Asymmetry of information in finance

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The old paradigm
for trading

Arrow-Debreu markets

. "*A contract for the transfer of a commodity now specifies, in addition to its physical properties, its location and its date, an event on the occurrence of which the transfer is conditional*" (Debreu)
The mathematical model then consists of specifying, at the initial time $t = 0$

- a finite set of possible states of the world $\Omega = \{\omega_1, ..., \omega_K\}$; an event is a subset $A \subset \Omega$
- a finite set of commodities, indexed from $i = 1$ to I; each commodity is available in any non-negative quantity
- a finite set of traders, indexed from $j = 1$ to J; each trader is characterized by his preferences over goods bundles and his initial allocation

Information does not matter

A goods bundle (also called a contingent claim) is a pair $(x \mid A)$, meaning that quantities $x = (x_1, .., x_I) \in R_+^I$ are to be delivered if the event A occurs. All trades occur at time $t = 0$, and traders are commited from then on. The market is *complete* if all contingent claims can be traded.
An equilibrium price is a price system (one for each contingent claim) such that the market clears (demand equals supply)
If the market is complete, and if every trader has convex preferences over contingent claims:

- there exists an (and possibly several) equilibrium
- every equilibrium is Pareto optimal, and every Pareto optimum can be realized as an equilibrium for some initial allocation $\left(x^1, ..., x^J\right) \in \left(R_+^I\right)^J$.

"*A model that I perceived is the critical functioning structure that defines how the world works*" Alan Greenspan, testimony to the US Congress, October 23, 2008

Information matters

Akerlof (1970). Consider a population of $2N$ people:

- N of them own a car and want to sell it
- N of them don't own a car and want to buy one

So there are N cars for sale. The quality of these cars is given by x, with $0 \leq x \leq 2$ is uniformly distributed. A car of quality x is worth $p = x$ for the seller, but $p = 3x/2$ for the buyer. Two cases:

- full information (buyers and sellers know the quality): cars of quality x are traded at any price p with $x \leq p \leq 3x/2$

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- full information (buyers and sellers know the quality): cars of quality x are traded at any price p with $x \leq p \leq 3x/2$
- asymmetric information (only the seller know the quality). Then all cars sell at the same price p. The average quality of cars on the market is $p/2$. Buying a car then costs p and is worth $3/2 \times p/2 = 3p/4 < p$. So there is no buyer

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- lack of information kills the market

A new paradigm for trading

Contract theory

The *decision structure* is as follows:

- there is a principal and a set of agents
- the principal moves first and offers one or several contracts to the agents
- each agent picks one or none

The *incentive structure* is as follows:

- a contract consists of an action by the agent and a payment from (or to) the principal
- the agent then performs the action and gets (or gives) the payment
- each accepted contract brings some profit to the principal

The *principal's problem* consists of devising the contract menu so as to maximize his expectedprofit

Contract theory

There are two types of *information structures:*

- Adverse selection (hidden information: poor driver)
  - each agent has a type x
  - each agent knows his type
  - the principal knows the distribution of types $d\mu(x)$
- Moral hazard (hidden action: risky driver)
  - the actions $a \in A$ of the agent cannot be directly observed by the principal
  - these actions will influence an outcome $z \in Z$ which can be observed by the principal and the agent

What to expect

If the principal knows the agent's type, or can observe the agent's actions, the latter will get her reservation utility. This is the *first-best* situation.
Asymmetry of information protects the agent. The principal then looks for a *second-best* outcome, which, from his point of view, will be inferior to the first-best
So there is an *informational rent* to the agent, which is higher for high-quality agents than for low-quality ones: good drivers pay less than they should, poor drivers are reduced to their reservation utility, or are not insured

Selling a security to an investor
with unknown risk aversion

There is a principal and a set of agents. They will trade risk, represented by a random variable $X \in L^2(\Omega)$.

The agents

Each agent holds an investment $Y = \sum_{k=1}^{K} \alpha_k B_k$ in securities $1, B_1, ..., B_K$. If she acquires (or sells) X at a price π, her utility is

$$\mathbb{E}[X+Y] - \lambda \mathbb{V}\mathrm{ar}[X+Y] - \pi$$

Without loss of generality, we assume that $\mathbb{E}[X] = \mathbb{E}[B_k] = 0$. The *type* of the agent is then:

$$\theta = (\lambda, \beta_1, ..., \beta_K) \text{ with } \beta_k = \frac{\alpha_k}{2\lambda}$$

The utility of an agent of type θ is:

$$U(\lambda, \beta_1, ..., \beta_k; X) = \mathbb{E}[X] - \lambda \mathbb{V}\mathrm{ar}[X] - \beta \cdot \mathbb{C}\mathrm{ov}(X, B) - 4\lambda^3 \|\beta\|^2$$

The constant term at the end plays no role in the optimisation. The reservation utility (no incentive to trade) is $U(\theta; 0) = 0$.

The contract

A *contract* is a pair (X, π) of maps $\theta \longmapsto (X(\theta), \pi(\theta))$ from Θ to $L^2 \times R$. A contract (X, π) is

- *individually rational* (IR) if

$$U(\theta, X(\theta)) - \pi(\theta) \geq 0$$

- *incentive-compatibe* (IC) if:

$$U(\theta, X(\theta)) - \pi(\theta) \geq U(\theta, X(\theta')) - \pi(\theta') \quad \forall \theta'$$

An *allocation* $\theta \rightarrow X_\theta$ is incentive-compatible if there exists some $\theta \rightarrow \pi(\theta)$ such that the contract (X, π) is incentive-compatible

We introduce the *indirect utility* of agent $\theta = (\lambda, \beta_1, ..., \beta_K)$:

$$\begin{aligned} v(\theta) &= \max_{\theta'} \{U(\theta, X(\theta')) - \pi(\theta')\} \\ &= \max_{\theta'} \{\mathbb{E}[X(\theta')] - \lambda \mathbb{V}\text{ar}[X(\theta')] - \beta \cdot \mathbb{C}\text{ov}(X(\theta'), B) - \pi(\theta' \end{aligned}$$

Incentive-compatibility and convexity

Theorem

v is a convex function of $\theta = (\lambda, \beta_1, ..., \beta_K)$, and an allocation $\theta \rightarrow X_\theta$ is IC if and only if

$$\forall \theta, \quad (-\mathbb{V}\mathrm{ar}\left[X(\theta)\right], -\mathbb{C}\mathrm{ov}\left(X(\theta), B\right)) \in \partial v(\theta) \tag{2}$$

Conversely, if v is a convex function and an allocation $\theta \rightarrow X_\theta$ satisfies (2), then it is incentive-compatible

Here $\partial v(\theta)$ denotes the subgradient of v at the point θ. It is defined by the condition:

$$\varphi \in \partial v(\theta) \Longleftrightarrow v(\theta') - v(\theta) \geq (\theta' - \theta, \varphi) \quad \forall \theta'$$

Note that v is differentiable a.e. At every point of differentiability, we have:

$$\partial v(\theta) = \{\nabla v(\theta)\}$$

Proof

The formula (1) defines $v(\theta)$ as the pointwise supremum of a family of affine functions. So $v(\theta)$ is a convex function. If $\theta \to X_\theta$ is IC, then there exists some $\theta \to \pi_\theta$ such that (X_θ, π_θ) is IC, so that the maximum on the right-hand side of (1) is attained for $\theta' = \theta$. This means precisely (2) Conversely, suppose v is convex and (2) holds. Set

$$\pi(\theta) = \mathbb{E}[X(\theta)] - \lambda \mathbb{V}\mathrm{ar}[X(\theta)] - \beta \mathbb{C}\mathrm{ov}(X(\theta), B) - v(\theta)$$

From (2) we have:

$$v(\theta) - v(\theta') \geq -\mathbb{C}\mathrm{ov}\left(X(\theta'), B\right) \cdot (\beta - \beta') - \mathbb{V}\mathrm{ar}\left[X(\theta')\right](\lambda - \lambda')$$

Plugging in the value for $\pi(\theta)$, we find that $X(\theta)$ is (IC):

$$\mathbb{E}[X(\theta)] - \lambda \mathbb{V}\mathrm{ar}[X(\theta)] - \beta \cdot \mathbb{C}\mathrm{ov}(X(\theta), B) - \pi(\theta) \geq$$
$$\mathbb{E}\left[X(\theta')\right] - \mathbb{C}\mathrm{ov}\left(X(\theta'), B\right) \cdot \beta - \mathbb{V}\mathrm{ar}\left[X(\theta')\right]\lambda - \pi(\theta')$$

The principal can produce any random variable X at a cost $C(X)$. If he sells $X(\theta)$ to type θ, he makes $\pi(\theta)$. He knows the density μ of types:

$$\mu(\theta) \geq 0 \text{ and } \int \mu(\theta)\, d\theta < \infty$$

He is risk-neutral, so he is maximizing his expected profit:

$$\Phi(X, \pi) = \sup \int (\pi_\theta - C(X_\theta))\, \mu(\theta)\, d\theta$$

over of all (IR) and (IC) contracts. Note that:

$$\begin{aligned}\pi_\theta - C(X_\theta) &= \mathbb{E}[X(\theta)] - \lambda \mathbb{V}\mathrm{ar}[X(\theta)] - \beta \cdot \mathbb{C}\mathrm{ov}(X(\theta), B) - v(\theta) - \\ &= \mathbb{E}[X(\theta)] + \lambda \frac{\partial v}{\partial \lambda} + \sum \beta_k \frac{\partial v}{\partial \beta_k} - v - C(\theta)\end{aligned}$$

For instance, if he has access to a financial market, he has $C(X) = \mathbb{E}[ZX]$ with $Z \geq 0$ and $\mathbb{E}[Z] = 1$.

Theorem

The principal's problem is:

$$\max \int \left[\begin{array}{c} \lambda \frac{\partial v}{\partial \lambda} + \sum (\beta_k + \xi_k) \frac{\partial v}{\partial \beta_k} - v + \\ + \sqrt{\mathbb{V}\mathrm{ar}[Z] - \sum \mathbb{C}\mathrm{ov}(Z, B_k)^2} \sqrt{-\frac{\partial v}{\partial \lambda} - \sum_k \left(\frac{\partial v}{\partial \beta_k}\right)^2} \end{array} \right] d\mu(\lambda, \beta$$

$v(\lambda, \beta_1, ..., \beta_K)$ *convex,* $v \geq 0$

The one-dimensional case: first-best

Suppose all agents have the same risk aversion $\lambda > 0$, but hold different amounts of a single asset B (the market portfolio) with $\mathbb{V}\mathrm{ar}[B] = 1$ and $\mathbb{E}[B] = 0$. The agent's type is then $\beta \in \left[\underline{\beta},\ \bar{\beta}\right]$, where $\beta B\ /2\lambda$ is the amount of asset she holds.

If the type of the agent is known to the principal, he will sell her X (with $\mathbb{E}[B] = 0$) and charge π, with a profit of $\pi - \mathbb{E}[XZ]$. So the principal's problem becomes:

$$\max \pi - \mathbb{E}[XZ]$$
$$\mathbb{E}[X] - \lambda \mathbb{V}\mathrm{ar}[X] - \beta \mathbb{C}\mathrm{ov}[B, X] \geq \pi$$

with a solution $X(\beta) = -\frac{1}{2\lambda}(Z - \mathbb{E}[Z]) - \frac{\beta}{2\lambda} B$ and $\pi(\beta) = \frac{1}{4\lambda}\left(2\beta^2 - \mathbb{V}\mathrm{ar}[Z]\right)$. Note that all agents now carry the same risk. The agent is indifferent between this and his original endowment, and the principal makes $\frac{1}{2\lambda}\mathbb{V}\mathrm{ar}[Z + \beta B]$.

The second-best

Suppose types are uniformly distributed. The principal's problem becomes:

$$\sup \int_{\underline{\beta}}^{\bar{\beta}} \left(\beta v' - v - \lambda v'^2 + \mathbb{C}\mathrm{ov}\left[B, Z\right] v\right)' d\beta$$
$$v \geq 0, \text{ convex}$$

The solution can be found explicitly, integrating by parts and using the fact that v is convex iff v' is non-decreasing. We get:

$$\begin{aligned} X(\beta) &= \frac{-1}{2\lambda} Z_0 - \frac{1}{2\lambda}\left(2\beta - \bar{\beta}\right) B_0 \text{ if } \beta \geq \frac{1}{2}\left(\bar{\beta} - \mathbb{C}\mathrm{ov}\left[B, Z\right]\right) \\ X(\beta) &= \frac{-1}{2\lambda} Z_0 + \frac{1}{2\lambda} \mathbb{C}\mathrm{ov}\left[B, Z\right] B_0 \text{ if } \beta \leq \frac{1}{2}\left(\bar{\beta} - \mathbb{C}\mathrm{ov}\left[B, Z\right]\right) \end{aligned}$$

The high types derive positive utility from the contract (informational rent), while the low types are at their reservation utility.

Multidimensional case

$$(\mathrm{P}) \quad \begin{cases} \sup \int_\Omega L\left(x, v, \nabla v\right) dx \\ v \geq 0,\ v \text{ convex} \end{cases}$$

Choose points $x_i, 1 \leq i \leq N$, in Ω, and consider the problem:

$$(\mathrm{P}_N) \quad \begin{cases} \sup \sum_{i=1}^N L\left(x_i, v_i, V_i^j\right) \\ v_i \geq 0 \quad \forall i \\ v_j - v_i \geq \sum_k V_i^k \left(x_j^k - x_i^k\right) \quad \forall i, j \end{cases}$$

Let $\left(\bar{v}_i, \bar{V}_i^j\right)$ solve (P_N). The approximate solution to (P) then is:

$$v_N(x) := \sup_i \left\{ v_i + \sum_k V_i^k \left(x^k - x_i^k\right) \right\}$$

Managing the management:
how to prevent excess risk-taking

The problem of limited liability

- Shareholders vs management
- The public vs the firm (BP)
- The government vs the banks (too big to fail)

Two possible answers:

- Regulation and overseeing. Generates bureaucracy, and shifts the problem: *quis custodiet ipsos custodes* ? Constitution design.
- Creating proper incentives. Bilateral contract design. Not always possible (soldiers), and even when possible has its own limits

Moral hazard in continuous time

The agent is in charge of a project which generates a stream of revenue, which accrue to the principal

Accidents occur, generating large losses, the cost of which will be borne by the principal

The risk (probability of losses) can be reduced by *due diligence* from the agent

Due diligence is costly to the agent, and *not directly observable* by the principal

The principal seeks to ensure due diligence from the agent by offering her a performance-based contract

Contracts must be based on *observables*, ie the stream of revenue and the occurence of accidents

The model

There is a project going on, with size X_t generating a stream of revenue R_t, and subject to accidents, which occur according to a Poisson process N_t :

$$\frac{dR_t}{X_t} = \mu dt - CdN_t$$

Revenues accrue to the principal (the owner), who also bears the cost of accidents. The frequency of accidents depends on the effort level of the agent. Between t and $t + dt$, she has two choices:

- either exterting effort, in which case the probability of an accident is λdt and her cost is 0
- or shirking, in which case the probability raises to $(\lambda + \Delta\lambda)\, dt$ and her private benefit is BX_t

The contract

The principal can decide on two things:

- the size of the project: at any time, he can downsize it costlessly, all the way to 0 or upsize it, at the maximum rate g_t, with $0 \leq g_t \leq \gamma$ and cost $c > 0$
- the salary of the agent, which depend on the past history of accidents

A contract will specify the rules for down/upsizing the project, the rules for terminating it, the agent's effort Λ_t, with $\Lambda_t \in \{\lambda,\ \lambda + \Delta\lambda\}$, and the salary L_t. The streams of revenue are then:

$$\text{(agent)} \quad \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(dL_t + \mathbf{1}_{\{\Lambda=\lambda+\Delta\lambda\}} BX_t dt\right)\right]$$

$$\text{(principal)} \quad \mathbb{E}\int_0^\infty e^{-rt}\left\{X_t\left[\mu - cg_t\right]dt - CdN_t - L_t dt\right\}$$

Maximum-effort behaviour

Assume the agent exerts effort Λ_t. Introduce her continuation utility at time t :

$$W_t = E\left[\int_t^\infty e^{-\rho s} dL_s \mid \mathcal{F}_t^N\right]$$

We want to see how Λ_t affects dW_t. This is done by introducing $U_t = E\left[\int_0^\infty e^{-\rho s} dL_s \mid \mathcal{F}_t^N\right]$, the utility garnered up to time t, and computing it in two different ways:

$$\begin{aligned} U_t &= \int_0^t e^{-\rho s} dL_s + e^{-\rho t} W_t \,(\Gamma) \\ U_t &= U_0 + \int_0^t e^{-\rho s} H_s \left(\Lambda_t ds - dN_s\right) \end{aligned}$$

where the last expression comes from the martingale representation theorem. Hence:

$$\begin{aligned} e^{-\rho t} H_t \left(\Lambda_t dt - dN_t\right) &= dU_t = e^{-\rho t} dL_t - \rho e^{-\rho t} W_t + e^{-\rho t} dW_t \\ dW_t &= \rho W_t dt + H_t \left(\Lambda_t dt - dN_t\right) - dL_t \end{aligned}$$

Incentive-compatible contracts

Suppose the agent has applied maximum effort $\Lambda_s = \lambda$ up to time t. Then $H_s, \ s \leq t$ is predictable (left-continuous) and $\mathbb{E}\left[H_t\left(\lambda dt - dN_t\right)\right] = 0$. What happens between t and $t + dt$?

- if she applies $\Lambda_t = \lambda$, then $\mathbb{E}\left[dW_t\right] = \rho W_t dt - dL_t$
- if she shirks, $\Lambda_t = \lambda + \Delta\lambda$, then $\mathbb{E}\left[dW_t\right] = \rho W_t dt - H_t \Delta\lambda dt + BX_t dt - dL_t$

For the contract to be IC, we need:

$$BX_t \leq H_t \Delta\lambda$$

Setting $b = \frac{B}{\Delta\lambda}$ we find that if there is an accident, the continuation utility of the agent must be reduced by bX at least. This is possible only if $W_t \geq bX_t$

The principal's problem

Consider the continuation value of the principal:

$$F(X, W) = \max_{\Gamma} E\left[\int_0^{\infty} e^{-rt}\left\{X_t\left[\mu - cg_t\right]dt - CdN_t - dL_t\right\} \mid X_0 = X, \ W_0$$

over all effort-inducing contracts It is defined for $X \geq 0$ and $W \geq bX$.
Recall that:

$$\begin{aligned} X_t &= X_0 + X_t^i + X_t^d \\ dX_t^i &= g_t X_t dt, \quad 0 \leq g_t \leq \gamma \\ dX_t^d &= (x_t - 1) X_t, \quad 0 \leq x_t \leq 1 \\ dW_t &= \rho W_t dt - dL_t + H_t(\lambda dt - dN_t) \end{aligned}$$

The HJB equation

The controls are

$$g_t, h_t = \frac{H_t}{X_t}, \ell_t = \frac{L_t}{X_t}, x_t$$

The corresponding HJB equation is:

$$\begin{aligned} rF = & \max_{g_t, h_t, \ell_t, x_t} \{ X_t [\mu - \lambda C - c g_t - \ell_t] + \\ & (\rho W_t dt + h_t X_t \lambda - \ell_t X_t) \frac{\partial F}{\partial W} + g_t X_t \frac{\partial F}{\partial X} \\ & - \lambda [F - F(x_t X_t, W_t - h_t X_t)] \} \end{aligned}$$

The reduced HJB equation

We will proceed by finding an (almost) explicit solution. This solution will have two properties:

- It will be homogeneous: $F(X, W) = Xf\left(\frac{W}{X}\right) = f(w)$.
- The size-adjusted value function $f(w)$ is concave

The system now becomes:

$$\begin{aligned} 0 &\leq g_t \leq \gamma, \quad b \leq h_t \leq \frac{W}{X} \\ 0 &\leq \ell_t, \quad 0 \leq x_t \leq 1 \end{aligned}$$

and we are looking for a function $f(w)$, which is concave and satisfies the following delay-differential equation:

$$\begin{aligned} rf &= \mu - \lambda\left(C + f(w)\right) + f'(w) + \max_{g,h,l,x}\left\{g\left(f(w) - wf'(w) - c\right)\right. \\ &\quad \left. -\ell\left(1 + f'(w)\right) + h\lambda f'(w) + \lambda x f\left(\frac{w-h}{x}\right)\right\} \end{aligned}$$

$$
\begin{aligned}
rf &= \mu - \lambda\left(C + f\left(w\right)\right) + f'\left(w\right) + \max_{g,h,l,x}\left\{g\left(f\left(w\right) - wf'\left(w\right) - c\right)\right. \\
&\quad \left. -\ell\left(1 + f'\left(w\right)\right) + h\lambda f'\left(w\right) + \lambda x f\left(\frac{w-h}{x}\right)\right\}
\end{aligned}
$$

The Sannikov model

The agent

The agent is in charge of a project which generates a stream of revenue for the principal:

$$dX_t = A_t dt + \sigma dZ_t$$

where Z_t is BM, $\sigma > 0$ is given, and A_t is the agent's effort. Her intertemporal utility is:

$$r\mathbb{E}\left[\int_0^\infty e^{-rt}\left(u\left(C_t\right) - h\left(A_t\right)\right) dt\right]$$

where C_t is the agent's salary, u her utility, and $h\left(A_t\right)$ her cost of effort, with $h\left(0\right) = 0$.

Contracts

A contract is a pair (C_t, A_t) adapted to (X_t, Z_t). As above, we look at the agent's continuation value:

$$W_t = r\mathbb{E}\left[\int_t^\infty e^{-r(s-t)}\left(u\left(C_s\right) - h\left(A_s\right)\right) ds \mid \mathcal{F}_t^Z\right]$$

Using the martingale representation theorem we find that there is a Z_t-adapted process Y_t (depending on C_t and A_t) such that:

$$\begin{aligned} dW_t &= r\left(W_t - u\left(C_t\right) + h\left(A_t\right)\right) dt + rY_t\sigma dZ_t \\ &= r\left(W_t - u\left(C_t\right) + h\left(A_t\right) - Y_t A_t\right) dt + rY_t dX_t \end{aligned}$$

Incentive compatible contracts

Suppose the agent has conformed to the contract (C_s, A_s) for $s \leq t$, and tries to shirk, by performing effort a in the following interval $[t, \ t + dt]$, and reverting to A_s for $s \geq t + dt$

- her cost on $[t, \ t + dt]$ is $rh(a)\,dt$
- her expected benefit on $[0, \ \infty]$ is $rY_t a dt$
- the balance is $r(aY_t - h(a))$

Theorem

Suppose:

$$Y_t A_t - h(A_t) = \max_{o \leq a \leq \bar{a}} \{aY_t - h(a)\} \qquad (3)$$

Then the contract (A_t, C_t) is (IC)

A beautiful proof

Suppose (A, C) does not satisfy condition (3). Then there is an alternative contract (A^*, C^*) with:

$$
\begin{aligned}
Y_t A_t^* - h(A_t^*) &\geq Y_t A_t - h(A_t) \quad \text{a.e} \\
\mathbb{P}\left[Y_t A_t^* - h(A_t^*) \geq Y_t A_t - h(A_t)\right] &> 0
\end{aligned}
$$

The agent picks $t > 0$ and plans to apply A^* for $s \leq t$ and A for $\geq t$. Expected utility at t :

$$
\begin{aligned}
V_t^* &= r \int_t^\infty e^{-rs} \left(u(C_s) - h(A_s^*)\right) ds + e^{-rt} W_t(A, C) \\
&= W_0(A, C) + r \int_0^t \left(h(A_s) - h(A_s^*) - A_s + A_s^*\right) ds \\
&\quad + r \int_0^t Y_s \left(dX - A_s^* ds\right)
\end{aligned}
$$

A beautiful proof, ct'd

The last term is a martingale. Hence:

$$\mathbb{E}\left[V_t^*\right] = W_0\left(A, C\right) + r\mathbb{E}\left[\int_0^t \left(h\left(A_s\right) - h\left(A_s^*\right) - A_s + A_s^*\right) ds\right]$$

The integrand is non-negative, and positive on a set of positive measure in $(t\,\omega)$. It follows that there is some $\bar{t}$ such that $\mathbb{E}\left[V_{\bar{t}}^*\right] > W_0\left(A, C\right)$. But this means that switching from A^* to A at time $\bar{t}$ is better than sticking with A from the beginning. So (A, C) cannot be (IC)