RESEARCH ARTICLE

The housing problem and revealed preference theory: duality and an application

Ivar Ekeland · Alfred Galichon

Abstract This paper exhibits a duality between the theory of revealed preference of Afriat and the housing allocation problem of Shapley and Scarf. In particular, it is shown that Afriat's theorem can be interpreted as a second welfare theorem in the housing problem. Using this duality, the revealed preference problem is connected to an optimal assignment problem, and a geometrical characterization of the rationalizability of experiment data is given. This allows in turn to give new indices of rationalizability of the data and to define weaker notions of rationalizability, in the spirit of Afriat's efficiency index.

Keywords Revealed preferences · Afriat's theorem · Optimal assignments · Indivisible allocations

JEL Classification   Primary D11; Secondary C60 · C78

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1 Introduction

The "revealed preference" problem and the "housing problem" are two classical problems in economic theory with a distinguished, but separate tradition. This paper is about connecting them and exploits the connection to obtain new results on revealed preference.

- The revealed preference (RP) problem, posed by Samuelson at the end of the 1930s, was solved by Houthakker in 1950 and was given an operational solution by Sidney Afriat in 1967. This classical problem asks whether, given the observation of n consumptions baskets and corresponding prices, one rationalizes these consumptions as the consumption of a single consumer facing different prices.
- The "housing problem" was investigated in 1974 by Shapley and Scarf. Given an initial allocation of n houses to n individuals and assuming individuals form preferences over houses and can trade houses, what is the core of the corresponding game? It is assumed that houses form no preferences over owners (in sharp contrast to the "Stable matching" problem of Gale and Shapley). In this setting, they showed nonemptiness of the core, as well as an algorithm to arrive to a core allocation: the method of "top-trading cycles," attributed to David Gale.

Although both problems have generated two well-established and distinct literatures, it will turn out that these problems are in fact dual in a precise sense. As we shall argue, the traditional expenditure/utility duality from consumer theory extends to the setting of revealed preference, and it is possible to show that the problem of Pareto efficiency in the housing problem and the issue of rationalizability of consumer data are both applications of a basic mathematical result characterizing cyclically consistent matrices proven by Fostel et al. (2004), generalizing Afriat's theorem. In particular, we show that *Afriat's theorem can be interpreted as a second welfare theorem in the housing problem*. Once we have established equivalence of both problems, we shall use the aforementioned results to give new characterization of both problems in terms of an optimal allocation problem. This will give us a very simple and intuitive characterization of the efficient outcomes in the indivisible allocation problem.

According to standard revealed preference theory, the data are not rationalizable when one can exhibit a preference cycle based on the direct revealed preference relation. A stronger violation of rationalizability occurs when there are subsets of the data where cycles exist. An even stronger departure from rationalizability occurs when all observations are part of such a cycle. We shall characterize these three situations in terms of indexes which will measure how strong the violations of rationalizability are. In the last part, we shall last use these results to provide a geometric interpretation of revealed preference theory, by showing that the problem of determining whether data can be rationalized or not can be recast as the problem of determining whether a particular point is on some part of the boundary of some convex body.

The connection between the housing problem and the revealed preference problem yields some insights for both theories. The important econometrics literature on revealed preference contains a wealth of tools to measure the intensity of departure from rationalizability. These tools may be successfully applied to the "housing problem" and its more applied versions such as the "kidney problem," which

deals with assigning organ transfers to patients. These tools may design indices to measure how far from Pareto optimality an actual assignment is, or what are the costs of a given constraint in terms of welfare. Conversely, the connection between the two problems sheds light on the role of matching theory into revealed preference theory. This leads to interesting research avenues, as matching theory has recently received a lot of attention, from both the empirical and computational points of view.

The literature on revealed preference in consumer demand traces back to Samuelson (1938), who formulated the problem and left it open. It was solved by Afriat (1967) using nontrivial combinatorial techniques. Diewert (1973) provided a linear programming proof and Varian (1982) an algorithmic solution. Fostel et al. (2004) and Chung-Piaw and Vohra (2003) provided alternative proofs. Matzkin (1991) and Forges and Minelli (2009) extended the theory to nonlinear budget constraints. Geanakoplos (2006) gives a proof of Afriat's theorem using a minmax theorem. Although minmax formulations will appear in our paper, these are distinct from Geanakoplos', as it will be explained below. The setting was recently extended to collective models of household consumptions by Cherchye et al. in a series of paper (see, e.g., 2007, 2010). A very recent paper by Quah (2012) extends Afriat's theorem in another direction n order to characterize datasets compatible with weak separability without concavity assumptions. The literature on the indivisible allocation problem was initiated by Shapley and Scarf (1974), who formulated as the "housing problem" and gave an abstract characterization of the core. Roth et al. (2004) study a related "kidney problem" and investigate mechanism design aspect. Finally, recent literature has extended revealed preference theory to classes of matching problems. Galichon and Salanié (2010) investigate the problem of revealed preferences in a matching game with transferable utility, with and without unobserved heterogeneity. Echenique et al. (2011) investigate the problem of revealed preferences in games with and without transferable utility, and without unobserved heterogeneity.

This paper will be organized as follows. We first study Pareto efficient allocations, and show the connection with the generalized theory of revealed preferences. We deduce a novel characterization of data rationalizability in terms of an optimal matching problem, and a geometric interpretation of revealed preference in terms of convex geometry. We then move on to providing indexes of increasingly weaker forms of rationalizability.

2 Revealed preference and the housing problem

2.1 The generalized Afriat's theorem

Assume as in Forges and Minelli (2009) that consumer has budget constraint $g_i(x) \leq 0$ in experiment i where x is an L-dimensional bundle of goods, and that bundle x_i is chosen where $g_i(x_i) = 0$. This is a nonlinear generalization of Afriat (1969) in which $g_i(x_j) = x_j \cdot p_i - x_i \cdot p_i$, where p_i is the price vector of the bundle of goods in experiment i. Forges and Minelli ask whether there exists a utility function $v(x)$ with appropriate properties such that

$$x_i \in \text{argmax}_x \{v(x) : g_i(x) \le 0\}$$

in which case v is said to *rationalize* the data.

As in the original Afriat's paper, the first part of their proof necessitates the existence of a utility level v_j associated with bundle j such that

$$g_i(x_j) < 0 \quad \text{implies} \quad v_j - v_i < 0. \tag{1}$$

Once the number v_i's have been determined, the second part of their proof consists in constructing a locally nonsatiated and continuous utility function $v(x)$ such that $v(x_j) = v_j$. The preference induced by v is said to rationalize the data in the sense that

$$i \in \text{argmax}_j \{v_j : g_i(x_j) \le 0\}.$$

The existence of real numbers v_i such that (1) holds is nontrivial, and it turns out to be equivalent to a property of matrix $R_{ij} = g_i(x_j)$, called "cyclical consistency." This is done by appealing to an extension of Afriat's original theorem beyond linear budget sets, proved by Fostel et al. (2004). This result can be slightly reformulated as follows:

Theorem 0 (Afriat's theorem). *The following conditions are equivalent:*

(i) *The matrix R_{ij} satisfies "cyclical consistency": for any cycle $i_1, \dots, i_{p+1} = i_1$,*

$$\forall k, \ R_{i_k i_{k+1}} \le 0 \ \text{implies} \ \forall k, \ R_{i_k i_{k+1}} = 0, \tag{2}$$

(ii) *There exist numbers (v_i, λ_i), $\lambda_i > 0$, such that*

$$v_j - v_i \le \lambda_i R_{ij},$$

(iii) *There exist numbers v_i such that*

$$R_{ij} \le 0 \quad \text{implies} \quad v_j - v_i \le 0, \ and$$
$$R_{ij} < 0 \quad \text{implies} \quad v_j - v_i < 0.$$

Before we give the proof of the result, let us briefly comment on it. Although we call it "Afriat's theorem," this result is in some sense a fundamental lemma upon which Afriat's analysis rests. In the original Afriat's work and in the subsequent literature, notably Forges and Minelli (2009), a full description of the budget sets is provided, that is the functions $g_i(x)$ defined on a larger domain than the set of observed consumption bundles $\{x_1, \dots, x_n\}$. Here, the result that we give only depends on the value $g_i(x_j)$ of the functions g_i over the set of the observed consumption bundles. Actually, even only the signs of these values matter. Here, the analysis is intrinsically discrete and the approach relies neither on a particular consumption space nor on explicit consumption bundles x_j.

We now turn to the proof of Theorem 0.

Proof of Theorem 0 (i) implies (ii) is proven in Fostel, Scarf and Todd (2004). (ii) immediately implies (iii). We now show that (iii) implies (i). Consider a cycle $i_1, \ldots, i_{p+1} = i_1$, such that

$$\forall k, \ R_{i_k i_{k+1}} \leq 0.$$

Then by (iii), there exist numbers v_i such that

$$R_{ij} \leq 0 \ \ \text{implies} \ \ v_j - v_i \leq 0, \ \text{and}$$
$$R_{ij} < 0 \ \ \text{implies} \ \ v_j - v_i < 0.$$

Thus, one has $v_{i_{k+1}} - v_{i_k} \leq 0$ for all k; hence, all the v_{i_k} are equal. Assume now that there is a k such that $R_{i_k i_{k+1}} < 0$. Then, $v_{i_{k+1}} - v_{i_k} < 0$, which is a contradiction. Therefore,

$$\forall k, \ R_{i_k i_{k+1}} = 0,$$

which proves the cyclical consistency of matrix R, that is (i). □

2.2 Pareto efficient allocation of indivisible goods

We now turn to the problem of allocation of indivisible goods, which was initially studied by Shapley and Scarf (1974). Consider n indivisible goods (eg. houses) $j = 1, \ldots, n$ to be allocated to n individuals. Cost of allocating (eg. transportation cost) house j to individual i is c_{ij}. An allocation is a permutation σ of the set $\{1, \ldots, n\}$ such that individual i gets house $j = \sigma(i)$. Let S be the set of such permutations. We assume for the moment that the initial allocation is given by the identity permutation: good i is allocated to individual i. The problem here is to decide whether this allocation is efficient in a Pareto sense.

If there are two individuals, say i_1 and i_2, that would both benefit from swapping houses (strictly for at least one), then this allocation is not efficient, as the swap would improve on the welfare of both individuals. Thus if allocation is efficient, then inequalities $c_{i_1 i_2} \leq c_{i_1 i_1}$ and $c_{i_2 i_1} \leq c_{i_2 i_2}$ cannot hold simultaneously unless they are both equalities. More generally, Pareto efficiency rules out the existence of exchange rings whose members would benefit (strictly for at least one) from a circular trade. In fact, we shall argue that *this problem is dual to the problem of generalized revealed preferences*.

2.3 A dual interpretation of revealed preference

Let us now formalize the notion of efficient allocation in the previous discussion. Allocation $\sigma_0(i) = i$ is Pareto efficient if and only if for any $\sigma \in S$, inequalities

$$c_{i\sigma(i)} \leq c_{ii}$$

cannot hold simultaneously unless these are all equalities. By the decomposition of a permutation into cycles, we see that this definition is equivalent to the fact that for every " trading cycle" $i_1, \ldots, i_{p+1} = i_1$,

$$\forall k, c_{i_k i_{k+1}} \leq c_{i_k i_k} \quad \text{implies} \quad \forall k, c_{i_k i_{k+1}} = c_{i_k i_k}$$

that is, introducing $R_{ij} = c_{ij} - c_{ii}$,

$$\forall k, R_{i_k i_{k+1}} \leq 0 \quad \text{implies} \quad \forall k, R_{i_k i_{k+1}} = 0,$$

which is to say that allocation is efficient if and only if the matrix R_{ij} is cyclically consistent.

By the equivalence between (i) and (ii) in Theorem 0 above, we have the following statement:

Proposition 1 *In the housing problem, allocation* $\sigma_0 (i) = i$ *is efficient if and only if*

$$\exists v_i \text{ and } \lambda_i > 0, \ v_j - v_i \leq \lambda_i R_{ij}. \qquad \text{(PARETO)}$$

Before giving an interpretation of this result, we would like to understand the link between efficiency and equilibrium in the housing problem. Assume we start from allocation $\sigma_0 (i) = i$, and we let people trade. Given a price system π where π_i is the price of house i , we assume that individual i can sell her house for price π_i and therefore can afford any house j whose price π_j is less than π_i. Therefore, individual i's budget set B_i is the set of houses that sell to a price lower than his

$$B_i = \{ j : \pi_j \leq \pi_i \}$$

We can now define the notion of a competitive equilibrium in this setting. Allocation $\sigma \in S$ is an equilibrium supported by price system π if for each individual i, (1) house $\sigma (i)$ is weakly preferred by individual i among all houses that she can afford, while (2) house $\sigma (i)$ is strictly preferred by i among all houses she can strictly afford (i.e., that trade for prices strictly less than her house i). While condition (1) is a necessary requirement, refinement (2) is natural from a behavioral point of view as it implies that if i is indifferent between two houses, then she is going to choose the cheapest house of the two.

In particular, $\sigma_0 (i) = i$ is a *no-trade equilibrium* if there is a system of prices π, where π_j is the price of house j, such that: (1) whenever house j is affordable for individual i, then it is not strictly preferred by i to i's house, that is $\pi_j \leq \pi_i$ implies $c_{ij} \geq c_{ii}$, and (2) for any house j in the strict interior of i's budget set, that is, that trades for strictly cheaper than house i, then individual i strictly prefers her own house i to house j, that is $\pi_j < \pi_i$ implies $c_{ij} > c_{ii}$.

Hence, $\sigma_0 (i) = i$ is a no-trade equilibrium supported by prices π when conditions (E1) and (E2) below are met, where:

(E1) if house j can be afforded by i, then individual i does not strictly prefer house j to house i, that is,

$$\pi_j \le \pi_i \quad \text{implies} \quad c_{ij} \ge c_{ii},$$

that is

$$\pi_j \le \pi_i \quad \text{implies} \quad R_{ij} \ge 0,$$

that is, yet equivalently:

$$R_{ij} < 0 \quad \text{implies} \quad \pi_j > \pi_i.$$

(E2) if house j is (strictly) cheaper than house i, then individual i strictly prefers house i to house j, that is

$$\pi_j < \pi_i \quad \text{implies} \quad c_{ij} > c_{ii},$$

that is

$$\pi_j < \pi_i \quad \text{implies} \quad R_{ij} > 0,$$

that is, yet equivalently:

$$R_{ij} \le 0 \quad \text{implies} \quad \pi_j \ge \pi_i.$$

Proposition 2 *In the housing problem, allocation $\sigma_0(i) = i$ is a no-trade equilibrium supported by prices π if and only if*

$$
\begin{aligned}
R_{ij} < 0 \quad &\text{implies} \quad \pi_j > \pi_i, \text{ and} \qquad\qquad \text{(EQUILIBRIUM)}\\
R_{ij} \le 0 \quad &\text{implies} \quad \pi_j \ge \pi_i.
\end{aligned}
$$

But (EQUILIBRIUM) is exactly formulation (iii) of Theorem 0 with $\pi_i = -v_i$. By Theorem 0, we know that this statement is equivalent to statement (PARETO). Hence, we get an interpretation of *the generalized Afriat's theorem as a second welfare theorem*

$$(EQUILIBRIUM) \iff (PARETO),$$

which we summarize in the following proposition:

Proposition 3 *In the allocation problem of indivisible goods, Pareto allocations are no-trade equilibria supported by prices, and conversely, no-trade equilibria are Pareto efficient.*

This is a "dual" interpretation of revealed preference, where v_i (utilities in generalized RP theory) become budgets here, and c_{ij} (budgets in generalized RP theory) become utilities here. To summarize this duality, let us give the following table:

	Revealed prefs.	Pareto indiv. allocs.
Setting	Consumer demand	Allocation problem
Budget sets	$\{j : c_{ij} \le c_{ii}\}$	$\{-v : -v \le -v_i\}$
Cardinal utilities to j	v_j	$-c_{ij}$
No of consumers	One	$n, i \in \{1, \ldots, n\}$
No of experiments	n	One
Unit of c_{ij}	Dollars	Utils
Unit of v_i	Utils	Dollars
Interpretation	Afriat's theorem	Welfare theorem

The table suggests that there is a single agent facing n choices in revealed preference theory, while there are n agents facing one choice in the housing problem. In revealed preference theory, the cardinal utility of alternative j of the consumer in revealed preference theory is v_j, while c_{ij} expresses a budget set. Conversely, in the housing problem, $-c_{ij}$ is the cardinal utility consumer i for alternative j, while $-v_j$ is interpreted as the price of bundle j. Hence, the roles of "prices" and "utilities" are exchanges in the two problems, which motivates our contention that revealed preference theory and the housing problem are "dual."

It is worth remarking that the notion of domination used in our definition of Pareto efficiency is different from the one used by Shapley and Scarf. Following a common use, our notion allows for weak domination for some individual, while Shapley and Scarf require strict domination for everybody. The two notions are not equivalent in a setting with indivisible goods. Indeed, using their notion of domination, Shapley and Scarf provide an example in which a Pareto efficient allocation cannot be sustained as a competitive allocation.

2.4 A characterization of rationalizability

The connection of both the revealed preference problem and the housing problem with assignment problems is clear; we shall now show that there is a useful connection with the optimal assignment problem (recalled below), where the sum of the individual utilities is maximized.

Recall that the data are called *rationalizable*; there are scalars v_i such that

$$R_{ij} = g_i(x_j) < 0 \quad \text{implies} \quad v_j < v_i, \text{ and} \qquad \text{(EQUILIBRIUM)}$$
$$R_{ij} = g_i(x_j) \le 0 \quad \text{implies} \quad v_j \le v_i.$$

With this in mind, we have the following novel characterization of rationalizability of the data, which extends Theorem 0:

Theorem 4 *In the revealed preference problem, the data are rationalizable if and only if there exist weights $\lambda_i > 0$ such that*

$$\min_{\sigma \in S} \sum_{i=1}^{n} \lambda_i R_{i\sigma(i)} = 0, \tag{3}$$

that is,

$$\min_{\sigma \in S} \sum_{i=1}^{n} \lambda_i c_{i\sigma(i)} = \sum_{i=1}^{n} \lambda_i c_{ii}. \tag{4}$$

Proof of Theorem 4 Start from (4): $\exists \lambda_i > 0$, $\min_{\sigma \in S} \sum_{i=1}^{n} \lambda_i R_{i\sigma(i)} = 0$
$\Longleftrightarrow \exists \lambda_i > 0$, $\min_{\sigma \in S} \sum_{i=1}^{n} \lambda_i R_{i\sigma(i)}$ is reached for $\sigma = Id$

This problem is to find the assignment $\sigma \in S$ which minimizes the utilitarian welfare loss computed as the sum of the individual costs $K_{ij} = \lambda_i R_{ij}$, setting weight one to each individual. This problem is therefore

$$\min_{\sigma \in S} \sum_{i=1}^{n} K_{i\sigma(i)}.$$

This problem was reformulated as a linear programming problem by Dantzig in the 1930s; see Shapley and Shubik (1972) for a game-theoretic interpretation, and Gretsky et al. (1992) for the continuous limit), and by the standard linear programming duality of the optimal assignment problem (Dantzig 1939; Shapley and Shubik 1972; see Ekeland 2010 for a recent review using the theory of optimal transportation)

$$\min_{\sigma \in S} \sum_{i=1}^{n} K_{i\sigma(i)} = \max_{u_i + v_j \leq K_{ij}} \sum_{i=1}^{n} u_i + \sum_{j=1}^{n} v_j, \tag{5}$$

where S is the set of permutations of $\{1, \ldots, n\}$. It is well known that for a $\sigma_0 \in S$ solution to the optimal assignment problem, there is a pair (u, v) solution to the dual problem such that

$$u_i + v_j \leq K_{ij}$$
$$\text{if } j = \sigma_0(i), \qquad \text{then } u_i + v_j = K_{ij}.$$

Hence, (4)
$\Longleftrightarrow \exists \lambda_i > 0, u, v \in \mathbb{R}^n$

$$u_i + v_j \leq \lambda_i R_{ij}$$
$$u_i + v_i = 0$$

$\Longleftrightarrow \exists \lambda_i > 0, v \in \mathbb{R}^n$

$$v_j - v_i \leq \lambda_i R_{ij},$$

which is (ii) in Theorem 0. □

The previous result leads to the following two remarks.

First, recall that matrix (M_{ij}) is called *cyclically monotone* if for any cycle $(i_1, \ldots, i_p)$, one has

$$\sum_{k=1}^{p} M_{i_k i_{k+1}} - M_{i_k i_k} \geq 0.$$

It is well known (see, e.g., Villani 2003) that cyclical monotonicity is equivalent to (3). Therefore, Theorem 4 states that (R_{ij}) is cyclically consistent if and only if there are weights $\lambda_i > 0$ such that $\lambda_i R_{ij}$ is cyclically monotone.

As a second remark, introduce

$$\mathcal{F} = \text{conv}\left(\left(c_{i\sigma(i)} \right)_{i=1,\ldots,n} : \sigma \in S \right) \subset \mathbb{R}^n$$

which is a convex polytope. Note that

$$\min_{\sigma \in S} \sum_{i=1}^{n} \lambda_i c_{i\sigma(i)} = \min_{C \in \mathcal{F}} \sum_{i=1}^{n} \lambda_i C_i.$$

We have the following geometric characterization of the fact that the data are rationalizable:

Proposition 5 *The data are rationalizable if and only if* 0 *is an extreme point of* $\mathcal{F}$ *with a componentwise positive vector in the normal cone.*

Proof Introduce $\mathcal{W}(\lambda) = \min_{C \in \mathcal{F}} \sum_{i=1}^{n} \lambda_i C_i$. This concave function is the support function of C. $C_i^0 = c_{i\sigma_0(i)}$ is an extreme point of $\mathcal{F}$ with a componentwise positive vector in the normal cone if and only 0 is in the superdifferential of $\mathcal{W}$ at such a vector λ. This holds if and only if $\mathcal{W}(\lambda) = \sum_{i=1}^{n} \lambda_i C_i^0$. □

3 Strong and weak rationalizability

3.1 Indices of rationalizability

Rationalizability of the data by a single consumer, as tested by Afriat's inequalities, is an important empirical question. Hence, it is of interest to introduce measures of how close the data are from being rationalizable. Since Afriat's original work on the topic, many authors have set out proposals to achieve this. Afriat's original "efficiency index" is the largest $e \leq 1$ such that $R_{ij}^e = R_{ij} + (1-e)b_i$ is cyclically consistent, where $b_i > 0$ is a fixed vector with positive components. In Afriat's setting, $R_{ij} = x_j \cdot p_i - x_i \cdot p_i$, and $R_{ij}^e = x_j \cdot p_i - e x_i \cdot p_i$, so $b_i = x_i \cdot p_i$. Many other tests and empirical approaches

have followed and are discussed in Varian (2006) review paper. A recent proposal is given in Echenique et al. (2011), who introduce the "money pump index" as the amount of money that could be extracted from a consumer with nonrationalizable preferences.

In the same spirit, we shall introduce indices that will measure how far the dataset is from being rationalizable. The indices we shall build are connected to the dual interpretation of revealed preferences we have outlined above. Our measures of departure from rationalizability will be connected to measures of departure from Pareto efficiency in that problem. One measure of departure from efficiency is the welfare gains that one would gain from moving to the efficient frontier; this is interpretable as Debreu (1951) *coefficient of resource utilization*,[1] in the case of a convex economy. This is only an analogy, as we are here in an indivisible setting, but this is exactly the idea we shall base the construction of our indices on.

It seems natural to introduce index A as

$$A = \max_{\lambda \in \Delta} \min_{\sigma \in S} \sum_{i=1}^{n} \lambda_i R_{i\sigma(i)}$$

where $\Delta = \left\{ \lambda \geq 0, \sum_{i=1}^{n} \lambda_i = 1 \right\}$ is the simplex of $\mathbb{R}^n$. Indeed, we have

$$A \leq 0,$$

and by compacity of Δ, equality holds if and only if there exist $\lambda \in \Delta$ such that

$$\min_{\sigma \in S} \sum_{i=1}^{n} \lambda_i R_{i\sigma(i)} = 0.$$

Of course, this differs from our characterization of rationalizability in Theorem 4, as there the weights λ_i's need to be all positive, not simply nonnegative. It is easy to construct examples where (4) holds with some zero λ_i's and $\sigma_0(i) = i$ is not efficient. For example, in the housing problem, if individual $i = 1$ has his most preferred option, then $\lambda_1 \neq 0$ and all the other λ_i's are zero, and $A = 0$, thus (4) holds. However, allocation may not be Pareto because there may be inefficiencies among the rest of the individuals.

Hence, imposing $\lambda > 0$ is crucial. Fortunately, it turns out that one can restrict the simplex Δ to a subset which is convex, compact and away from zero, as shown in the next lemma.

Lemma 6 *There is $\varepsilon > 0$ (dependent only on the entries of matrix c, but independent on the matching considered) such that the following disjuction holds*

- *either there exist no scalars $\lambda_i > 0$ satisfying the conditions in Theorem 4*

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[1] We thank Don Brown for suggesting this interpretation to us.

- *or there exist scalars $\lambda_i > 0$ satisfying the conditions in Theorem 4 and such that*

$$\begin{cases} \lambda_i \geq \varepsilon \text{ for all } i, \\ \sum_{i=1}^{n} \lambda_i = 1. \end{cases}$$

Proof Standard construction (see Fostel et al. 2004) of the λ_i's and the v_i's provides a deterministic procedure that returns strictly positive $\lambda_i \geq 1$ within a finite and bounded number of steps, with only the entries of R_{ij} as input; hence, λ, if it exists, is bounded, so there exists M depending only on R such that $\sum_{i=1}^{n} \lambda_i \leq M$. Normalizing λ so that $\sum_{i=1}^{n} \lambda_i = 1$, one sees that one can choose $\varepsilon = 1/M$. □

We denote Δ_ε the set of such vectors λ, and we recall that Δ is the set of λ such that $\lambda_i \geq 0$ for all i and $\sum_{i=1}^{n} \lambda_i = 1$. Recall $R_{ij} = c_{ij} - c_{ii}$ and introduce

$$A^* = \max_{\lambda \in \Delta_\varepsilon} \min_{\sigma \in S} \sum_{i=1}^{n} \lambda_i R_{i\sigma(i)},$$

so that we have

$$A^* \leq 0$$

and equality if and only if the data are rationalizable (as in this case there exist $\lambda_i > 0$ such that the characterization of rationalizability in 4 is met). Further, as $\Delta_\varepsilon \subset \Delta$, one gets

$$\underbrace{\max_{\lambda \in \Delta_\varepsilon} \min_{\sigma \in S} \sum_{i=1}^{n} \lambda_i R_{i\sigma(i)}}_{A^*} \leq \underbrace{\max_{\lambda \in \Delta} \min_{\sigma \in S} \sum_{i=1}^{n} \lambda_i R_{i\sigma(i)}}_{A} \leq 0.$$

The max-min formulation for index A leads naturally to introduce a new index which comes from the dual minmax program

$$B = \min_{\sigma \in S} \max_{\lambda \in \Delta} \sum_{i=1}^{n} \lambda_i R_{i\sigma(i)}$$
$$= \min_{\sigma \in S} \max_{i \in \{1,\dots,n\}} R_{i\sigma(i)}$$

Note that the inequality max min $\leq$ min max always holds, so $A \leq B$, and further, we have

$$B \leq \max_{i \in \{1,\dots,n\}} R_{ii} = 0.$$

Therefore, we have

$$\underbrace{\max_{\lambda \in \Delta_\varepsilon} \min_{\sigma \in S} \sum_{i=1}^{n} \lambda_i R_{i\sigma(i)}}_{A^*} \leq \underbrace{\max_{\lambda \in \Delta} \min_{\sigma \in S} \sum_{i=1}^{n} \lambda_i R_{i\sigma(i)}}_{A} \leq \underbrace{\min_{\sigma \in S} \max_{\lambda \in \Delta} \sum_{i=1}^{n} \lambda_i R_{i\sigma(i)}}_{B} \leq 0.$$

We shall come back to the interpretation of $A^* = 0$, $A = 0$ and $B = 0$ as stronger or weaker forms of rationalizability of the data. Before we do that, we summarize the above results.

Proposition 7 *We have:*

(i) $A^* = 0$ *if and only if there exist scalars v_i and weights $\lambda_i > 0$ such that*

$$v_j - v_i \leq \lambda_i R_{ij}.$$

(ii) $A = 0$ *if and only if there exist scalars v_i and weights $\lambda_i \geq 0$, not all zero, such that*

$$v_j - v_i \leq \lambda_i R_{ij}.$$

(iii) $B = 0$ *if and only if*

$$\forall \sigma \in S, \; \exists i \in \{1, \ldots, n\} : R_{i\sigma(i)} \geq 0.$$

(iv) *One has*

$$A^* \leq A \leq B \leq 0. \tag{6}$$

Note that it is easy to find examples where these inequalities hold strict for $n \geq 3$.

Proof (i) follows from Lemma 6. To see (ii), note that $A = 0$ is equivalent to the existence of $\lambda \in \Delta$ such that $\min_{\sigma \in S} \sum_{i=1}^{n} \lambda_i R_{i\sigma(i)} = 0$, and the rest follows from Theorem 4. The condition in (iii) is equivalent to the fact that for all $\sigma \in S$, $\max_{i \in \{1, \ldots, n\}} R_{i\sigma(i)} \geq 0$, that is for all $\sigma \in S$, there exists $i \in \{1, \ldots, n\}$ such that $R_{i\sigma(i)} \geq 0$. The inequalities in (iv) were explained above. □

In an unpublished manuscript (Geanakoplos 2006) that he kindly communicated to us on our request, John Geanakoplos introduces the following minmax problem, which in our notations can be defined as

$$G = \max_{\lambda \in \Delta} \min_{\sigma \in C} \sum_{i=1}^{n} \lambda_i R_{i\sigma(i)}$$

where $C \subset S$ is the set of permutations that have only one cycle, that is, such that there exist a cycle $i_1, \ldots, i_{p+1} = i_1$ such that $\sigma(i_k) = i_{k+1}$ and $\sigma(i) = i$ if $i \notin \{i_1, \ldots, i_p\}$. As $C \subset S$ one has

$$A \leq G \leq 0.$$

Geanakoplos uses index G and von Neuman's minmax theorem in order to provide an interesting alternative proof of Afriat's theorem. However, it seems that index G is not directly connected with the assignment problem.

3.2 Interpretation of indexes A^*, A, B

As seen above, the index A^* was constructed so that $A^* = 0$ if and only if the data are rationalizable. The indexes A and B will both be equal to 0 if the data are rationalizable; hence, $A < 0$ or $B < 0$ implies that the data are not rationalizable; however, the converse is not true, so these indexes can be interpreted as measures of weaker form of rationalizability. Hence, we would like to give a meaningful interpretation of the situations where $A = 0$ and $B = 0$. It turns out that $A = 0$ is equivalent to the fact that a subset of the observations can be rationalized, the subset having a property of *coherence* that we now define. $B = 0$ is equivalent to the fact that one cannot partition the set of observations into *increasing cycles*, a notion we will now define. In other words, the condition $B < 0$ means that any observation is part of a preference cycles—indeed a very strong departure from rationalizability. These indexes may be used to compute identified regions in models with partial identification; see Ekeland et al. (2010) and references therein.

Throughout this subsection, it will be assumed that no individual is indifferent between two distinct consumptions for the direct revealed preference relation, that is:

Assumption A *In this subsection, R is assumed to verify $R_{ij} \neq 0$ for $i \neq j$.*

We first define the notion of coherent subset of observations.

Definition 8 (*Coherent subset*) In the revealed preference problem, a subset of observations included in $\{1, \ldots, n\}$ is said to be coherent when $i \in I$ and i directly revealed preferred to j implies $j \in I$. Namely, I is coherent when

$$i \in I \text{ and } R_{ij} < 0 \text{ implies } j \in I.$$

In particular, $\{1, \ldots, n\}$ is coherent; any subset of observations where each observation is directly revealed preferred to no other one is also coherent. Next, we define the notion of increasing cycles.

Definition 9 (*Increasing cycles*) A cycle $i_1, \ldots, i_{p+1} = i_1$ is called *increasing* when each observation is strictly directly revealed preferred to its predecessor. Namely, cycle $i_1, \ldots, i_{p+1} = i_1$ is increasing when

$$R_{i_k i_{k+1}} < 0 \quad \text{for all} \quad k \in \{1, \ldots, p\}.$$

Of course, the existence of an increasing cycle implies that the matrix R is not cyclically consistent; hence, it implies in the revealed preference problem that the data are not rationalizable. It results from the definition that an increasing cycle has length greater than one.

We now state the main result of this section, which provides an economic interpretation for the indexes A^*, A and B.

Theorem 10 *We have:*

(i) $A^* = 0$ *iff the data are rationalizable,*
(ii) $A = 0$ *iff a coherent subset of the data is rationalizable,*
(iii) $B = 0$ *iff there is no partition of* $\{1, \ldots, n\}$ *in increasing cycles. and (i) implies (ii), which implies (iii).*

Proof (i) was proved in Theorem 4 above.

Let us show the equivalence in (ii). From Proposition 7, $A = 0$ is equivalent to the existence of $\exists \lambda_i \geq 0$, $\sum_{i=1}^{n} \lambda_i = 1$ and $v \in \mathbb{R}^n$ such that

$$v_j - v_i \leq \lambda_i R_{ij},$$

so defining I as the set of $i \in \{1, \ldots, n\}$ such that $\lambda_i > 0$, this implies a subset I of the data is rationalizable. We now show that I is coherent. Indeed, for any two k and l not in I and i in I, one has $v_k = v_{l'} \geq v_i$; thus if $i \in I$ and $R_{ij} < 0$, then $v_j < v_i$, hence $j \in I$, which show that I is coherent.

Conversely, assume a coherent subset of the data I is rationalizable. Then, there exist $(u_i)_{i \in I}$ and $(\mu_i)_{i \in I}$ such that $\mu_i > 0$ and

$$u_j - u_i \leq \mu_i R_{ij}$$

for $i, j \in I$. Complete by $u_i = \max_{k \in I} u_k$ for $i \notin I$, and introduce $\tilde{R}_{ij} = 1_{\{i \in I\}} R_{ij}$. One has $\tilde{R}_{ij} < 0$, which implies $i \in I$ and $R_{ij} < 0$; hence, $j \in I$ by the coherence property of I, thus $u_j - u_i < 0$. Now, $\tilde{R}_{ij} = 0$ implies either $i \notin I$ or $R_{ij} = 0$, thus $i = j$; in both cases, $u_j \leq u_i$. Therefore, one has

$$\tilde{R}_{ij} < 0 \quad \text{implies} \quad u_j - u_i < 0, \text{and}$$
$$\tilde{R}_{ij} \leq 0 \quad \text{implies} \quad u_j - u_i \leq 0.$$

Hence, by Theorem 0, there exist v_i and $\bar{\lambda}_i > 0$ such that

$$v_j - v_i \leq \bar{\lambda}_i \tilde{R}_{ij}$$

and defining $\lambda_i = \bar{\lambda}_i 1_{\{i \in I\}}$, one has

$$v_j - v_i \leq \lambda_i R_{ij}$$

which is equivalent to $A = 0$.

(iii) Now Proposition 7 implies that $B < 0$ implies that there is $\sigma \in S$ such that $\forall i \in \{1, \ldots, n\}$, $R_{i\sigma(i)} < 0$. The decomposition of σ in cycles gives a partition of $\{1, \ldots, n\}$ in increasing cycles.

(iv) The implication (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) results from inequality $A^* \leq A \leq B \leq 0$.
□

4 Concluding remarks

To conclude, we make a series of remarks.

First, we have shown how our dual interpretation of Afriat's theorem in its original revealed preference context as well as in a less traditional interpretation of efficiency in the housing problem could shed new light and give new tools for both problems. It would be interesting to understand if there is a similar duality for problems of revealed preferences in matching markets, recently investigated by Galichon and Salanié (2010) and Echenique et al. (2011).

Also, we argue that it makes sense to investigate "weak rationalizability" (ie. $A = 0$ or $B = 0$) instead of "strong rationalizability" (ie. $A^* = 0$), or equivalently, it may make sense to allow some λ_i's to be zero. In the case of $A = 0$, recall the λ_i's are interpreted in Afriat's theory as the Lagrange multiplier of the budget constraints. Allowing for $\lambda = 0$ corresponds to excluding wealthiest individuals as outliers. It is well known that when taken to the data, strong rationalizability is most often rejected. It would be interesting to test econometrically for weak rationalizability, namely whether $A = 0$.

$B < 0$ is a very strong measure of nonrationalizability, as $B < 0$ means that one can find a partition of the observation set into increasing cycles, which seems a very strong violations of the Generalized Axiom of Revealed Preference. But this may arise in some cases, especially with a limited number of observations.

We should emphasize on the fact that indexes A^*, A and B provide a measure of how close the data are from being rationalizable, in the spirit of Afriat's efficiency index. It is clear for every empirical researcher that the relevant question about revealed preference in consumer demand is not whether the data satisfy GARP, but it is how much they violate it. These indexes are an answer to that question. Also the geometric interpretation of rationalizable is likely to provide useful insights for handling unobserved heterogeneity. We plan to investigate this question in further research.

Last, it seems that an interesting research avenue (not pursued in the present paper) deals with exploring similar connections as in this paper to understand what would be the analogue of variants of the theories discussed here. For instance, in the light of the duality exposed in the present paper, it would be interesting to characterize the dual theory to the revealed preference theory for collective models of household consumption as in Cherchye et al. (2007), Cherchye (2010). Similarly, it would also be of interest to understand the revealed preference dual of the *two-sided* version of the housing problem, namely the marriage problem, where both sides of the market have preferences over the other side. In these two cases as in others, what becomes of the connection stressed in the current paper? This question, too, is left for future research.

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