How to deal with cheaters::

Moral hazard in continuous time

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Guide to the litterature:

I am going to describe the original model of Sannikov:

- Sannikov, "*A continuous-time version of the principal-agent problem*", RES (2008) 75, 957-984
- Sannikov, "*Contracts: the theory of dynamic principa-agent relationships and the continuous-time approach*", Working paper, 2012

Sannikov uses the PDE approach (HJB). Cvitanic uses the stochastic maximum principle approach (BSDE):

- Cvitanic and Zhang, "*Contract theory in continuous-time models*", Springer, 2012

In discrete time, there is a series of remarkable papers by the Toulouse school:

- Biais, Mariotti, Rochet, Villeneuve, "*Large risks, limited liability and dynamic moral hazard*", EMA (2010), 73-118

The model

The agent is in charge of a project which generates a stream of revenue for the principal:

$$dX_t = A_t dt + \sigma dZ_t$$

where Z_t is BM, $\sigma > 0$ is given, and A_t is the agent's effort. If the project is allowed to continue up to $t = \infty$, the intertemporal utilities are::

$$\begin{aligned} &(\text{principal}) \; r\mathbb{E}\left[\int_0^\infty e^{-rt}\left(dX_t - C_t dt\right)\right] \\ &(\text{agent}) \quad r\mathbb{E}\left[\int_0^\infty e^{-rt}\left(u\left(C_t\right) - h\left(A_t\right)\right) dt\right] \end{aligned}$$

where C_t is the agent's compensation (salary + bonuses), u her utility (concave, increasing) and $h\left(A_t\right)$ her cost of effort, (increasing, concave, $h\left(0\right) = 0$).

The problem

The principal observes X_t, $0 \leq t \leq T$, but not A_t . This is the *moral hazard* problem. So C_T is conditional on $X_t, 0 \leq t \leq T$, not on A_t or Z_t
The principal can reward the agent, but cannot punish her. This is the *limited liability* problem. So $C_t \geq 0$.
What incentive scheme can the principal devise so that the agent finds it in her own interest to exert effort ?

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- $C_t = X_t - c$ (fermage)

Contracts

A *contract* is a pair (C_t, A_t) adapted to (X_t, Z_t). The part concerning C_t is enforceable in the courts. The agent is allowed to interrupt the contract at any time T and faces no penalty for doing so. The principal can retire the agent at any time but must compensate by offering her the certainty equivalent of her expected gains:

$$\begin{aligned} W_T &= \mathbb{E}\left[\int_T^\infty e^{-r(s-T)}\left(u\left(C_s\right)-h\left(A_s\right)\right)ds \mid \mathcal{F}_t^Z\right] \\ c &= u^{-1}\left(W_T\right) \end{aligned}$$

A contract is *incentive-compatible* if the agent finds it in its own interest to exert effort A_t at every t. It is *individually rational* if both the principal and the agent find it in their own interest to enter the contract at $t=0$.

$$\text{(principal)}\ r\mathbb{E}\left[\int_0^T e^{-rt}\left(A_t - C_t\right)dt - e^{-rT}u^{-1}\left(W_T\right)\right] \geq 0$$

$$\text{(agent)}\quad r\mathbb{E}\left[\int_0^\infty e^{-rt}\left(u\left(C_t\right)-h\left(A_t\right)\right)dt\right] \geq 0$$

Finding incentive-compatible contracts:

We look at the agent's *continuation value*:

$$\begin{aligned} W_t &= r\mathbb{E}\left[\int_t^\infty e^{-r(s-t)}\left(u\left(C_s\right) - h\left(A_s\right)\right) ds \mid \mathcal{F}_t^Z\right] \\ &= re^{rt}\mathbb{E}\left[\int_0^\infty e^{-rs}\left(u\left(C_s\right) - h\left(A_s\right)\right) ds \mid \mathcal{F}_t^Z\right] \\ &\quad -re^{rt}\int_0^t e^{-rs}\left(u\left(C_s\right) - h\left(A_s\right)\right) ds \end{aligned}$$

Using the martingale representation theorem we find that there is a Z_t-adapted process Y_t (depending on C_t and A_t) such that:

$$\begin{aligned} \frac{1}{r} dW_t &= \left(W_t - u\left(C_t\right) + h\left(A_t\right)\right) dt + Y_t \sigma dZ_t \\ &= \left(W_t - u\left(C_t\right) + h\left(A_t\right) - Y_t A_t\right) dt + Y_t dX_t \end{aligned}$$

Incentive compatible contracts

Suppose the agent has conformed to the contract (C_s, A_s) for $s \leq t$, and tries to cheat, by performing effort a in the following interval $[t,\ t + dt]$, and reverting to A_s for $s \geq t + dt$

- her cost on $[t,\ t + dt]$ is $rh(a)\,dt$
- her expected benefit on $[0,\ \infty]$ is $rY_t a dt$
- the balance is $r(aY_t - h(a))$

It turns out that testing for such small deviations is enough:

Theorem (One-shot rule)

Suppose:

$$Y_t A_t - h(A_t) = \max_{0 \leq a \leq \bar{a}} \{aY_t - h(a)\} \quad a.e \tag{1}$$

Then the contract (A_t, C_t) is (IC)

A beautiful proof

Suppose (A, C) does not satisfy condition (1). Then there is an alternative contract (A^*, C^*) with:

$$\begin{aligned} Y_t A_t^* - h(A_t^*) &\geq Y_t A_t - h(A_t) \quad \text{a.e} \\ P[Y_t A_t^* - h(A_t^*)] &> P[Y_t A_t - h(A_t)] \end{aligned}$$

The agent picks $t > 0$ and plans to apply A^* for $s \leq t$ and A for $\geq t$. Expected utility at t, conditional on $Z_{:t}$:

$$\begin{aligned} \frac{1}{r} V_t^* &= E\left[\int_0^\infty e^{-rt}(u(C_t) - h_t)\, ds \mid \mathcal{F}_t^Z\right] \\ & \int_0^t e^{-rs}(u(C_s) - h(A_s^*))\, ds + e^{-rt} W_t(A, C) \\ &= W_0(A, C) + \int_0^t e^{-rs}(h(A_s) - h(A_s^*) - Y_s A_s + Y_s A_s^*)\, ds \\ & + \int_0^t e^{-rs} Y_s (dX - A_s^* ds) \end{aligned}$$

A beautiful proof, ct'd

The last term is a martingale. Hence:

$$\mathbb{E}\left[V_t^*\right] = W_0\left(A, C\right) + \mathbb{E}\left[\int_0^t e^{-rs}\left(h\left(A_s\right) - h\left(A_s^*\right) - Y_s A_s + Y_s A_s^*\right) ds\right]$$

The integrand is non-negative, and positive on a set of positive measure in $(t\,\omega)$. It follows that there is some $\bar{t}$ such that $\mathbb{E}\left[V_t^*\right] > W_0\left(A, C\right)$. But this means that switching from A^* to A at time $\bar{t}$ is better than sticking with A from the beginning. So (A, C) cannot be (IC)

The optimal control problem

If $Y_t = h'(A_t)$, the contract is incentive-compatible. The principal can now devise the optimal contract (C_t, A_t), subject to this constraint. Sannikov's idea consist of considering W_t as a *performance index* (to be constructed along the trajectory), and on conditioning the contract on W_t

$$\max_{C_t, A_t} \mathbb{E}\left[\int_0^T e^{-rt}(A_t - C_t)\,dt - e^{-rT}u^{-1}(W_T)\right]$$

$$\begin{aligned} \frac{1}{r}dW_t &= (W_t - u(C_t) + h(A_t))\,dt + h'(A_t)\,\sigma dZ_t \\ W_t &\geq 0 \quad 0 \leq t \leq T \end{aligned}$$

The *initial value* W_0 is part of the contract.
Mathematically speaking, the state is W_t, and the controls are C_t and A_t

The HJB equation

Introduce the value function:

$$F(w) = \sup E\left[\int_0^T e^{-rt}(A_t - C_t)\,dt - e^{-rT}u^{-1}(W_T) \mid W_0 = w\right]$$

$F : [0, \ \infty) \to R$ is continuous and $F(w) \geq -u^{-1}(w)$ everywhere.
T is the first time when $F(W_t) \geq -u^{-1}(w)$
The HJB is in fact a *quasi-variational inequality*:

$$\max_{a,c}\left\{\begin{array}{c} u^{-1}(w) - F(w), \\ a - c + F'(w)(w - u(c) + h(a)) + \frac{r}{2}F''(w)\,h'(a)^2\sigma^2 - F(w) \end{array}\right\} =$$

Set

$$a_{\max} = A(w) \quad \text{and} \quad c_{\max} = C(w)$$

The verification theorem.

There is an optimal contract, which is *Markovian* (in terms of $W_t m$)

Theorem

Suppose F solves (IQV) with $F(0) = 0$. Pick some w_0 and define W_t as follows:

$$
\begin{aligned}
\frac{1}{r} dW_t &= W_t - u\left(C(W_t) + h(A(W)) - h'(A(W_t))A(W_t)\right) dt + h'(A(
\\
W_0 &= w_0
\end{aligned}
$$

Then the contract $C_t = C(W_t)$, $A_t = A(W_t)$ is (IC), (IR), and has value w_0 for the agent and $F(w_0)$ for the principal. The principal buys off the agent at time $T := \inf\left\{t \mid F(W_t) \geq -u^{-1}(W_t)\right\}$. Any (IC) (IR) contract starting from $W_0 = w$ yields to the principal a profit less than or equal to $F(w)$

Note that the stopping time T occurs either when $W_t = 0$ or when $W_t = \bar{w}$, where $\bar{w}$ is the smallest positive solution of $F(w) = u^{-1}(w)$

Proof

It is clear that the continuation value of that contract is W_t. It is (IC) and (IR) by construction. Let us compute its value for the principal. Define a r.v. G_t by:

$$G_t := \int_0^t e^{-rs} \left(A_s - C_s ds \right) + e^{-rt} F \left(W_t \right)$$

It is a diffusion, and for $t < T$ its drift vanishes. By the optional stopping theorem,

$$E \left[G_T \right] = G_0 = rF \left(W_0 \right)$$

On the other hand, the value of the contract to the principal is:

$$E \left[\int_0^t e^{-rs} \left(A_s - C_s ds \right) - e^{-rt} u^{-1} \left(W_T \right) \right]$$

We have $F \left(W_T \right) = -u^{-1} \left(W_T \right)$ so this coincides with G_t and the result follows

Proof (cont'd)

Let (C_t^*, A_t^*) be another (IC) (IR) contract. Define a r.v. G_t^* by:

$$G_t^* := \int_0^t e^{-rs} (A_s - C_s ds) + e^{-rt} F(W_t)$$

By the one-step rule, its drift is negative, so it is a supermartingale, and by the optional stopping theorem:

$$E[G_T^*] \leq G_0 = F(W_0)$$

Conclusions

There are two remarkable facts:

- the optimal contract is Markovian, depending only on the current value of an appropriate index

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There are two remarkable facts:

- the optimal contract is Markovian, depending only on the current value of an appropriate index
- the one-shot deviation principle: the agent's incentive constraints hold for all alternative strategies A_t^* if they hold for all strategies which differ from A_t for an infinitesimally small time