optimal to consume when the reserves are above this critical level and explore otherwise. This critical reserve level is a smooth increasing function of the unexplored area. For given unexplored area x, there is a one-to-one correspondence between reserves and price, hence one can also define the critical price level, such that it is optimal to consume when price is below the critical level and explore otherwise. The optimal strategy is represented in Figure 4, where the critical levels are shown in blue. If the initial reserves are above the critical reserve level (point A), the agent consumes until both price and reserve level hit their respective critical levels (point B), after which an exploration period starts (segment BC). Upon finding a new deposit, the reserve level increases by amount a (segment CD). If the new deposit is found quickly, the new reserve level (point D) will be well above the critical level, consequently the new price level (point D) will be well below the critical level and below its previous value (point B), so that new find leads to a price drop. If the new deposit takes a long time to find (segment EF), the new reserve level (point G) may end up just above the critical level and the new price (point G) will be just below the critical level and above its previous value (point E), so that the new find leads to a price rise. It may also happen that the new reserve level will still be below the critical level; in this case exploration will continue without consumption. Finally, it may happen that the entire area is explored and no new deposit is found (segment HI): in this case the new price (point I) will be above the critical level, thus the price always jumps upward at the end of the exploration regime.

In finite time the entire interval [0, 1] is explored, and society recognizes that there are no undiscovered reserves. After an upward price jump the model transitions from the exploratory regime to the deterministic regime in which the price path follows the Hotelling rule.

This regime-ending jump occurs with positive probability in any exploratory episode. This fact has important implications for price behavior within the exploratory regime in which $x < 1$. In this regime, the expected price path follows the Hotelling rule as the horizon recedes past the next exploratory episode. Since price jumps upward if the regime ends at that episode, it must jump down in expectation, conditional on remaining within the exploratory regime, and then rise smoothly at the discount rate to the next exploratory episode. In general, realized price in the exploratory regime has this sawtooth pattern. However, if a new deposit is discovered after a very large interval of unsuccessful exploration, price may also jump up upon discovery.

When the probability of finding a deposit is high, we can further distinguish two stages in the exploration regime. First, when the unexplored area is likely to still contain many deposits, reserves are abundant, the cost of not knowing the exact amount of reserves is less severe and it is optimal to defer costly exploration. In this stage, the critical reserve level is very close to zero and all new discoveries are immediately consumed (see Figure 15). As the uncertainty over the remaining amount of resources grows, at some point it is no longer optimal to consume all discoveries immediately and the reserves start to accumulate, as a sign of increasing scarcity.

Starting in a state with resources above an endogenous resource threshold, the realized price path might jump above the initial Hotelling path of expected prices if multiple deposits are discovered in a single exploratory episode, a perhaps counter-intuitive negative signal of total consumable reserves, reflecting at least one unusually large interval of unsuccessful exploration in that episode. Starting with resources below this threshold, the fluctuating price path is bounded above by the expected path, until resource exhaustion. Taken together, these two results falsify a conjecture of Arrow and Chang that the upward trend in realized price in a similar model is strongest as unexplored resources approach exhaustion.

This paragraph remains unclear

The structure of the paper is as follows. In section 2 we define the value function $V(x,R)$ and we characterize it as the solution of a suitable HJB equation (Theorem 1). In section 3, we prove the existence of a free boundary $R^*(x)$ separating the exploration region below from the consumption region above (Proposition 2) and we prove that $0 < R^*(x) < \infty$ for all $x < 1$ (as long as there is an unexplored region, there is a level of reserves above which one consumes and below which one explores). We also prove that $R^*(x)$ is an increasing function of x and that the smooth pasting condition holds across the free boundary. It is this result that implies that the price path realized in the exploratory regime becomes bounded above by the path of price expectations when exploratory resources pass below an endogenous fixed threshold. In section 4, we investigate the intersection of the free boundary $R^*(x)$ with $x = 1$. We compute the point of intersection $R^*(1)$ and the tangent $\frac{dR^*}{dx}(1)$: we find that it is positive, in line with our general result that R^* is increasing. These data allow us to check the numerics, that is, the accuracy of the algorithm we are using to compute $V(x,R)$.

Finally, we check that $\mathbb{E}[p_t] = p_0 e^{rt}$, the conditional expectation of price follows the Hotelling rule. This does NOT mean that every realization will follow the Hotelling rule, even loosely, as our simulations show. Indeed, the last section is devoted to numerical simulations, and shows a wide range of behaviours with minimum reserves increasing, and sawtooth price paths realized in the exploration region falling below the the path of price expectations as explorable resources decline.

In a concluding section, we relate our results to the literature and address their generality.

Throughout the paper, we will take $u(c) = \frac{1}{\alpha}c^{\alpha}$, with $0 < \alpha < 1$, and sometimes $\alpha = 1/2$. This is to make computations easier, but there is no reason our results would not hold for more general u with $u''(c) > 0$ for $c > 0$.

2 Solving the Model

2.1 The HJB equation

In this section we characterize the value function of our optimal stopping problem as the solution of a variant of the Hamilton-Jacobi-Bellman equation. Let N denote a Poisson process on the interval $[0,1]$ with intensity λ, which models our stochastic exploration process. The jump times of N, which correspond to the locations of the deposits, will be denoted by $(x_k)_{k\geq 0}$, with $x_0 = 0$, and the interarrival times of N, which correspond to the area that has to be explored before finding the next deposit, will be denoted by $\Lambda_n := x_n - x_{n-1}$ for $n \geq 1$. Finally, we let $(\mathcal{F}_k)_{k\geq 0}$ be the discrete filtration generated by $(x_k)_{k\geq 0}$. A consumption-exploration strategy is an increasing $(\mathcal{F}_k)$-adapted sequence of random variables $(\theta_k)_{k\geq 0}$ (moments where the exploration starts) and a sequence of $\mathcal{F}_k \times \mathcal{B}(\mathbb{R}_+)$-measurable maps $c_k : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$, where the consumption process of the agent is given by

$$c_t = \sum_{k\geq 0} c_t^k \mathbf{1}_{\theta_k < t \leq \theta_{k+1}}.$$

We say that the consumption-exploration strategy (c, θ) is admissible for initial data (x, R) if it satisfies the budget constraint

$$R - \int_0^t c_s ds + a \sum_{n=1}^{N_{1-x}} \mathbf{1}_{\theta_n \leq t} \geq 0 \tag{3}$$

a.s. for all $t \geq 0$. Equation (3) ensures that the reserve level remains positive at all times.

The value function is defined by

$$V(x, R) = \sup_{c,\theta} \mathbb{E}\left[\int_0^{\infty} e^{-rt} u(c_t) dt - k \sum_{n=1}^{\infty} e^{-r\theta_n} \Lambda_n \wedge (1 - x(\theta_n))\right] \tag{4}$$

where the supremum is evaluated over admissible consumption-exploration strategies.

The following lemma provides an priori upper bound for V, instrumental in the proof of the HJB equation.

Lemma 1. *Assume that $u(c) = \frac{c^{\alpha}}{\alpha}$ with $0 < \alpha < 1$. Then, for all $x, R \in [0,1] \times \mathbb{R}_+$ and all $h > 0$,*

$$V(x, R+h) \leq V(x, R) + \left(1 - \left(1 - \frac{h}{R}\right)^{\alpha}\right) \mathbb{E}[U(R + h + aN_{1-x})], \quad h \geq 0,$$

and in particular,

$$V(x, R+h) \leq V(x, R) + \frac{\alpha h}{R} \mathbb{E}[U(R + h + aN_{1-x})], \quad h \geq 0.$$

Proof. Let (c, θ) be a consumption-exploration strategy with

$$R + h - \int_0^t c_s ds + a \sum_{n=0}^{N_{1-x}-1} \mathbf{1}_{\theta_n \le t} \ge 0 \tag{5}$$

a.s. for all $t \ge 0$. Then, clearly,

$$R - \int_0^t \frac{R}{R+h} c_s ds + a \sum_{n=0}^{N_{1-x}-1} \mathbf{1}_{\theta_n \le t} \ge 0$$

a.s. for all $t \ge 0$, so that $\left(\frac{R}{R+h} c, \theta\right)$ is an admissible consumption-exploration strategy for (x, R). Then,

$$\begin{aligned} V(x, R+h) - V(x, R) &\le \sup_{c,\theta} \mathbb{E}\left[\int_0^\infty e^{-rt}\left(u(c_t) - u\left(\frac{R}{R+h} c_t\right)\right) dt\right] \\ &= \left(1 - \left(\frac{R}{R+h}\right)^\alpha\right) \sup_{c,\theta} \mathbb{E}\left[\int_0^\infty e^{-rt} u(c_t) dt\right], \end{aligned}$$

where the supremum is taken over consumption-stopping strategies satisfying equation (5). Since in this optimization problem there is no cost, it is optimal to explore all remaining area immediately, and we have

$$\sup_{c,\theta} \mathbb{E}\left[\int_0^\infty e^{-rt} u(c_t) dt\right] = \mathbb{E}[U(R + h + aN_{1-x})].$$

□

We now proceed as usual to show that the value function satisfies a variant of the HJB equation. First we formulate without proof the following dynamic programming principle for our value function. The proof can be done along the lines of Theorem A.1 in Pham and Tankov (2009).

Lemma 2 (Dynamic programming principle)**.** *The value function satisfies:*

$$V(x, R) = \sup_{c, \theta_1 \ge 0} \int_0^{\theta_1} e^{-rt} u(c_t) dt + e^{-r\theta_1} MV(x, R - \int_0^{\theta_1} c_s\, ds),$$

where the sup *is taken over all measurable deterministic functions* $c : \mathbb{R}_+ \to \mathbb{R}_+$ *and constants* $\theta_1 \in \mathbb{R}_+$ *such that* $\int_0^{\theta_1} c_s ds \le R$*, and the operator* M *is defined by*

$$Mf(x, R) = \int_0^{1-x} f(x + h, R + a)\lambda e^{-\lambda h} dh + f(1, R)e^{-\lambda(1-x)} - k\frac{1 - e^{-\lambda(1-x)}}{\lambda}.$$

Corollary 1. *For all* $h > 0$*,* $\hat{c} : [0, h] \to \mathbb{R}_+$*, such that* $\int_0^h \hat{c}_s ds \le R$*,*

$$V(x, R) \ge \int_0^h e^{-rt} u(\hat{c}_t) dt + e^{-rh} V\left(x, R - \int_0^h \hat{c}_s ds\right).$$

Proof. Let $\tilde{c} : \mathbb{R}_+ \to \mathbb{R}_+$ and $\tilde{\theta}_1 \in \mathbb{R}_+$. Applying the dynamic programming principle to

$$\begin{aligned} c: &\quad \mathbb{R}_+ \to \mathbb{R}_+, \quad t \mapsto \hat{c}_t \mathbf{1}_{t<h} + \tilde{c}_{t+h} \mathbf{1}_{t\geq h} \\ \theta_1 &\quad = h + \tilde{\theta}_1, \end{aligned}$$

we get:

$$\begin{aligned} V(x,R) &\geq \int_0^h e^{-rt} u(\hat{c}_t) dt + \int_h^{h+\tilde{\theta}_1} e^{-rt} u(\tilde{c}_{t+h}) dt + e^{-r(h+\theta_1)} MV \left(x, R - \int_0^h \hat{c}_t dt + \int_h^{h+\tilde{\theta}_1} \tilde{c}_{t+h} dt \right) \\ &= \int_0^h e^{-rt} u(\hat{c}_t) dt + e^{-rh} \left\{ \int_0^{\tilde{\theta}_1} e^{-rt} u(\tilde{c}_t) dt + e^{-r\theta_1} MV \left(x, R - \int_0^h \hat{c}_t dt + \int_0^{\tilde{\theta}_1} \tilde{c}_t dt \right) \right\}. \end{aligned}$$

Taking the sup over $\tilde{c}$ and $\tilde{\theta}_1$, we get the statement of the corollary. □

Corollary 2. *The value function satisfies:*

$$V(x,R) = \sup_{0\leq Q\leq R, \theta_1 \geq 0} \{\widetilde{U}(\theta_1, Q) + e^{-r\theta_1} MV(x, R-Q)\},$$

where

$$\widetilde{U}(\theta_1, Q) = \frac{Q^\alpha}{\alpha} \left(\frac{1-\alpha}{r} \right)^{1-\alpha} (1 - e^{-\frac{r\theta_1}{1-\alpha}})^{1-\alpha}.$$

Proposition 1. *The value function $V(x,R)$ is the solution of the HJB equation*

$$\max \left\{ u^* \left(\frac{\partial V}{\partial R} \right) - rV, MV - V \right\} = 0, \quad V(1,R) = U(R),$$

meaning that

i. For all $R \geq 0$, $x \in [0,1]$, $V(x,R) \geq MV(x,R)$.

ii. For all $R > 0$, $x \in [0,1]$,

$$u^* \left(\liminf_{h\to 0} \frac{V(x,R+h) - V(x,R)}{h} \right) \leq rV(x,R).$$

iii. At all points (x,R) such that $R > 0$ and $V(x,R) > MV(x,R)$, V is differentiable in R and satisfies

$$u^* \left(\frac{\partial V}{\partial R} \right) = rV.$$

Conversely, if a function $\widetilde{V}(x,R)$ is locally Lipschitz continuous in R on $(0,\infty)$ for every $x \in [0,1]$, satisfies the properties i.-iii. above, and admits the bound

$$\widetilde{V}(x,R) \leq C(1 + U(R)), \quad R \geq 0,$$

for some $C < \infty$, it is given by equation (4).

Proof. *First part.* It follows from Lemma 2 that $V \geq MV$. Let us prove the property ii. In other words, we need to show that for all $c > 0$,

$$u(c) - c \liminf_{h \to 0} \frac{V(x, R+h) - V(x,R)}{h} \leq rV. \tag{6}$$

Fix $h < R$ and $\bar{c} > 0$. From Corollary 1,

$$V(x,R) \geq \int_0^{h/\bar{c}} e^{-rt} u(\bar{c}) dt + e^{-rh/\bar{c}} V(x, R-h),$$

and therefore

$$\frac{V(x,R+h) - V(x,R)}{h} \geq u(\bar{c}) \frac{1 - e^{-rh/\bar{c}}}{hr} - \frac{1 - e^{-rh/\bar{c}}}{h} V(x,R),$$

from which equation (6) follows by passing to the limit.

Let us now turn to property iii. By Corollary 2 and continuity of V and MV, there exists $\varepsilon > 0$ such that $V(x,R') > MV(x,R')$ for all R' with $|R - R'| < \varepsilon$ and

$$V(x,R) = \sup_{\varepsilon \leq Q \leq R, \theta_1 \geq 0} \{\widetilde{U}(\theta_1, Q) + e^{-r\theta_1} MV(x, R-Q)\}.$$

Then,

$$\begin{aligned}
V(x,R) &= \sup_{c, \theta_1 \geq 0 : \int_0^{\theta_1} c_s ds > \varepsilon} \int_0^{\theta_1} e^{-rt} u(c_t) dt + e^{-r\theta_1} MV(x, R - \int_0^{\theta_1} c_s \, ds) \\
&= \sup_{c, \theta_1 \geq \tau : \int_0^{\tau} c_s ds = \varepsilon} \int_0^{\tau} e^{-rt} u(c_t) dt + \int_\tau^{\theta_1} e^{-rt} u(c_t) dt + e^{-r\theta_1} MV(x, R - \varepsilon - \int_\tau^{\theta_1} c_s \, ds) \\
&= \sup_{c, \tau \tau : \int_0^{\tau} c_s ds = \varepsilon} \int_0^{\tau} e^{-rt} u(c_t) dt + e^{-r\tau} V(x, R-\varepsilon) \\
&= \sup_{\tau \geq 0} \{\widetilde{U}(\tau, \varepsilon) + e^{-r\tau} V(x, R - \varepsilon)).
\end{aligned}$$

Remark that

$$\widetilde{U}(\tau, Q) \leq \frac{Q^\alpha}{\alpha r^{1-\alpha}} (1 - e^{-r\tau})^{1-\alpha}.$$

Thus,

$$V(x,R) \leq \sup_\tau \{ \frac{\varepsilon^\alpha}{\alpha r^{1-\alpha}} (1 - e^{-r\tau})^{1-\alpha} + e^{-r\tau} V(x, R-\varepsilon)\}$$

The first-order condition for the maximization in the RHS writes:

$$1 - e^{-r\tau} = \varepsilon \left(\frac{V(x, R-\varepsilon) \alpha r^{1-\alpha}}{1-\alpha} \right)^{-\frac{1}{\alpha}},$$

which provides an upper bound:

$$\frac{V(x,R)-V(x,R-\varepsilon)}{\varepsilon} \leq \left(\frac{\alpha}{1-\alpha} rV(x,R-\varepsilon)\right)^{\frac{\alpha-1}{\alpha}}.$$

On the other hand, from part ii., we have

$$\liminf_{\varepsilon\to 0} \frac{V(x,R)-V(x,R-\varepsilon)}{h} \geq \left(\frac{\alpha}{1-\alpha} rV(x,R-\varepsilon)\right)^{\frac{\alpha-1}{\alpha}}.$$

Together, the two inequalities show the differentiability of V and complete the proof of the first part.

Second part. Assume now that the function satisfies the assumptions of the proposition and let us show that it coincides with the value function. Let (θ, c) be an admissible consumption and stopping strategy. We define the dynamics of the state variables:

$$\begin{aligned} x(t) &= x + N^{-1}(n_t) \wedge (1-x), \quad \text{where} \quad n_t = \max\{n : \theta_n \leq t\}, \\ R(t) &= R - \int_0^t c_s ds + a(n_t \wedge N_{1-x}). \end{aligned}$$

$$e^{-rT}\widetilde{V}(x(T),R(T)) - \widetilde{V}(x,R) = \int_0^T e^{-rt}(-r\widetilde{V}(x(t),R(t)) - \frac{\partial \widetilde{V}}{\partial R} c_t)dt$$
$$+\sum_{n=1}^{\infty} \mathbf{1}_{\theta_n \leq T} e^{-r\theta_n}\{\widetilde{V}((x(\theta_n-)+\Lambda_n)\wedge 1, R(\theta_n-)+a\mathbf{1}_{x(\theta_n-)<1}) - \widetilde{V}(x(\theta_n-),R(\theta_n-))\}$$

so that

$$\begin{aligned} \widetilde{V}(x,R) &\geq \mathbb{E}\left[\int_0^T e^{-rt}u(c_t)dt\right] + \mathbb{E}[e^{-rT}\widetilde{V}(x(T),R(T))] \\ &\quad -\mathbb{E}\left[\sum_{n=1}^{\infty} \mathbf{1}_{\theta_n\leq T} e^{-r\theta_n} \mathbb{E}[\widetilde{V}((x(\theta_n-)+\Lambda_n)\wedge 1, R(\theta_n-)+a\mathbf{1}_{x(\theta_n-)<1}) - \widetilde{V}(x(\theta_n-),R(\theta_n-))|\mathcal{F}_{n-1}]\right] \\ &= \mathbb{E}\left[\int_0^T e^{-rt}u(c_t)dt\right] + \mathbb{E}[e^{-rT}\widetilde{V}(x(T),R(T))] \\ &\quad -\mathbb{E}\left[\sum_{n=1}^{\infty} \mathbf{1}_{\theta_n\leq T} e^{-r\theta_n}\{M\widetilde{V}(x(\theta_n-),R(\theta_n-)) - \widetilde{V}(x(\theta_n-),R(\theta_n-)) + k\frac{1-e^{-\lambda(1-x(\theta_n-))}}{\lambda}\}\right] \\ &\geq \mathbb{E}\left[\int_0^T e^{-rt}u(c_t)dt - k\sum_{n=1}^{\infty} \mathbf{1}_{\theta_n\leq T}\Lambda_n \wedge (1-x(\theta_n-))\right] + \mathbb{E}[e^{-rT}\widetilde{V}(x(T),R(T))], \end{aligned}$$

where we have used the fact that $\theta_n \in \mathcal{F}_{n-1}$ and Λ_n is independent from $\mathcal{F}_{n-1}$. As $T \to \infty$,

$$\begin{aligned} 0 &\leq \lim_{T\to\infty} \mathbb{E}[e^{-rT}\widetilde{V}(x(T),R(T))] \\ &\leq \lim_{T\to\infty} C\mathbb{E}[e^{-rT}(1+U(R(T)))] \leq \lim_{T\to\infty} C\mathbb{E}[e^{-rT}(1+U(R+aN_{1-x}))] = 0, \end{aligned}$$

so that

$$\widetilde{V}(x,R) \geq \mathbb{E}\left[\int_0^\infty e^{-rt}u(c_t)dt - k\sum_{n=1}^{\infty} \mathbf{\Lambda}_n \wedge (1 - x(\theta_n-))\right],$$

and since the consumption-exploration strategy was arbitrary,

$$\widetilde{V}(x,R) \geq V(x,R).$$

Now consider the specific consumption-exploration strategy defined as follows:

$$\begin{aligned} c_t &= I\left(\frac{\partial \widetilde{V}}{\partial R}(x(t),R(t))\right)\mathbf{1}_{\widetilde{V}(x(t),R(t))>M\widetilde{V}(x(t),R(t))} \\ \theta_n &= \inf\{t \geq \theta_{n-1} : \widetilde{V}(x(t),R(t)) = M\widetilde{V}(x(t),R(t))\}. \end{aligned}$$

Then the inequalities above hold as equalities and we conclude that

$$\widetilde{V}(x,R) = V(x,R).$$

□

2.2 Characterization of Consumption and Exploration Regions

We denote by u^* the convex conjugate of u and by u_1 the inverse of u^*. Recall that the operator M is defined by

$$Mf(x,R) = \int_0^{1-x} f(x+h,R+a)\lambda e^{-\lambda h}dh + f(1,R)e^{-\lambda(1-x)} - k\frac{1-e^{-\lambda(1-x)}}{\lambda}.$$

In particular,

$$MV(x,0) \geq (1-e^{-\lambda(1-x)})\left(U(a) - \frac{k}{\lambda}\right).$$

In the following we make the natural assumption that $U(a) > \frac{k}{\lambda}$. This assumption guarantees that it is optimal to explore at zero reserve level, and therefore the set $\{(x,R) : R = 0\}$ belongs to the exploration region and $V(x,0) = MV(x,0) > 0$ for all $x \in [0,1)$.

Lemma 3. *For all $(x,R) \in [0,1] \times \mathbb{R}_+$, $MV(x,R)$ is infinitely differentiable and satisfies*

$$\begin{aligned} MV(x,R) &= \mathbb{E}[V(x(\tau_C),R(\tau_C)) - k(x(\tau_C) - x)] \\ MV_R^{(n)}(x,R) &= \mathbb{E}[V_R^{(n)}(x(\tau_C),R(\tau_C))], \quad n \geq 1, \end{aligned}$$

where $R(k) = R + a(k \wedge N_{1-x})$, $x(k) = x + \theta_k \wedge (1-x)$, $(\theta_k)_{k\geq 1}$ is the sequence of jump times of a Poisson process $(N_t)_{t\geq 0}$ with intensity λ and $\tau_C = \inf\{k \geq 1 : V(x(k),R(k)) > MV(x(k),R(k))\}$.

Proposition 2. *Assume that* $u(c) = \frac{c^\alpha}{\alpha}$*. Then for every* $x \in [0,1)$ *there exists* $R^*(x) \in [0,\infty]$ *such that* $V(x,R) > MV(x,R)$ *for all* $R > R^*(x)$ *and* $V(x,R) = MV(x,R)$ *for all* $R \le R^*(x)$*.*

Proof. Let $\mathcal{C} = \{R : V(x,R) > MV(x,R)\}$. Since V and MV are continuous in R, the set $\mathcal{C}$ is open, and is therefore a union of disjoint open intervals. To prove the proposition it is enough to show that none of these intervals is bounded. By way of contradiction, assume that $(a,b) \subset \mathcal{C}$ is a bounded interval such that $V(x,a) = MV(x,a)$ and $V(x,b) = MV(x,b)$. Consider the function $f : [a,b] \to \mathbb{R}_+$ defined by $f(R) = V(x,R) - MV(x,R)$. By Proposition 1 and Lemma 3, f is infinitely differentiable on (a,b), hence there exists $\bar{R} \in (a,b)$ such that $f'(\bar{R}) = 0$ and $f''(\bar{R}) \le 0$. By Lemma 3, this means that

$$\begin{aligned} V_R'(x,R) &= \mathbb{E}[V_R'(x(\tau_C), R(\tau_C))] \\ V_R''(x,R) &\le \mathbb{E}[V_R''(x(\tau_C), R(\tau_C))], \end{aligned}$$

and by Proposition 1, this is equivalent to

$$\begin{aligned} V(x,R)^{1-\frac{1}{\alpha}} &= \mathbb{E}[V(x(\tau_C), R(\tau_C))^{1-\frac{1}{\alpha}}] \\ V(x,R)^{1-\frac{2}{\alpha}} &\ge \mathbb{E}[V(x(\tau_C), R(\tau_C))^{1-\frac{2}{\alpha}}]. \end{aligned}$$

Let $Z = V(x(\tau_C), R(\tau_C))^{1-\frac{1}{\alpha}}$. The above estimates imply that

$$\mathbb{E}[Z^{\frac{2-\alpha}{1-\alpha}}] \le \mathbb{E}[Z]^{\frac{2-\alpha}{1-\alpha}},$$

and since Z is positive and the function $x \mapsto x^{\frac{2-\alpha}{1-\alpha}}$ is convex on $\mathbb{R}_+$, Jensen's inequality implies that Z is deterministic, which is a contradiction unless $x = 1$. □

Proposition 3. *Assume that* $u(c) = \frac{c^\alpha}{\alpha}$ *and that* $U(a) > \frac{k}{\lambda}$*. There exists a function* $R^* : [0,1) \mapsto (0,\infty)$*, such that for every* $x \in [0,1)$*, the points* $\{(x,R) : R \le R^*(x)\}$ *belong to the exploration region.*

Proof. In view of our assumption $V(x,0) = MV(x,0) > 0$ for $x < 1$: when there are no reserves left, it is optimal to explore since exploration ensures positive utility. Let us define

$$R^*(x) = \max\{R > 0 : (1 - e^{-\lambda(1-x)})(U(R+a) - k/\lambda) - \{e^{\frac{\alpha\lambda}{1-\alpha}(1-x)} - e^{-\lambda(1-x)}\}U(R) > 0\}$$

It is easy to see that $R^*(x) > 0$. When x converges to 1, $R^*(x)$ converges to the nonzero limit given by

$$R^*(0) = \max\{R > 0 : (1-\alpha)(U(R+a) - k/\lambda) > U(R)\}.$$

With the aim of arriving to a contradiction, assume that there exists a point $(x,\hat{R})$ with $\hat{R} \le R^*(x)$ and $V(x,\hat{R}) > MV(x,\hat{R})$. In other words, this point belongs to the consumption region. Let $\overline{R} = \max\{R < \hat{R} : V(x,R) = MV(x,R)\}$. In view of the above remark, $\overline{R} \ge 0$.

The points between $\overline{R}$ and $\hat{R}$ belong to the consumption region, and therefore, the value function satisfies the equation

$$\frac{\partial V(x,R)}{\partial R} = u_1(rV(x,R))$$

on $(\overline{R},\hat{R})$. Since u_1 is decreasing, for $R \in (\overline{R},\hat{R})$,

$$\begin{aligned} u_1(rV(x,R)) &\leq u_1(rV(x,\overline{R})) \\ &= u_1(rMV(x,\overline{R})) \\ &\leq u_1\left(r(1-e^{-\lambda(1-x)})\left(U(\overline{R}+a)-\frac{k}{\lambda}\right)+re^{-\lambda(1-x)}U(\overline{R})\right), \end{aligned}$$

and we have that

$$V(x,R) \leq MV(x,\overline{R}) + (R-\overline{R})u_1\left(r(1-e^{-\lambda(1-x)})\left(U(\overline{R}+a)-\frac{k}{\lambda}\right)+re^{-\lambda(1-x)}U(\overline{R})\right).$$

On the other hand, since V is increasing in R,

$$\begin{aligned} MV(x,R) - MV(x,\overline{R}) &= \lambda\int_0^{1-x} e^{-\lambda h}dh\left\{V(x+h,R+a)-V(x+h,\overline{R}+a)\right\} + e^{-\lambda(1-x)}(U(R)-U(\overline{R})) \\ &\geq e^{-\lambda(1-x)}(U(R)-U(\overline{R})) \end{aligned}$$

Combining the above estimates, passing to the limit $R \downarrow \overline{R}$, we get

$$\liminf_{R\downarrow\overline{R}} \frac{MV(x,R)-V(x,R)}{R-\overline{R}} \geq e^{-\lambda(1-x)}U'(\overline{R}) - u_1\left(r(1-e^{-\lambda(1-x)})\left(U(\overline{R}+a)-\frac{k}{\lambda}\right)+re^{-\lambda(1-x)}U(\overline{R})\right).$$

On the other hand,

$$U'(\overline{R}) = u_1(rU(\overline{R})),$$

and by definition of $R^*(x)$,

$$e^{-\lambda(1-x)}u_1(rU(\overline{R})) > u_1\left(r(1-e^{-\lambda(1-x)})\left(U(\overline{R}+a)-\frac{k}{\lambda}\right)+re^{-\lambda(1-x)}U(\overline{R})\right),$$

which contradicts the assumption that $V(x,\hat{R}) > MV(x,\hat{R})$. $\square$

The following proposition shows that under the assumption of power utility, the agent always consumes at large reserve level.

Proposition 4. *Assume that $u(c) = \frac{c^\alpha}{\alpha}$ with $0 < \alpha < 1$. Then there exists a constant $\check{R} < \infty$ such that the set $\{(x,R) : x \in [0,1], R \geq \check{R}\}$ belongs to the consumption region.*

Proof. It it enough to show that for all $R > \check{R}$, $MV(x,R) < V(x,R)$. Since V is decreasing in the first argument, the following estimate holds true.

$$MV(x,R) \leq V(x,R) + (1-e^{-\lambda(1-x)})\left\{V(x,R+a)-V(x,R)-\frac{k}{\lambda}\right\}.$$

From Lemma 1 and concavity of U, it follows then that,

$$\begin{aligned} MV(x,R) &\leq V(x,R) + (1-e^{-\lambda(1-x)})\left\{\frac{h}{R}\mathbb{E}[U(R+a+aN_{1-x})] - \frac{k}{\lambda}\right\} \\ &\leq V(x,R) + (1-e^{-\lambda(1-x)})\left\{\frac{h}{R}U(R+a+a\lambda(1-x)) - \frac{k}{\lambda}\right\} \\ &\leq V(x,R) + (1-e^{-\lambda(1-x)})\left\{\frac{h}{R}U(R+a+a\lambda) - \frac{k}{\lambda}\right\}. \end{aligned}$$

Since $U(x) = Cx^\alpha$ for some constant C, there exists $\check{R}$ such that for all $R \geq \check{R}$,

$$\frac{\alpha a}{R}U(R+a+a\lambda) < \frac{k}{\lambda},$$

and the proof is complete. □

Proposition 5. *Assume that $u(c) = \frac{c^\alpha}{\alpha}$ with $0 < \alpha < 1$. Then the value function $V(x,R)$ is concave and continuously differentiable in R on $[0,1] \times (0,\infty)$.*

Proof. The statement is obviously true for $x = 1$; fix $x < 1$ and let $R^*(x)$ be the boundary between consumption and exploration regions whose existence was shown in Proposition 2. By Proposition 1 and Lemma 3, the value function is concave and continuously differentiable on $(0, R^*(x))$ and on $(R^*(x), \infty)$. It remains then to check that the right and left derivatives at $R^*(x)$ coincide. We may assume without loss of generality that $R^*(x) \in (0,\infty)$. Since, on $[0, R^*(x)]$, $V(x,R) = MV(x,R)$, the left-hand derivative satisfies

$$V'_-(x,R) = MV'(x,R),$$

while for the right-hand derivative we have,

$$V'_+(x,R) = u_1(rV(x,R)).$$

Remark that, by the HJB equation,

$$MV'(x,R) \geq u_1(rV(x,R)),$$

and since $V'_+(x,R) \geq MV'(x,R)$, we also have

$$u_1(rV(x,R)) \geq MV'(x,R),$$

so that

$$MV'(x,R) = u_1(rV(x,R)) = V'_+(x,R).$$

□

Proposition 6. *Assume that $u(c) = \frac{c^\alpha}{\alpha}$ and $U(a) > \frac{k}{\lambda}$. Then the function $R^*(x)$ defined in Proposition 2 is increasing and continuously differentiable.*

Proof. Fix $x \in [0,1)$. By Proposition 1, $MV'_R(x,R) \geq u_1(MV(x,R))$ for $R \leq R^*(x)$. On the other hand, for $R > R^*(x)$, the argument used in the proof of Proposition 2 shows that $V'_R(x,R) > MV'_R(x,R)$. Since, in this region, $V'_R(x,R) = u_1(rV(x,R)) < u_1(rMV(x,R))$, we have that $MV'_R(x,R) < u_1(rMV(x,R))$ and thus the consumption region is characterized as follows.

$$R^*(x) = \inf\{R : MV'_R(x,R) < u_1(rMV(x,R))\}.$$

Letting $g(x,R) = \frac{d}{dR}(MV(x,R)^{\frac{1}{\alpha}})$, we can also write $R^*(x)$ as follows.

$$R^*(x) = \inf\{R : g(x,R) < c\}, \qquad c = \frac{1}{\alpha}\left(\frac{\alpha r}{1-\alpha}\right)^{\frac{1}{\alpha}-1}.$$

Therefore, $R^*(x)$ is increasing if and only if g is increasing in x and, by the implicit function theorem, $R^*(x)$ is continuously differentiable at the point x if g is continuously differentiable in the two variables at the point $(x, R^*(x))$ and

$$\frac{\partial g(x,R^*(x)}{\partial R} < 0.$$

We now proceed to compute the derivatives of g. A direct computation using the formula for MV shows:

$$\begin{aligned}
\frac{\partial MV(x,R)}{\partial x} &= k + \lambda(MV(x,R) - V(x,R+a)) \\
\frac{\partial MV'_R(x,R)}{\partial x} &= \lambda(MV'_R(x,R) - V'_R(x,R+a)),
\end{aligned}$$

and in particular,

$$\begin{aligned}
\frac{\partial MV(x,R^*(x))}{\partial x} &= k + \lambda(V(x,R) - V(x,R+a)) \\
\frac{\partial MV'_R(x,R^*(x))}{\partial x} &= \lambda(V'_R(x,R) - V'_R(x,R+a)).
\end{aligned}$$

Denoting by $\sim$ equality up to a multiplicative constant, we then have

$$\begin{aligned}
\frac{\partial g(x,R^*(x))}{\partial x} &\sim \frac{1-\alpha}{\alpha} V'_R(x,R^*(x)) V^{\frac{1-2\alpha}{\alpha}}(x,R^*(x)) \frac{\partial MV(x,R^*(x))}{\partial x} \\
&\qquad + V^{\frac{1-\alpha}{\alpha}}(x,R^*(x)) \frac{\partial MV'_R(x,R^*(x))}{\partial x} \\
&\sim \frac{1-\alpha}{\alpha} V^{-1}(x,R^*(x))\{k + \lambda(V(x,R^*(x)) - V(x,R^*(x)+a))\} \\
&\qquad + V^{\frac{1-\alpha}{\alpha}}(x,R^*(x))\lambda(V^{\frac{\alpha-1}{\alpha}}(x,R^*(x)) - V^{\frac{\alpha-1}{\alpha}}(x,R^*(x)+a)) \\
&\sim \frac{1-\alpha}{\alpha}\left\{\frac{k}{\lambda V(x,R^*(x))} + 1 - \frac{V(x,R^*(x)+a)}{V(x,R^*(x))}\right\} \\
&\qquad + 1 - \left(\frac{V(x,R^*(x)+a))}{V(x,R^*(x))}\right)^{\frac{\alpha-1}{\alpha}}.
\end{aligned}$$

Recall that

$$V(x, R^*(x) + a) = \{V(x, R^*(x)) + ca\}^\alpha, \quad c = \frac{1}{\alpha}\left(\frac{1-\alpha}{r\alpha}\right)^{\frac{1-\alpha}{\alpha}},$$

so that

$$\begin{aligned}
\frac{\partial g(x, R^*(x))}{\partial x} &\sim (1-\alpha)\frac{k}{\lambda V(x, R^*(x))} + 1 - (1-\alpha)\left\{1 + \frac{ca}{V(x, R^*(x))^{\frac{1}{\alpha}}}\right\}^\alpha - \alpha\left\{1 + \frac{ca}{V(x, R^*(x))^{\frac{1}{\alpha}}}\right\}^{\alpha-1} \\
&\geq (1-\alpha)\left\{\frac{k}{\lambda V(x, R^*(x))} + 1 - \left\{1 + \frac{ca}{V(x, R^*(x))^{\frac{1}{\alpha}}}\right\}^\alpha\right\} \\
&\geq \frac{1-\alpha}{V(x, R^*(x))}\left\{\frac{k}{\lambda} - (ca)^\alpha\right\} = \frac{1-\alpha}{V(x, R^*(x))}\left\{\frac{k}{\lambda} - U(a)\right\} > 0.
\end{aligned}$$

This shows that $R^*(x)$ is strictly increasing. On the other hand, the derivative of g with respect to R satisfies

$$\frac{\partial g(x, R)}{\partial R} = \frac{1}{\alpha} MV^{\frac{1-2\alpha}{\alpha}}(x, R)\left\{\frac{1-\alpha}{\alpha}(MV_R'(x, R))^2 + MV(x, R)MV_R''(x, R)\right\}.$$

At the point $(x, R^*(x))$, $MV_R'(x, R) = V_R'(x, R)$, and by the same argument as in Proposition 2, $MV''(x, R) < V''(x, R)$. Together with the smooth pasting, this leads to the following estimate.

$$\frac{\partial g(x, R^*(x))}{\partial R} < \frac{1}{\alpha} V^{\frac{1-2\alpha}{\alpha}}(x, R)\left\{\frac{1-\alpha}{\alpha}(V_R'(x, R))^2 + V(x, R)V_R''(x, R)\right\} = 0,$$

since in the consumption region,

$$V_R'(x, R) = \left(\frac{\alpha r V(x, R)}{1-\alpha}\right)^{\frac{\alpha-1}{\alpha}}.$$

□

3 Behavior Near Exhaustion

This section provides a more in-depth analysis when the unexplored area approaches zero.

3.1 Computing the Point of Intersection

In this subsection, we will study the particular case:

$$u(c) = 2\sqrt{c}$$

The results can be extended immediately to $u(c)=\frac{1}{\alpha}c^{\alpha}$ with $0<\alpha<1$. We then have $u^{*}(p)=\frac{1}{p}$ and the problem becomes:

$$\max\left\{1-rV\frac{\partial V}{\partial R},\ MV-V\right\}=0 \qquad (7)$$
$$V(1,R)=U(R)$$

We have $U'(R)U(R)=1/r$ and it follows immediately that, for some constant C, we have $U(R)=\sqrt{C+\frac{2}{r}R}$. Normalizing by $U(0)=0$, we get:

$$U(R)=\sqrt{\frac{2}{r}R}$$

We wish to investigate the solution on the band $(1-\varepsilon\leq x\leq 1,\ R\geq 0)$. To do this, we will assume that the solution $V(x,R)$ is C^1, which is the classical smooth pasting condition. We have $V(1,R)=U(R)$ and both sides of 7 are satisfied simultaneously at $x=1$, namely $rV\frac{\partial V}{\partial R}=1$and $V=MV$. Expanding to the first order in $y=1-x>0$, we define two regions in $[0,\infty)$:

3.2 The Exploration Region

. This is the set of R where $MV(1-y,\ R)=V(1-y,\ R)$ and $rV\frac{\partial V}{\partial R}\geq 1$ for all $y>0$ small enough.

Expanding MV to the first order wrt to y at $y=0$, and taking into account that the probability of hitting oil between $1-y$ and y is λy , we get:

$$MV(1-y,\ R)=(1-\lambda y)U(R)+\lambda yU(R+a)-ky$$

The equation $MV(1-y,\ R)=V(1-y,\ R)$ then becomes:

$$V(1-y,\ R)=(1-\lambda y)U(R)+\lambda yU(R+a)-ky$$

Differentiating wrt R, we get:

$$\frac{\partial}{\partial R}MV=(1-\lambda y)U'(R)+\lambda yU'(R+a)$$

We have to check that $rV\frac{\partial V}{\partial R}\geq 1$. This becomes:

$$1\leq r\left((1-\lambda y)U(R)+\lambda yU(R+a)-ky\right)\left((1-\lambda y)U'(R)+\lambda yU'(R+a)\right)$$

Substituting $U'(R)=(rU(R))^{-1}$, this becomes:

$$U(R)\left(1-\frac{U(R)}{U(R+a)}\right)\leq U(R+a)-U(R)-\frac{k}{\lambda}$$

Simplifying, we get:

$$2 - \frac{U(R)}{U(R+a)} - \frac{U(R+a) - k/\lambda}{U(R)} \leq 0$$

Finally, substituting $U(R) = \sqrt{\frac{2}{r}R}$, this becomes:

$$\frac{\sqrt{R}}{\sqrt{R+a}} + \frac{\sqrt{R+a}}{\sqrt{R}} - \frac{k}{\lambda\sqrt{R}}\sqrt{\frac{r}{2}} \geq 2 \tag{8}$$

Setting $t = \sqrt{\frac{2}{r}R}$,the inequality becomes:

$$\begin{aligned} \frac{t}{\sqrt{t^2 + \frac{2a}{r}}} + \frac{\sqrt{t^2 + \frac{2a}{r}}}{t} - \frac{k}{\lambda t} &\geq 2 \\ \frac{t^2}{\sqrt{t^2 + \frac{2a}{r}}} + t^2 + \frac{2a}{r} - \frac{k}{\lambda} &\geq 2t \end{aligned}$$

So $\left(2t^2 + \frac{2a}{r}\right)^2 \geq \left(\frac{k}{\lambda} + 2t\right)^2 \left(t^2 + \frac{2a}{r}\right)$ and we end up with the algebraic inequality:

$$4kr\lambda t^3 + k^2 rt^2 + 8ak\lambda t \leq 2a\left(\frac{2a}{r}\lambda^2 - k^2\right) \tag{9}$$

If $\sqrt{a} \leq \frac{k}{\lambda}\sqrt{\frac{r}{2}}$, the inequality is never satisfied. If $\sqrt{a} \geq \frac{k}{\lambda}\sqrt{\frac{r}{2}}$ denote by t_0 the non-negative root of the equation:

$$4kr\lambda t^3 + k^2 rt^2 + 8ak\lambda t = 2a\left(\frac{2a}{r}\lambda^2 - k^2\right) \tag{10}$$

Then inequality (9) is satisfied for $0 \leq t \leq t_0$ and inequality (8) is satisfied for $0 \leq R \leq R_0 = \frac{r}{2}t^2$.

3.3 The Consumption Region

This is the set of R such that where $MV(1-y,\, R) \leq V(1-y,\, R)$ and $rV\frac{\partial V}{\partial R} = 1$ for all $y > 0$ small enough.

As before, we have:

$$MV(1-y,\, R) = (1-\lambda y)\, U(R) + \lambda y U(R+a) - ky$$

Take $R > R_1$ with both in the consumption region. Integrating $rV\frac{\partial V}{\partial R} = 1$ wrt R, we get, for x close enough to 1:

$$V(x, R) = \sqrt{f(x) + \frac{2}{r}R}$$

with $f(1) = 0$. Note that, as a consequence, $U(R) = \sqrt{\frac{2}{r}R} = \sqrt{U(R_1)^2 + \frac{2}{r}(R - R_1)}$ We can estimate $f(x)$ by using the fact that $MV(1-y\,,R_1) \leq V(1-y\,,R_1)$, namely:

$$(1-\lambda y)\,U(R_1) + \lambda y U(R_1+a) - ky \leq \sqrt{f(1-y) + \frac{2}{r}R_1}$$

This yields $f(1-y) \geq ((1-\lambda y)\,U(R_1) + \lambda y U(R_1+a) - ky)^2 - \frac{2}{r}R_1$ and hence:

$$V(1-y,R) \geq \sqrt{((1-\lambda y)\,U(R_1) + \lambda y U(R_1+a) - ky)^2 + \frac{2}{r}(R - R_1)}$$

Equality holds for $y = 0$. Since $MV(1-y,\,R) - V(1-y,\,R) \leq 0$, we have:

$$(1-\lambda y)\,U(R) + \lambda y U(R+a) - ky - \sqrt{((1-\lambda y)\,U(R_1) + \lambda y U(R_1+a) - ky)^2 + \frac{2}{r}(R - R_1)} \leq 0$$

Since equality holds for $y = 0$, the derivative wrt y at $y = 0$ must be non-positive, namely:

$$\lambda\,(U(R+a) - U(R)) - k \leq \frac{U(R_1)}{\sqrt{U(R_1)^2 + \frac{2}{r}(R - R_0)}}\,[\lambda\,(U(R_0+a) - U(R_0)) - k] \quad (11)$$

Remembering that $U(R)^2 = U(R_1)^2 + \frac{2}{r}(R - R_0)$ this becomes

$$U(R)\left(U(R+a) - U(R) - \frac{k}{\lambda}\right) \leq U(R_0)\left(U(R_0+a) - U(R_0) - \frac{k}{\lambda}\right)$$

Equality holds for $R = R_0$. For the inequality to hold for $R \geq R_1$, it is sufficient to prove that the function:

$$R \to\leq \sqrt{\frac{2}{r}R}\left(\sqrt{\frac{2}{r}(R+a)} - \sqrt{\frac{2}{r}R} - \frac{k}{\lambda}\right)$$

is decreasing. Changing variables, by setting $t = \sqrt{\frac{2}{r}R}$, we obtain the function:

$$f(t) = t\sqrt{t^2 + \frac{2a}{r}} - t^2 - \frac{k}{\lambda}t$$

Its derivative is computed to be:

$$f'(t) = \frac{2t^2 + \frac{2a}{r}}{\sqrt{t^2 + \frac{2a}{r}}} - 2t - \frac{k}{\lambda}$$

Writing $f'(t) \leq 0$ we obtain the algebraic inequality:

$$4kr\lambda t^3 + k^2 r t^2 + 8ak\lambda t \geq 2a\left(\frac{2a}{r}\lambda^2 - k^2\right) \quad (12)$$

We recognize the reverse of inequality 9. It is satisfied for all $t \geq 0$ when $\sqrt{a} \leq \frac{k}{\lambda}\sqrt{\frac{r}{2}}$, and for $t \geq t_0$ when $\sqrt{a} > \frac{k}{\lambda}\sqrt{\frac{r}{2}}$

Let us summarize our findings:

Lemma. *Suppose* $\sqrt{a} > \frac{k}{\lambda}\sqrt{\frac{r}{2}}$ *and the solution* V *of the HJB equation with* $V(1,R) = U(R)$ *is* C. *Let* R_0 *be the positive solution of the equation*:

$$\frac{\sqrt{R}}{\sqrt{R+a}} + \frac{\sqrt{R+a}}{\sqrt{R}} - \frac{k}{\lambda\sqrt{R}}\sqrt{\frac{r}{2}} = 2 \tag{13}$$

Then the consumption region is $R \geq R_0$ *and the exploration region is* $0 \leq R < R_0$.

3.4 Computing the Tangent

Proposition 7. *As* $x \to 1$, *the exercise frontier satisfies*

$$R^*(x) = R_0 + (1-x)R_1 + O((1-x)^2),$$

where R_0 *is the solution of*

$$\frac{U(R_0)}{U(R_0+a)} + \frac{U(R_0+a)}{U(R_0)} - \frac{k/\lambda}{U(R_0)} - 2 = 0.$$

and R_1 *is negative and given by*

$$\frac{R_1}{r}\left\{\frac{\xi-\xi^{-1}}{U^2(R_0)} - \frac{\xi-\xi^{-1}}{U^2(R_0+a)} + \frac{k/\lambda}{U^3(R_0)}\right\} + \frac{\lambda}{2}\{\xi-1\}^3\{\xi+2\} = 0,$$

where $\xi = \frac{U(R_0)}{U(R_0+a)}$.

Proof. Let $R^*(y)$ be the exercise frontier. Then,

$$V(y,R^*(y)) = MV(y,R^*(y)) = \int_0^y \lambda e^{-\lambda(y-h)}V(h,R^*(y)+a)dh + U(R^*(y))e^{-\lambda y} - \frac{k}{\lambda}(1-e^{-\lambda y}) \tag{14}$$

This implies that $V(y,R^*(y)) = U(R^*(y)) + O(h)$. On the other hand, for $R > R^*(y)$,

$$V(y,R) = \sqrt{V^2(y,R^*(y)) + \frac{2}{r}(R-R^*(y))} = \sqrt{U^2(R^*(y)) + \frac{2}{r}(R-R^*(y))} + O(h) = U(R) + O(y).$$

Substituting this into equation (14) yields

$$\begin{aligned} V(y,R^*(y)) &= \int_0^y \lambda e^{-\lambda(y-h)}U(R^*(y)+a)dh + U(R^*(y))e^{-\lambda y} - \frac{k}{\lambda}(1-e^{-\lambda y}) + O(y^2) \\ &= U(R^*(y)) + \lambda y(U(R^*(y)+a) - U(R^*(y)) - k/\lambda) + O(y^2), \end{aligned}$$

and also

$$\begin{aligned} V(y,R) &= \sqrt{(U(R^*(y)) + \lambda y(U(R^*(y)+a) - U(R^*(y)) - k/\lambda))^2 + \frac{2}{r}(R-R^*(y))} \\ &= \sqrt{U(R)^2 + 2\lambda y U(R^*(y))(U(R^*(y)+a) - U(R^*(y)) - k/\lambda)} \\ &= U(R) + \lambda y U(R^*(y))\frac{(U(R^*(y)+a) - U(R^*(y)) - k/\lambda}{U(R)} + O(y^2). \end{aligned} \tag{15}$$

and

$$V(h,R^*(y)+a)=U(R^*(y)+a)+\lambda hU(R^*(y))\frac{(U(R^*(y)+a)-U(R^*(y))-k/\lambda}{U(R^*(y)+a)}+O(y^2). \quad (16)$$

Substituting this back into equation (14) now yields

$$V(y,R^*(y))=U(R^*(y))+(U(R^*(y)+a)-U(R^*(y))-k/\lambda)\left[\lambda y+\frac{\lambda^2y^2}{2}\left\{\frac{U(R^*(y))}{U(R^*(y)+a)}-1\right\}\right]+O(y^3).$$

On the other hand, by the smooth pasting property,

$$V'(y,R^*(y))=MV'(y,R^*(y))=\int_0^y \lambda e^{-\lambda(y-h)}V'(h,R^*(y)+a)dh+U'(R^*(y))e^{-\lambda y},$$

which is equivalent to

$$\frac{1}{V(y,R^*(y))}=\int_0^y \frac{\lambda e^{-\lambda(y-h)}}{V(h,R^*(y)+a)}dh+\frac{e^{-\lambda y}}{U(R^*(y))}.$$

From equation (15),

$$\begin{aligned}\frac{1}{V(h,R^*(y)+a)} &= \frac{1}{U(R^*(y)+a)+\lambda hU(R^*(y))\frac{(U(R^*(y)+a)-U(R^*(y))-k/\lambda}{U(R^*(y)+a)}}+O(y^2)\\ &= \frac{1}{U(R^*(y)+a)}-\lambda hU(R^*(y))\frac{(U(R^*(y)+a)-U(R^*(y))-k/\lambda}{U^3(R^*(y)+a)}+O(y^2),\end{aligned}$$

and therefore

$$\frac{1}{V(y,R^*(y))}=\frac{\lambda y-\frac{\lambda^2y^2}{2}}{U(R^*(y)+a)}-\frac{\lambda^2y^2}{2}U(R^*(y))\frac{(U(R^*(y)+a)-U(R^*(y))-k/\lambda}{U^3(R^*(y)+a)}+\frac{1-\lambda y+\frac{\lambda^2y^2}{2}}{U(R^*(y))}+O(y^3).$$

Computing the product of the two second order expansions, and performing some simplifications, we get:

$$\begin{aligned}&U(R^*(y))\left\{\frac{1-\frac{\lambda y}{2}}{U(R^*(y)+a)}-\frac{\lambda y}{2}U(R^*(y))\frac{(U(R^*(y)+a)-U(R^*(y))-k/\lambda}{U^3(R^*(y)+a)}+\frac{-1+\frac{\lambda y}{2}}{U(R^*(y))}\right\}\\ &\quad+(U(R^*(y)+a)-U(R^*(y))-k/\lambda)\left\{\frac{\frac{3}{2}\lambda y}{U(R^*(y)+a)}+\frac{1-\frac{3}{2}\lambda y}{U(R^*(y))}\right\}=O(y^2)\end{aligned}$$

or, in other words,

$$\begin{aligned}&\frac{U(R^*(y))}{U(R^*(y)+a)}-1+\frac{U(R^*(y)+a)-U(R^*(y))-k/\lambda}{U(R^*(y))}+\frac{\lambda y}{2}\left\{1-\frac{U(R^*(y))}{U(R^*(y)+a)}\right\}\\ &\quad+\frac{\lambda y}{2}(U(R^*(y)+a)-U(R^*(y))-k/\lambda)\left\{-\frac{U^2(R^*(y))}{U^3(R^*(y)+a)}+\frac{3}{U(R^*(y)+a)}-\frac{3}{U(R^*(y))}\right\}=O(y^2).\end{aligned}$$

Let $R^*(y) = R_0 + R_1 y + O(y^2)$. Then, R_0 may be found from the following equation.

$$\frac{U(R_0)}{U(R_0+a)} + \frac{U(R_0+a)}{U(R_0)} - \frac{k/\lambda}{U(R_0)} - 2 = 0.$$

To identify R_1 we compute the first order term in the Taylor series.

$$R_1 \frac{d}{dR}\left\{\frac{U(R)}{U(R+a)} - 1 + \frac{U(R+a) - U(R) - k/\lambda}{U(R)}\right\}\Big|_{R=R_0} + \frac{\lambda}{2}\left\{1 - \frac{U(R_0)}{U(R_0+a)}\right\}$$
$$+\frac{\lambda}{2}\frac{U(R_0+a) - U(R_0) - k/\lambda}{U(R_0)}\left\{-\frac{U^3(R_0)}{U^3(R_0+a)} + \frac{3U(R_0)}{U(R_0+a)} - 3\right\} = 0,$$

which is equivalent to

$$\frac{R_1}{r}\left\{\frac{\xi - \xi^{-1}}{U^2(R_0)} - \frac{\xi - \xi^{-1}}{U^2(R_0+a)} + \frac{k/\lambda}{U^3(R_0)}\right\} \qquad + \frac{\lambda}{2}\{\xi - 1\}^3\{\xi + 2\} = 0,$$

where $\xi = \frac{U(R_0)}{U(R_0+a)}$. Since U is increasing, we conclude that $\xi > 1$ and therefore $R_1 < 0$. $\square$

Zero-order expansion for general α As before, the expansion of the value function and its derivative up to $O(y^2)$ yield

$$\begin{aligned} V(y, R^*(y)) &= U(R^*(y)) + \lambda y(U(R^*(y)+a) - U(R^*(y)) - k/\lambda) + O(y^2) \\ V(y, R^*(y))^{\frac{\alpha-1}{\alpha}} &= U(R^*(y))^{\frac{\alpha-1}{\alpha}} + \lambda y(U(R^*(y)+a)^{\frac{\alpha-1}{\alpha}} - U(R^*(y))^{\frac{\alpha-1}{\alpha}}) + O(y^2) \end{aligned}$$

Combining these equations and passing to the limit $y \to 0$ then leads to the following equation for $R_0 := R^*(0)$.

$$\alpha\left(\frac{U(R_0+a)}{U(R_0)}\right)^{\frac{\alpha-1}{\alpha}} + (1-\alpha)\frac{U(R_0+a)}{U(R_0)} - \frac{k(1-\alpha)}{\lambda U(R_0)} = 1$$

3.5 Hotelling in Expectations

In the proof of Proposition 1, we have seen that the optimal consumption-stopping strategy may be defined in feeedback form by

$$\begin{aligned} c_t &= I\left(\frac{\partial V}{\partial R}(x(t), R(t))\right)\mathbf{1}_{V(x(t),R(t))>MV(x(t),R(t))} \\ \theta_n &= \inf\{t \geq \theta_{n-1} : V(x(t), R(t)) = MV(x(t), R(t))\}. \end{aligned}$$

where

$$\begin{aligned} x(t) &= x + N^{-1}(n_t) \wedge (1-x), \quad \text{with} \quad n_t = \max\{n : \theta_n \leq t\}, \\ R(t) &= R - \int_0^t c_s ds + a(n_t \wedge N_{1-x}). \end{aligned}$$

Considering an agent with utility u, this consumption trajectory is optimal for the price trajectory given by

$$p_t = \frac{\partial V}{\partial R}(x(t), R(t))\mathbf{1}_{V(x(t),R(t))>MV(x(t),R(t))}.$$

We therefore shall use this formula as the definition of the price.

Remark 1. *Since, during consumption, the explored area variable $x(t)$ does not change, the strategy consists in consuming until the reserve level $R(t)$ hits the critical reserve level $R^*(x(t))$. Since, as we have seen above, the critical reserve level $R^*(x)$ is increasing in x, as the unexplored are shrinks, the agents start exploring at higher reserve levels. Alternatively, one can argue that during consumption the price p_t goes up, and hence the strategy consists in consuming until the price p_t hits the critical price given by $\frac{\partial V}{\partial R}(x(t), R^*(x(t)))$. As the unexplored area shrinks, the critical price level at which agents start to explore goes up. Indeed, by definition of $R^*(x)$ in the proof of Proposition 6, the function $g(x,R)$ is constant on the exercise frontier. However, this function satisfies*

$$\begin{aligned} g(x, R^*(x)) &= \frac{1}{\alpha} MV^{\frac{1}{\alpha}-1}(x, R^*(x)) MV'_R(x, R^*(x)) \\ &= \frac{1}{\alpha} V^{\frac{1}{\alpha}-1}(x, R^*(x)) MV'_R(x, R^*(x)) \sim \frac{MV'_R(x, R^*(x))}{p(x, R^*(x))}, \end{aligned}$$

where $p(x,R)$ is the price. Therefore, on the exercise frontier $p(x, R^*(x)) \sim MV'_R(x, R^*(x))$ and so

$$\frac{d}{dx} p(x, R^*(x)) \sim \frac{d}{dx} MV'_R(x, R^*(x)) = \lambda \left(\frac{\partial V}{\partial R}(x, R^*(x)) - \frac{\partial V}{\partial R}(x, R^*(x) + a) \right).$$

This last term is clearly positive since V is concave in R.

Proposition 8. *Assume that $u(c) = \frac{c^\alpha}{\alpha}$ with $0 < \alpha < 1$. Then $p_t e^{-rt}$ is a martingale and in particular for all $t \geq 0$, we have $\mathbb{E}[p_t] = p_0 e^{rt}$.*

Proof. The price process is continuous within the consumption region and jumps immediately when $R(t)$ hits the boundary of the exploration region. Inside the consumption region the value function is smooth and its second derivative satisfies

$$\frac{\partial^2 V}{\partial R^2}(x(t), R(t)) = -r \frac{\partial V}{\partial R}(x(t), R(t)) I \left(\frac{\partial V}{\partial R}(x(t), R(t)) \right)^{-1} = -r p_t c_t^{-1}$$

The price therefore can then be written as follows.

$$\begin{aligned} e^{-rT} p_T &= p_0 - r p_t e^{rt} dt + \int_0^T e^{-rt} \frac{\partial^2 V}{\partial R^2}(x(t), R(t)) dR_t \\ &\quad + \sum_{i=1}^{n_t} e^{-r\theta_i} \{ \frac{\partial V}{\partial R}(x(\theta_i), R(\theta_i)) - \frac{\partial V}{\partial R}(x(\theta_i -), R(\theta_i -)) \} \\ &= p_0 + \sum_{i=1}^{n_t} e^{-r\theta_i} \{ \frac{\partial V}{\partial R}(x(\theta_i), R(\theta_i)) - \frac{\partial MV}{\partial R}(x(\theta_i -), R(\theta_i -)) \}, \end{aligned}$$

where we have used the smooth pasting principle in the last line. From the proof of Proposition 5 it follows that

$$\frac{\partial MV}{\partial R}(x,R) = \int_0^{1-x} \lambda e^{-\lambda h} \frac{\partial V(x+h, R+a)}{\partial R} dh + e^{-\lambda h} U'(R).$$

Therefore,

$$\mathbb{E}\left[\frac{\partial V}{\partial R}(x(\theta_i), R(\theta_i)) \Big| \mathcal{F}_{\theta_i-}\right] = \frac{\partial MV}{\partial R}(x(\theta_i-), R(\theta_i-))$$

and we conclude that $e^{rt}p_t$ is a martingale. □

4 Numerical Experiments

In the first example we use the following parameters: discovered amount $a = 1.5$, discovery intensity $\lambda = 1$ and exploration cost $k = 3$. The utility parameter is $\alpha = 0.5$ for both examples and the interest rate is $r = 0.02$. Figure 6 plots the exploration/consumption region and the contour of the price function. We see that the agreement between the theoretical value of R_0, the limit of the exercise frontier as $x \to 1$, and the corresponding numerical value is very good.

Figure 7 plots the exercise boundary and the critical price level as function of the explored area x. We see that both functions are increasing, as predicted by the theoretical analysis.

Figure 8 plots the evolution of a single representative trajectory of various variables. We see in particular that finds correspond to downward jumps in the price while the last jump, when no oil is found, is upward.

Lastly, Figure 9 plots the average evolution and the quantiles of various quantities computed over 1000 simulations. We see that in agreement with the theoretical findings, the average price grows at the interest rate.

Figures 6–9 plot the same quantities for our second example where we have taken find size $a = 0.5$, find intensity $\lambda = 10$ and exploration cost $k = 1$.

5 Discussion

We next compare our findings to early literature that examined costly exploration with stochastic discoveries. Some of our findings are at odds with earlier publications, and this section sets out to discuss whether differences in modeling assumptions are responsible for these discrepancies.

First, we found that the reserve level is decreasing as the unexplored area decreases. An earlier paper argues that the reverse is true (Quyen 1991). The paper has a slightly different setup where he discritices the unexplored area into a finite number of cells. Hence the exploration decision becomes to either explore a cell or not, while in Arrow and Chang (1982) and our approach, the choice of the exploration area is continuous. However, when the number of discrete cells approaches infinity, the two solutions should become identical. The proof in (Quyen 1991) proof has a sign error and the results are hence not due to different modelling assumptions. The problem with the proof is that $B(\xi^*(x,2),\tau^*(x,2),1) < 0$, not greater than zero as written in equation (10), page 787. To see this note that

$$\begin{aligned} B(\xi^*(x,2),\tau^*(x,2),1) &= V_\xi(\xi^*(x,2),\tau^*(x,2)) - e^{-\delta\tau^*(x,2)}W_z(x-\xi^*(2,1),1) \\ &< V_\xi(\xi^*(x,2),\tau^*(x,2)) - e^{-\delta\tau^*(x,2)}W_z(x-\xi^*(2,1),2) \\ &= B(\xi^*(x,2),\tau^*(x,2),2) \\ &= 0 \end{aligned}$$

The second line follows from the fact that $W_z(z,2) < W_z(z,1)$.

Second, Lasserre (1984) follows the same setup as (Arrow and Chang 1982) and us, yet asserts that the "price of reserves is expected to drop upon exploration." (Equation (4), page 197). On the other hand, we have argued that the prices always rises at the rate of interest in expectation, and since exploration occurs instantaneously, the price is the same in expectation. However, the proof in Lasserre (1984) uses a strict inequality obtained for a discrete change in the unexplored area X. When taking the limit for $X \to 0$, this strict inequality reverses to a weak inequality. Specifically, note equation (21) in (Arrow and Chang 1982) establishes that $V_X(R,X) = \lambda\Delta V(R,X) - P$. Hence $V_{XR}(R,X) = \lambda\Delta V_R(R,X)$ and *not* $V_{XR}(R,X) > \lambda\Delta V_R(R,X)$ as Lasserre asserts in equation (4). Hence, the expected price is not dropping but remaining constant as exploration happens in zero time.

6 Conclusion

New stochastic discoveries are an important component of many exhaustible resources - we don't know how much of a finite resource is buried in the ground. We are the first to fully solve the problem of an exhaustible resource with stochastic discoveries that can be made using an exploration process that has constant marginal cost. The optimal solution balances delaying exploration cost into the future and getting earlier information on the exact amount of reserves in the ground. We find that the optimal policy follows a discrete solution. Once the known the reserve stock drops below a critical level that depends on the size of the remaining unexplored area, exploration starts at infinite speed until either the proven reserve stock again exceeds the critical level or the entire remaining unexplored area is exhausted. We show that the critical reserve level is increasing in the unexplored area. If proven reserves are above the critical level through new discoveries that follow a Poisson process, exploration stops and price follows a classical price path that rises at the rate of interest.

The paper provides several new insights into the price process of exhaustible resources, many of who have not risen as predicted in the simple deterministic Hotelling model. We show that while the price path always rises at the rate of interest in expectation, a realized price path conditional on not having run out of unexplored area rises on average at less than the rate of interest. This helps explain why forward-looking test cannot refute the Hotel ling rule, why backward-looking tests generally have. Moreover, since the critical reserve stock is increasing as the unexplored area decreases, the level of proven resource is a poor indicator of the underlying scarcity. An increasing reserve level can just be the result of the decrease in unexplored area and rather signal an increase in scarcity,

References

Arrow, Kenneth J. and Sheldon Chang, "Optimal Pricing, Use, and Exploration of Uncertain Natural Resource Stocks," *Journal of Environmental Economics and Management*, March 1982, *9* (1), 1–10.

Asker, John, Allan Collard-Wexler, and Jan De Loecker, "Mis-Allocation, Market Power, and Global Oil Extraction," *American Economic Review*, 2019, *109* (4), 1568–1615.

Berck, RobertPeter and Michael J. Roberts, "Natural Resource Prices: Will They Ever Turn Up?," *Journal of Environmental Economics and Management*, July 1996, *31* (1), 65–78.

Halvorsen, Robert and Tim R. Smith, "A Test of the Theory of Exhaustible Resources," *Quarterly Journal of Economics*, February 1991, *106* (1), 123–140.

Heal, Geoffrey and Wolfram Schlenker, "Coase, Hotelling and Pigou: The Incidence of a Carbon Tax and CO2 Emissions," *NBER Working Paper No. 26086*, June 2020.

Hotelling, Harold, "Economics of Exhaustible Resources," *Journal of Political Economy*, 1931, *39* (April), 137–175.

Lasserre, Pierre, "Reserve and Land Prices with Exploration under Uncertainty," *Journal of Environmental Economics and Management*, September 1984, *11* (3), 191–201.

Miller, Merton H. and Charles W. Upton, "A Test of the Hotelling Valuation Principle," *Journal of Political Economy*, February 1985, *93* (1), 1–25.

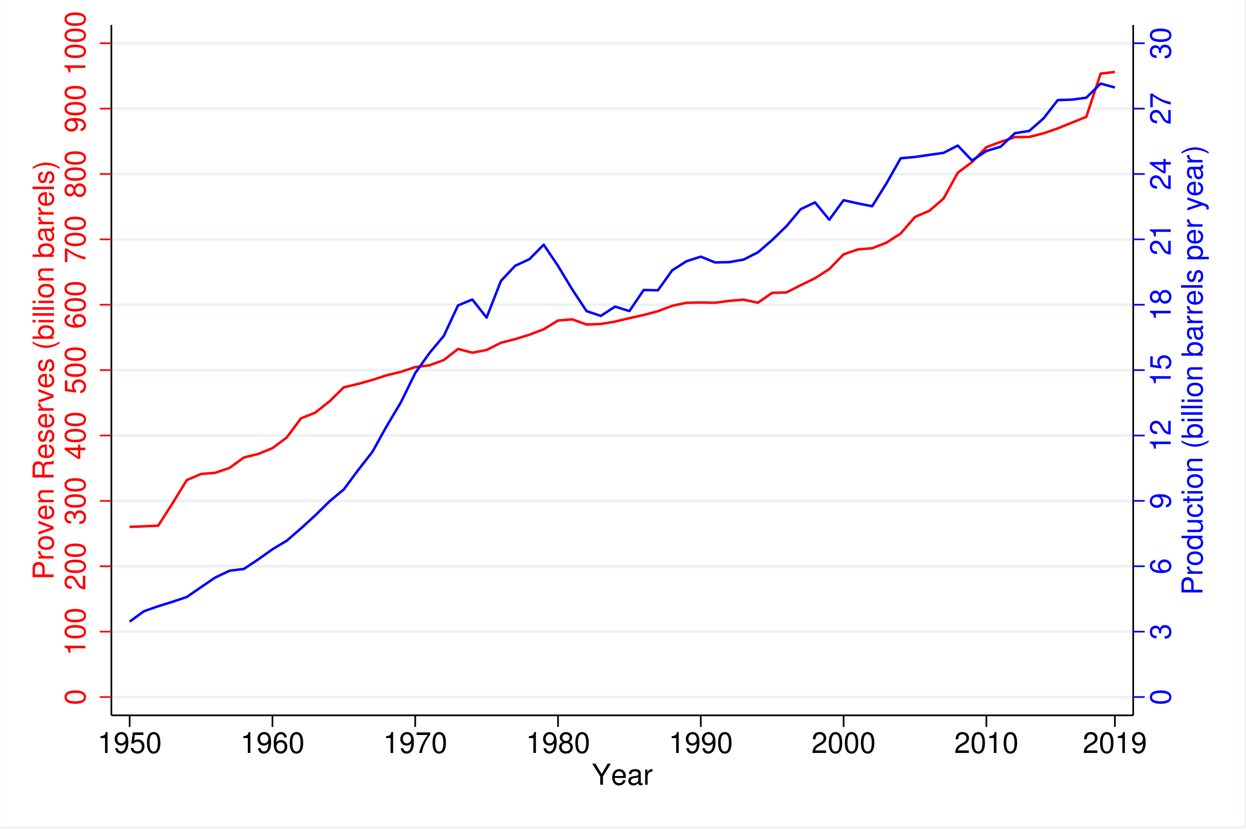
Pham, Huyên and Peter Tankov, "A coupled system of integrodifferential equations arising in liquidity risk model," *Applied Mathematics and Optimization*, 2009, *59* (2), 147.

Pindyck, Robert S., "The Optimal Exploration and Production of Nonrenewable Resources," *Journal of Political Economy*, October 1978, *86* (5), 841–861.

Quyen, "Exhaustible Resources: A Theory of Exploration," *The Review of Economic Studies*, June 1991, *58* (4), 777–789.

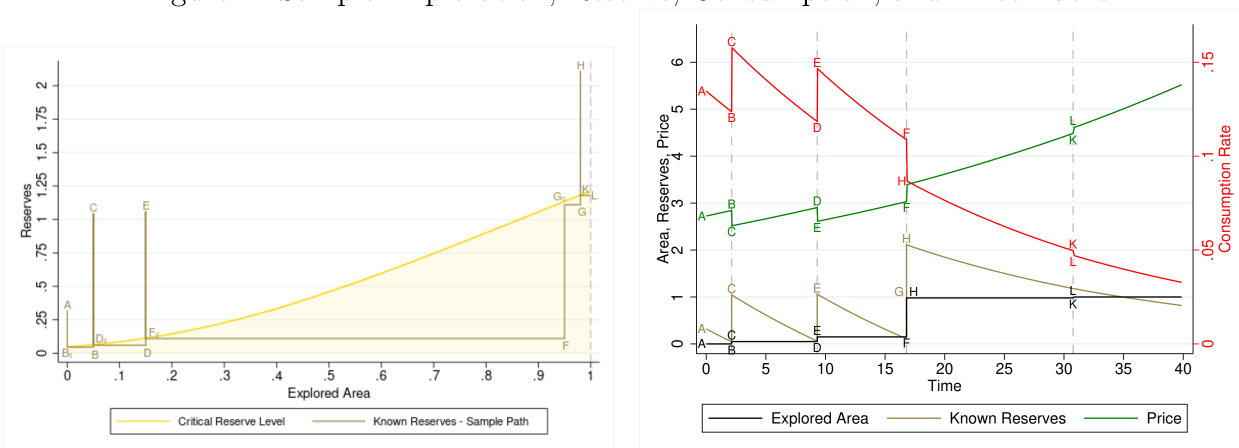
Slade, Margaret E., "Trends in Natural-Resource Commodity Prices: An Analysis of the Time Domain," *Journal of Environmental Economics and Management*, 1982, *9* (1), 122–137.

Figure 1: Proven Crude Oil Reserves and Annual Production Over Time



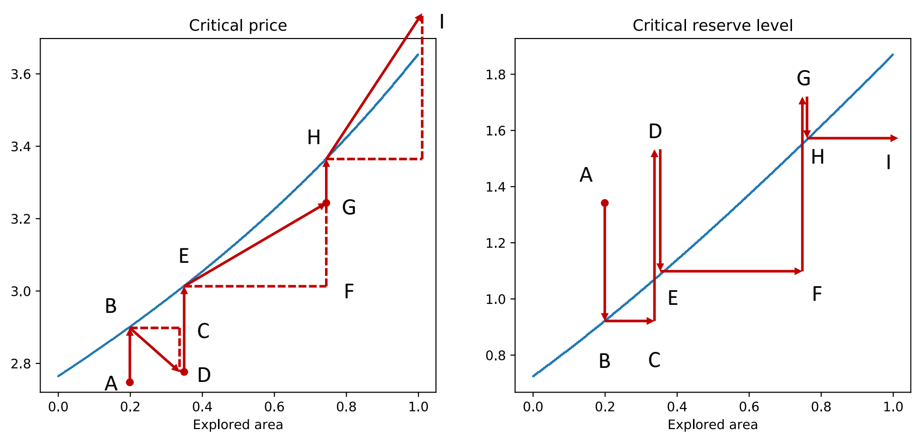
Notes: Proven Reserves are plotted in red (left vertical axis) by summing all assets in Rystad's database that are "Producing," "Discovery," or "Under development," while excluding "Undiscovered" assets. Annual production of crude oil is plotted in blue (right vertical axis) on a different scale.

Figure 2: Sample Exploration, Reserve, Consumption, and Price Paths



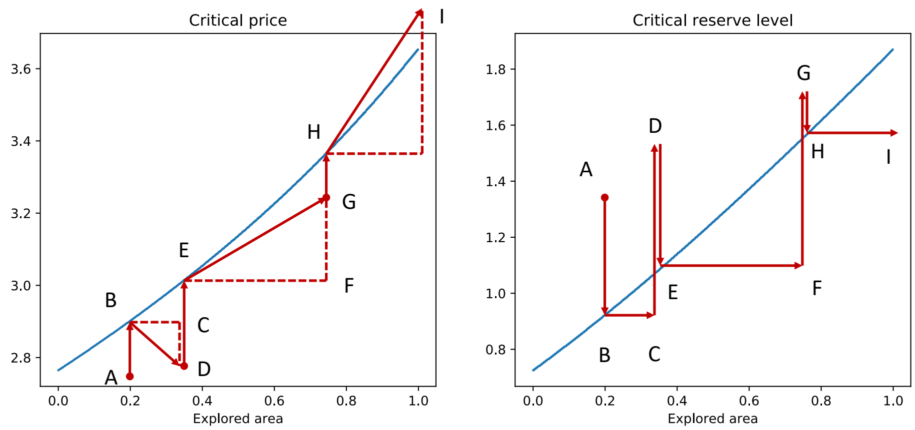
Notes: The left graph motivates the optimal exploration policy by plotting known reserves (y-axis) against the explored area (x-axis). There exists a critical reserve level shown in yellow that increases in the explored area. If known reserves exceed the critical level, exploration is zero. As soon as consumption decreases the stock of known reserve level, exploration starts at infinite speed (zero time) until either enough new discoveries are found to push the stock of known reserves above the critical level or all unexplored area is exhausted. A sample path is shown by $A \to B_0 \to B \to C \to D_0 \to D \to E \to F_0 \to F \to G_0 \to G \to H \to K \to L$. The right graph plots explored area, known reserves, price and consumption rate (y-axes) against time (x-axis), using the same letters to mark events. The sample path start at point A when known reserves are large enough so exploration is zero, price rises at the rate of interest, and the consumption rate decreases in time until known reserves hit level B_0. The exploration activity in zero time results in a finding (B) which pushes known reserves up by the size of the discovery (C). Note on the right graph how all variables jump in zero time to their new levels between B and C. The size of the jumps depend on the random amount of explored area that is required for the next discovery. The process of no exploration, price rising at the rate of interest, consumption rate declining than repeats itself $C \to D_0$ before the next exploration starts $D_0 \to E$. The exploration that starts at point F_0 requires a large amount of area until the next discovery is made (G_0), which is below the critical reserve level, and hence the next exploration is immediately started until another discovery is made (H).

Figure 3: Representation of the optimal consumption-exploration strategy



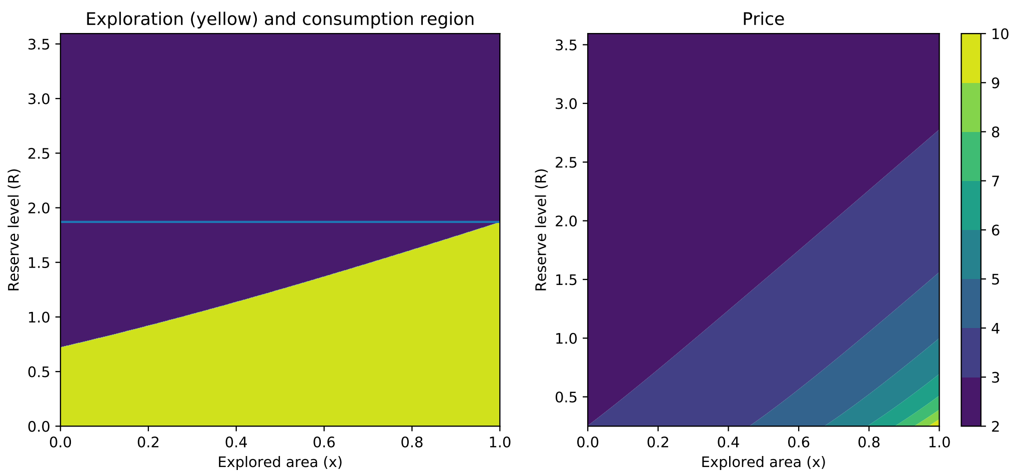
Notes: The strategy is determined by the critical price / critical reserve level (shown in blue). If the initial reserves are above the critical reserve level (point A), the agent consumes until both price and reserve level hit their respective critical levels (point B), after which an exploration period starts (segment BC). Upon finding a new deposit, the reserve level increases by amount a (segment CD). If the new deposit is found quickly, the new reserve level (point D) will be well above the critical level, consequently the new price level (point D) will be well below the critical level and below its previous value (point B), so that new find leads to a price drop. If the new deposit takes a long time to find (segment EF), the new reserve level (point G) may end up just above the critical level and the new price (point G) will be just below the critical level and above its previous value (point E), so that the new find leads to a price rise. It may also happen that the new reserve level will still be below the critical level; in this case exploration will continue without consumption. Finally, it may happen that the entire area is explored and no new deposit is found (segment HI): in this case the new price (point I) will be above the critical level, thus the price always jumps upward at the end of the exploration regime.

Figure 4: Representation of the optimal consumption-exploration strategy



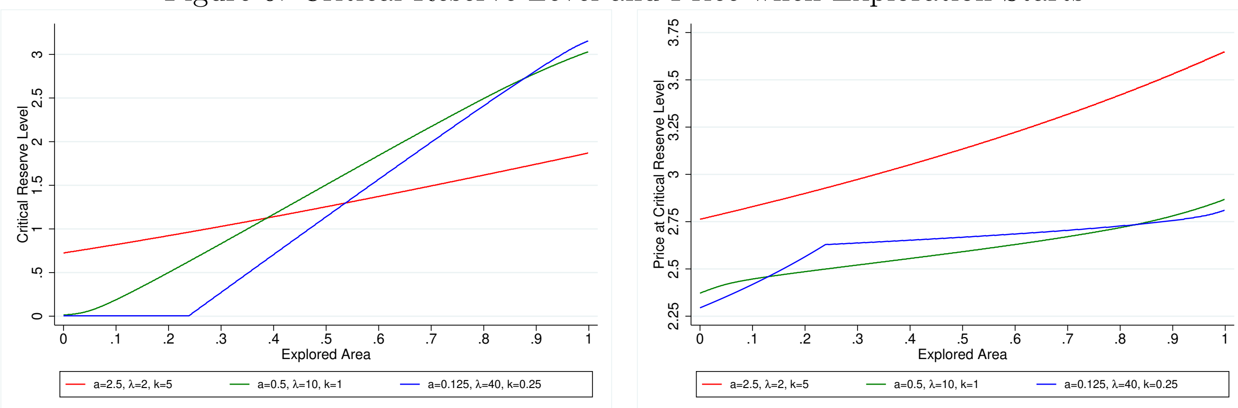
Notes: The strategy is determined by the critical price / critical reserve level (shown in blue). If the initial reserves are above the critical reserve level (point A), the agent consumes until both price and reserve level hit their respective critical levels (point B), after which an exploration period starts (segment BC). Upon finding a new deposit, the reserve level increases by amount a (segment CD). If the new deposit is found quickly, the new reserve level (point D) will be well above the critical level, consequently the new price level (point D) will be well below the critical level and below its previous value (point B), so that new find leads to a price drop. If the new deposit takes a long time to find (segment EF), the new reserve level (point G) may end up just above the critical level and the new price (point G) will be just below the critical level and above its previous value (point E), so that the new find leads to a price rise. It may also happen that the new reserve level will still be below the critical level; in this case exploration will continue without consumption. Finally, it may happen that the entire area is explored and no new deposit is found (segment HI): in this case the new price (point I) will be above the critical level, thus the price always jumps upward at the end of the exploration regime.

Figure 5: Separating Phases of Exploration and Consumption



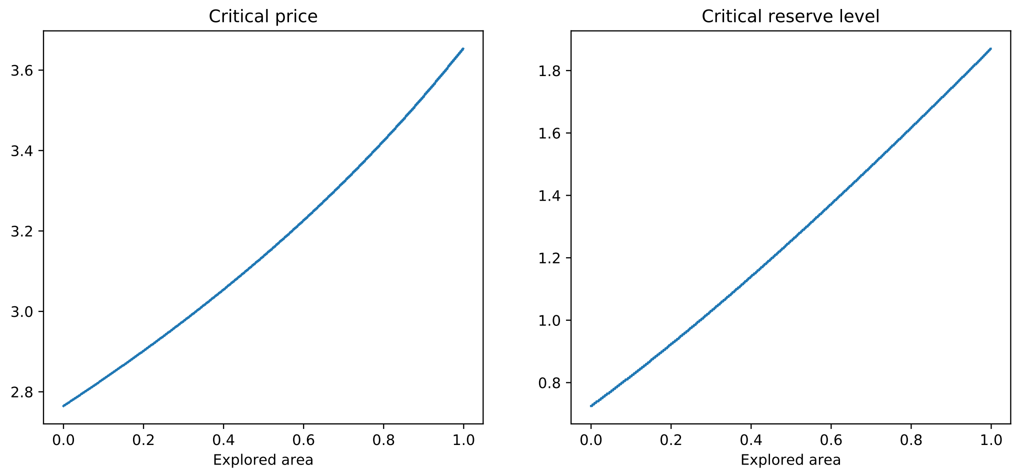
Notes: The left graph distinguishes the exploration region (yellow) and consumption region (violet). The blue vertical line shows the theoretical limit of the exploration frontier as $x \to 1$. The right graph shows contour plot of the price function. The horizontal axis corresponds to the reserve level and the vertical axis corresponds to the unexplored area x. Parameter values are: demand elasticity $\alpha - 1 = -0.5$, interest rate $r = 0.02$, discovery size $a = 2.5$, discovery intensity $\lambda = 2$ and marginal exploration cost $k = 5$.

Figure 6: Critical Reserve Level and Price when Exploration Starts



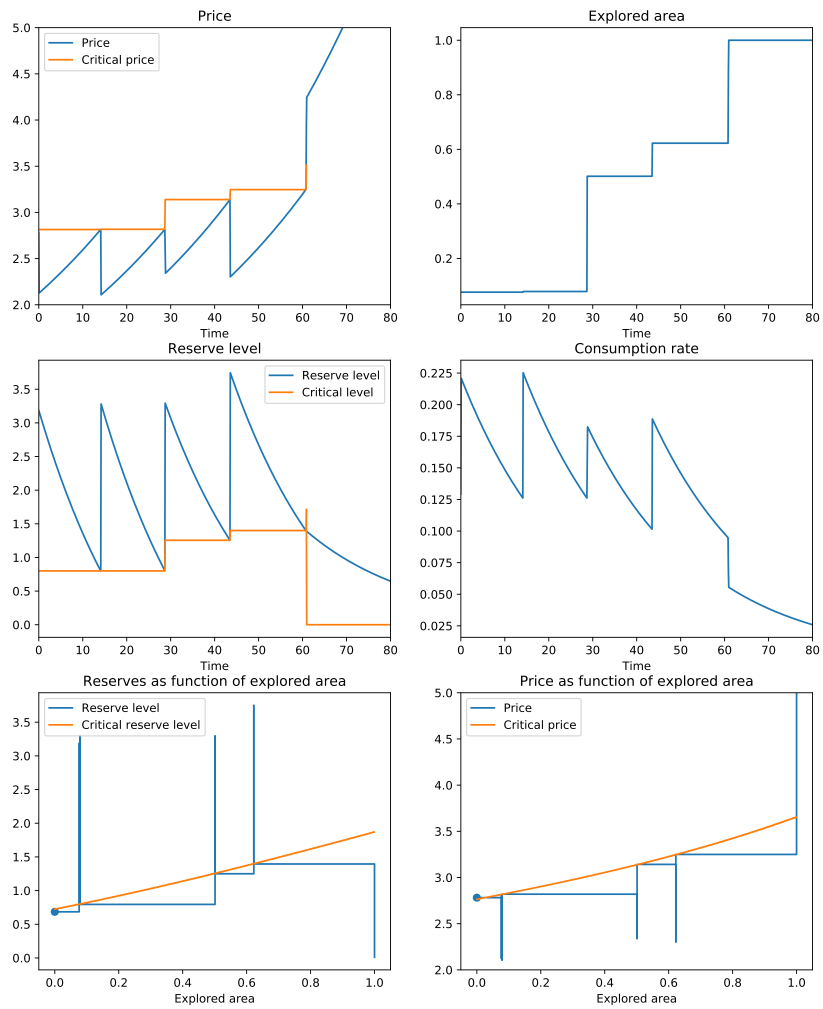
Notes: The left graph displays the critical reserve stock when exploration starts. Optimal exploration follows a bang-bang solution: it is zero if reserves exceed the critical reserve level, but start at infinite speed as soon as consumption decreases proven reserves to the critical level. Exploration stops either if new discoveries bring the known reserves again above the critical reserve level or if all unexplored area is exhausted. The graph highlights the crucial role of the form of uncertainty by picking parameters where the expected number of discoveries $a \times \lambda = 5$ and the cost per unit of discovery $\frac{k}{\lambda} = 0.1$ are the same in each case. The only difference is that the uncertainty (standard deviation) of discoveries $a\sqrt{\lambda} = \frac{5}{\sqrt{\lambda}}$ is decreasing in λ. The right graph displays the price of the resource at the critical level. The other constant parameters are a demand elasticity $\alpha - 1 = -0.5$ and interest rate $r = 0.02$.

Figure 7: Critical Price and Reserve Level



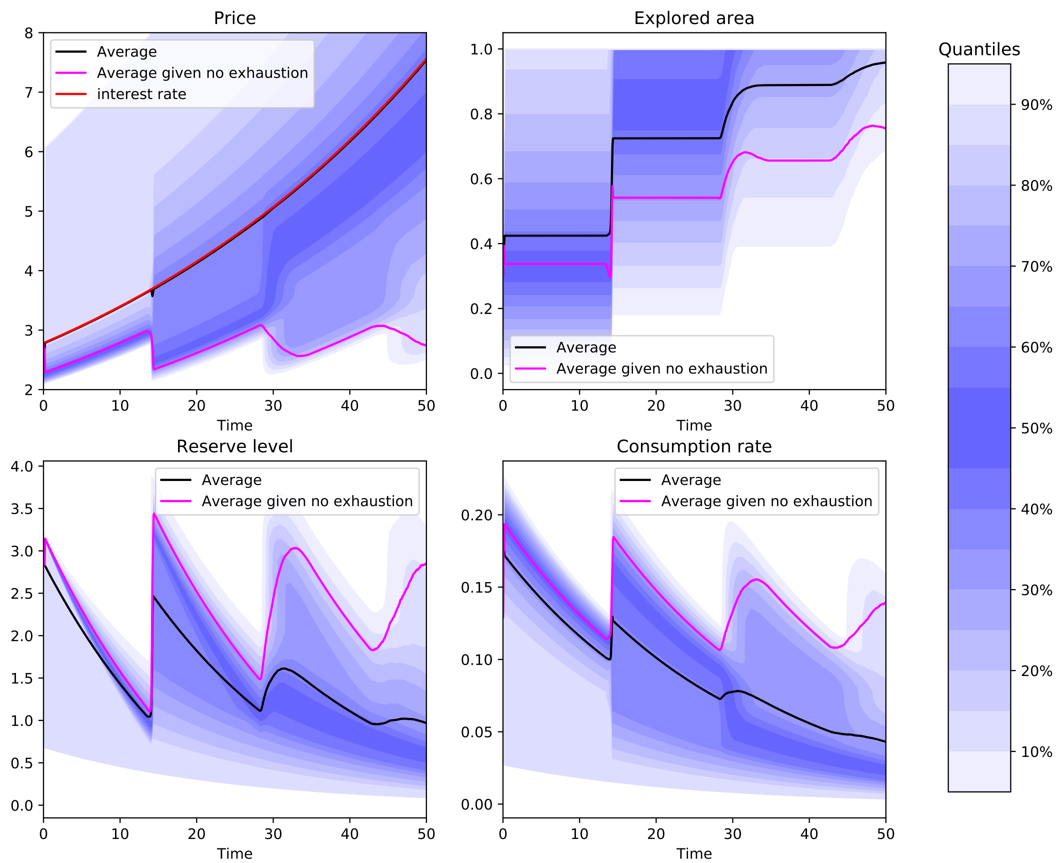
Notes: Right graph displays the critical reserve level (exploration frontier) as a function of the explored area. The left graph shows the corresponding price of the resource when reserves equal the critical reserve level and exploration starts. Parameter values are: demand elasticity $\alpha - 1 = -0.5$, interest rate $r = 0.02$, discovery size $a = 2.5$, discovery intensity $\lambda = 2$ and marginal exploration cost $k = 5$.

Figure 8: Evolution of Variables under One Model Simulation



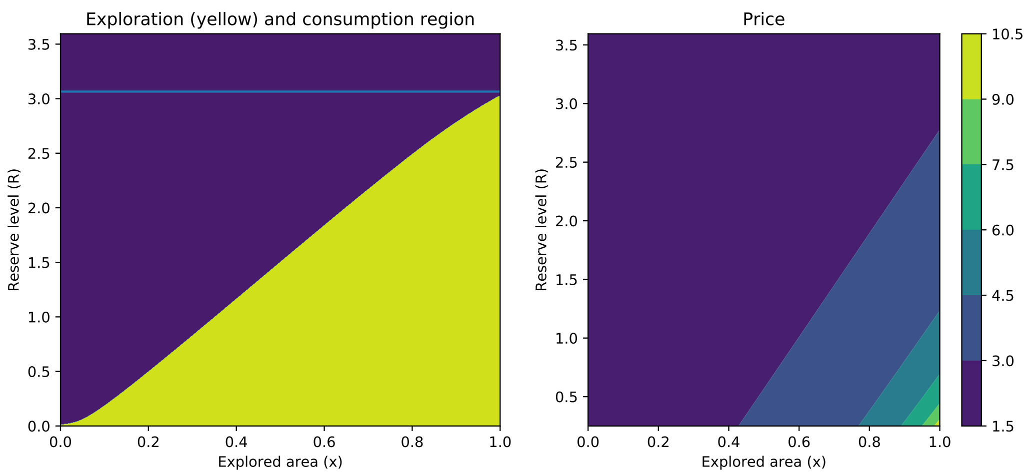
Notes: The four upper graphs display the evolution of the resource price, explored area, reserve level and the consumption rate over time under one model run as blue lines. The two lower graphs display the evolution of price and reserves as function of explored area. Whenever the price is lower than the critical price (shown in orange), or equivalently reserves are lager than the critical price (also shown in orange) in the left column, there is no exploration, price rises at the rate of interest, and consumption decreases. Once the price (or reserve level) hit the critical level, exploration occurs in zero time until the new discovery (or multiple discoveries) increases the reserve above the critical level or all area is explored. The different vertical jumps in the top right graph are the random amount of area that is required for the next discovery. Discoveries are all of equal size a, the vertical jumps in the bottom left graph. Once all area is explored (here at $t = 52$), the price always jumps upward. Parameter values are: demand elasticity $\alpha - 1 = -0.5$, interest rate $r = 0.02$, discovery size $a = 2.5$, discovery intensity $\lambda = 2$ and marginal exploration cost $k = 5$.

Figure 9: Distribution of Variables under 1000 Model Simulations



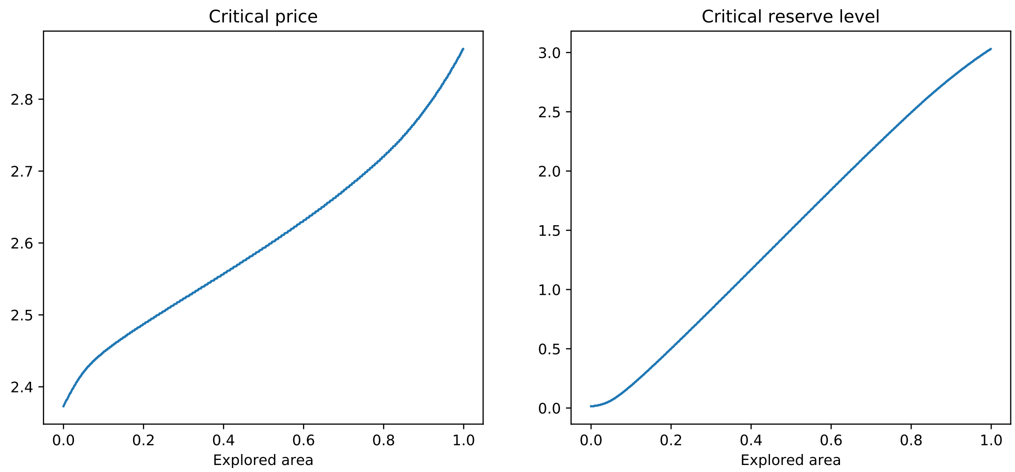
Notes: Figure replicates 1000 model runs of Figure 8. Black lines display the average resource price, explored area, reserve level and the consumption rate over time. The price graph in the top left also displays a path that rises at the rate of interest in red, which equals the average price path. Shaded areas display the distribution of outcomes. Note how the median price is below the average price due to the asymmetry of the distribution, i.e., in most cases the price will rise at less than the rate of interest. There is no uncertainty until $t = 3$, the first time the price equals the critical price and the stochastic exploration starts. Parameter values are: demand elasticity $\alpha - 1 = -0.5$, interest rate $r = 0.02$, discovery size $a = 2.5$, discovery intensity $\lambda = 2$ and marginal exploration cost $k = 5$.

Figure 10: Exploration and Consumption - High Likelihood of Discoveries



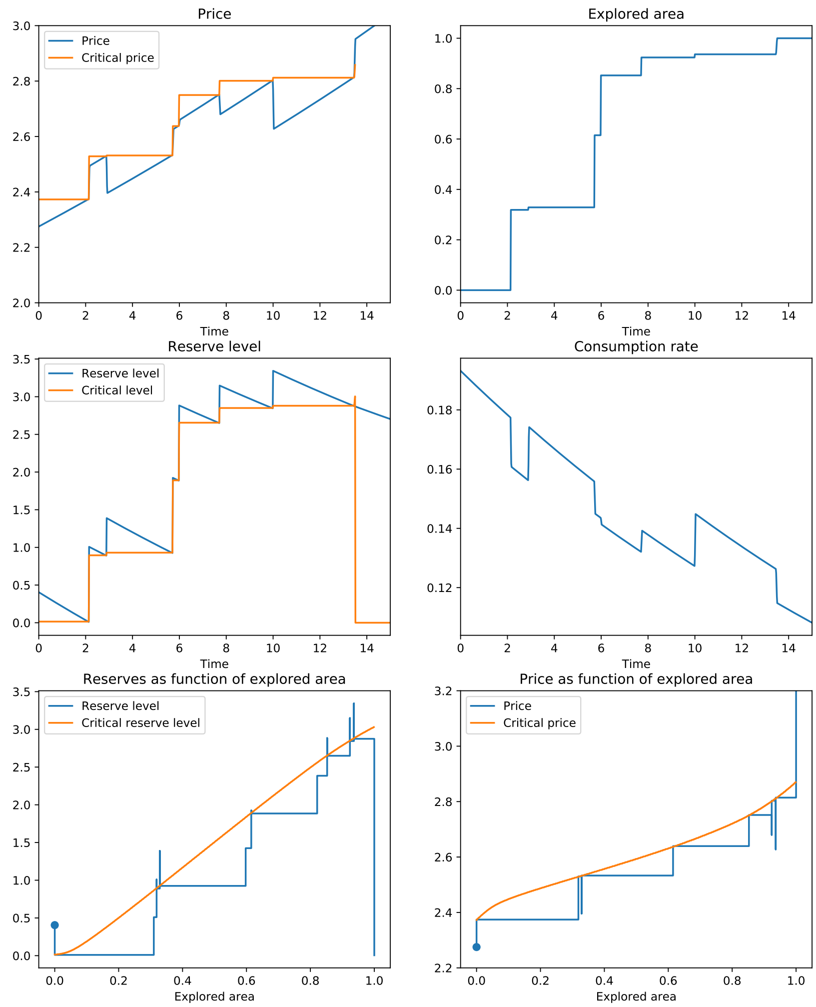
Notes: Figure replicates Figure 6 for different parameter values of the costly stochastic discovery process. The left graph again distinguishes the exploration region (yellow) and consumption region (violet). The blue vertical line shows the theoretical limit of the exploration frontier as $x \to 1$. The right graph shows contour plot of the price function. The horizontal axis corresponds to the reserve level and the vertical axis corresponds to the unexplored area x. Parameter values are: demand elasticity $\alpha - 1 = -0.5$, interest rate $r = 0.02$, discovery size $a = 0.5$, discovery intensity $\lambda = 10$ and marginal exploration cost $k = 1$.

Figure 11: Critical Price and Reserve Level - High Likelihood of Discoveries



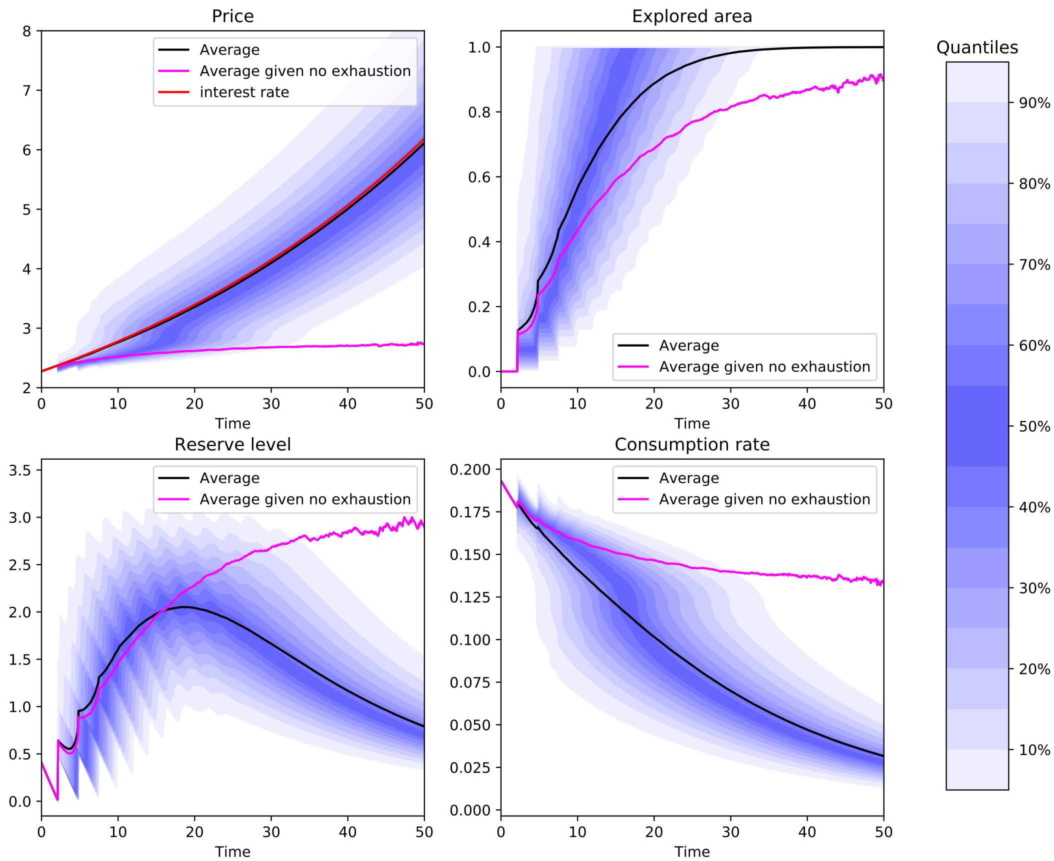
Notes: Figure replicates Figure 7 for different parameter values of the costly stochastic discovery process. The right graph again displays the critical reserve level (exploration frontier) as a function of the explored area. The left graph shows the corresponding price of the resource when reserves equal the critical reserve level and exploration starts. Parameter values are: demand elasticity $\alpha - 1 = -0.5$, interest rate $r = 0.02$, discovery size $a = 0.5$, discovery intensity $\lambda = 10$ and marginal exploration cost $k = 1$.

Figure 12: Evolution of Variables - High Likelihood of Discoveries



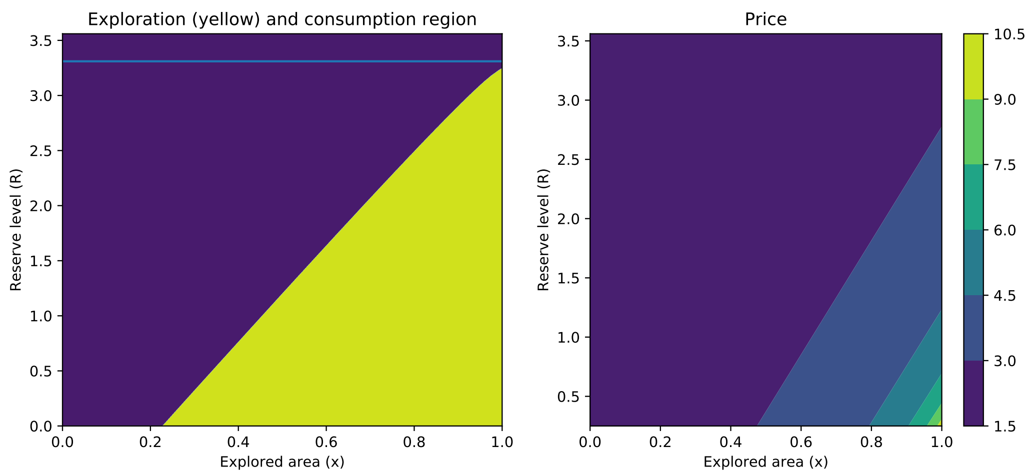
Notes: Figure replicates Figure 8 for different parameter values of the costly stochastic discovery process. The four upper graphs display the evolution of the resource price, explored area, reserve level and the consumption rate over time under one model run as blue lines, and the two lower graphs display the evolution of reserves and the price as function of explored area. Whenever the price is lower than the critical price (shown in orange), or equivalently reserves are lager than the critical price (also shown in orange) in the left column, there is no exploration, price rises at the rate of interest, and consumption decreases. Once the price (or reserve level) hit the critical level, exploration occurs in zero time until the new discovery (or multiple discoveries) increases the reserve above the critical level or all area is explored. Parameter values are: demand elasticity $\alpha - 1 = -0.5$, interest rate $r = 0.02$, discovery size $a = 0.5$, discovery intensity $\lambda = 10$ and marginal exploration cost $k = 1$.

Figure 13: Distribution of Variables - High Likelihood of Discoveries



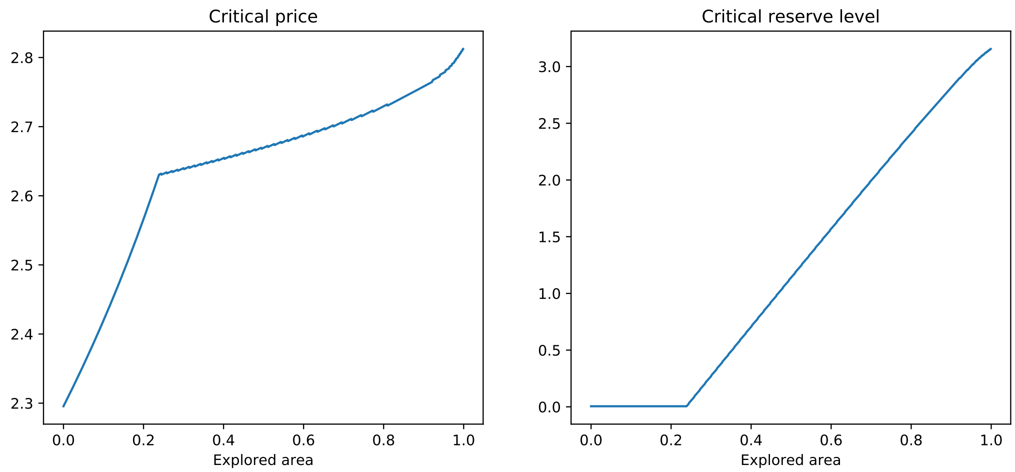
Notes: Figure replicates Figure 9 for different parameter values of the costly stochastic discovery process. Figure replicates 1000 model runs of Figure 12. Black lines display the average resource price, explored area, reserve level and the consumption rate over time. The price graph in the top left also displays a path that rises at the rate of interest in red, which equals the average price path. Shaded areas display the distribution of outcomes. There is no uncertainty until $t = 3$, the first time the price equals the critical price and the stochastic exploration starts. Parameter values are: demand elasticity $\alpha - 1 = -0.5$, interest rate $r = 0.02$, discovery size $a = 0.5$, discovery intensity $\lambda = 10$ and marginal exploration cost $k = 1$.

Figure 14: Exploration and Consumption - Very High Likelihood of Discoveries



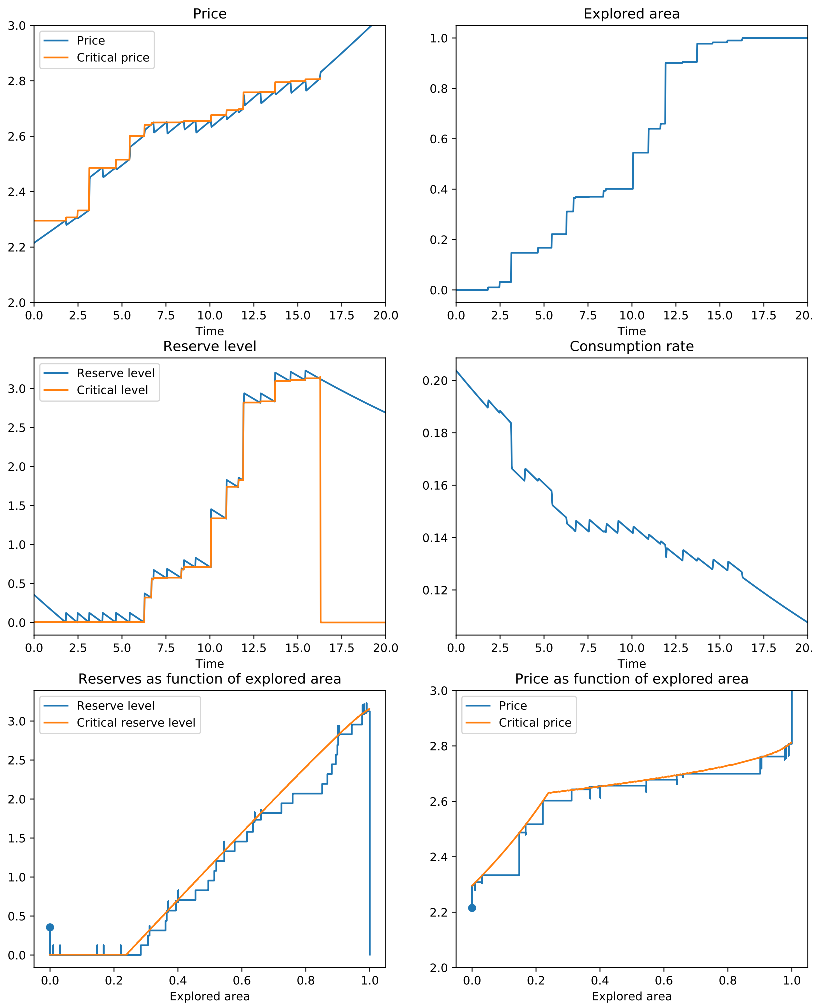
Notes: Figure replicates Figure 6 for different parameter values of the costly stochastic discovery process. The left graph again distinguishes the exploration region (yellow) and consumption region (violet). The blue vertical line shows the theoretical limit of the exploration frontier as $x \to 1$. The right graph shows contour plot of the price function. The horizontal axis corresponds to the reserve level and the vertical axis corresponds to the unexplored area x. Parameter values are: demand elasticity $\alpha - 1 = -0.5$, interest rate $r = 0.02$, discovery size $a = 0.125$, discovery intensity $\lambda = 40$ and marginal exploration cost $k = 0.25$.

Figure 15: Critical Price and Reserve Level - Very High Likelihood of Discoveries



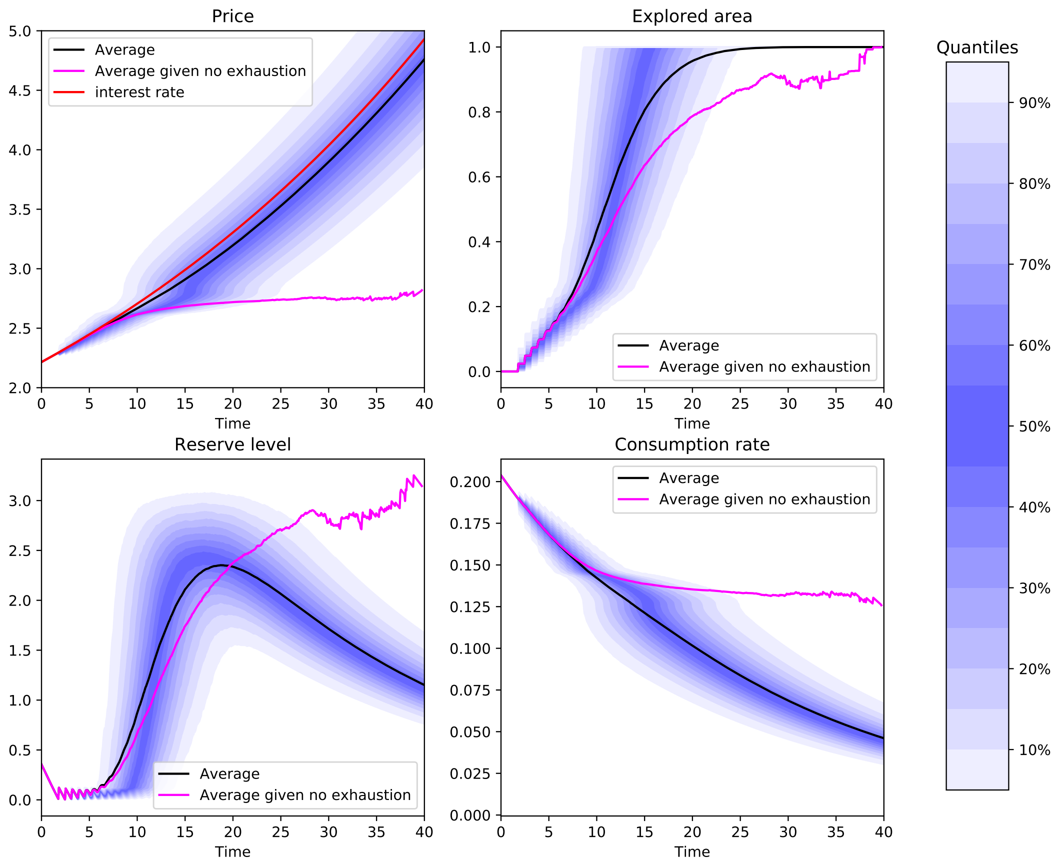
Notes: Figure replicates Figure 7 for different parameter values of the costly stochastic discovery process. The right graph again displays the critical reserve level (exploration frontier) as a function of the explored area. The left graph shows the corresponding price of the resource when reserves equal the critical reserve level and exploration starts. Parameter values are: demand elasticity $\alpha - 1 = -0.5$, interest rate $r = 0.02$, discovery size $a = 0.125$, discovery intensity $\lambda = 40$ and marginal exploration cost $k = 0.25$.

Figure 16: Evolution of Variables - Very High Likelihood of Discoveries



Notes: Figure replicates Figure 8 for different parameter values of the costly stochastic discovery process. The four upper graphs display the evolution of the resource price, explored area, reserve level and the consumption rate over time under one model run as blue lines, and the two lower graphs display the evolution of reserves and the price as function of explored area. Whenever the price is lower than the critical price (shown in orange), or equivalently reserves are lager than the critical price (also shown in orange) in the left column, there is no exploration, price rises at the rate of interest, and consumption decreases. Once the price (or reserve level) hit the critical level, exploration occurs in zero time until the new discovery (or multiple discoveries) increases the reserve above the critical level or all area is explored. Parameter values are: demand elasticity $\alpha - 1 = -0.5$, interest rate $r = 0.02$, discovery size $a = 0.125$, discovery intensity $\lambda = 40$ and marginal exploration cost $k = 0.25$.

Figure 17: Distribution of Variables - Very High Likelihood of Discoveries



Notes: Figure replicates Figure 9 for different parameter values of the costly stochastic discovery process. Figure replicates 1000 model runs of Figure 12. Black lines display the average resource price, explored area, reserve level and the consumption rate over time. The price graph in the top left also displays a path that rises at the rate of interest in red, which equals the average price path. Shaded areas display the distribution of outcomes. There is no uncertainty until $t = 3$, the first time the price equals the critical price and the stochastic exploration starts. Parameter values are: demand elasticity $\alpha - 1 = -0.5$, interest rate $r = 0.02$, discovery size $a = 0.125$, discovery intensity $\lambda = 40$ and marginal exploration cost $k = 0.25$.