SECOND-ORDER EVOLUTION EQUATIONS ASSOCIATED WITH CONVEX HAMILTONIANS

BY

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§0. Introduction. Many problems in mathematical physics can be formulated as differential equations of second order in time:

\[ \dot{x} = -\nabla V(x) \]

with \( V \) a convex functional. This is the Euler equation for the Lagrangian

\[ \frac{1}{2}|\dot{x}|^2 - V(x) = \lambda(x, \dot{x}) \]

which is convex with respect to \( \dot{x} \), and concave with respect to \( x \). On the other hand, the associated Hamiltonian, by the Legendre transform, is seen to be:

\[ \frac{1}{2}|p|^2 + V(x) = \Gamma(x, p) \]

It is convex in both variables. It is the purpose of this paper to show how the convexity of the Hamiltonian can be systematically used in the study of equations (1). In the first part, we shall show that the solutions of (1), although they are only extremal for the original Lagrangian \( \lambda \), are actually minimizing for another, more complex, Lagrangian \( K \). In the second part, we shall show how this characterization can be used to prove the existence of solutions to (1) satisfying various initial or boundary conditions.

§1. Characterization. Let \( H \) be some Hilbert space; for the sake of convenience, it will be assumed to contain a countable dense subset. Denote by \( H^1(0, T; H) \) the Sobolev space of all functions \( x \) in \( L^2(0, T; H) \) with derivative \( \dot{x} = dx/dt \) also in \( L^2(0, T; H) \). When the simpler notations \( H^1 \) and \( L^2 \) are used, they will always refer to these particular spaces.

Let \( \Gamma \) be a lower semi-continuous (l.s.c.) function on \( L^2 \times L^2 \), with values in \( R \cup \{+\infty\} \). It will be assumed to be jointly convex in the variables \( (x, p) \), and to be proper (i.e. there exists \( (x_0, p_0) \) such that \( \Gamma(x_0, p_0) < \infty \)). We shall refer to \( \Gamma \) as the Hamiltonian.

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We define the Lagrangian \( \wedge \) from the Hamiltonian by taking the Fenchel conjugate with respect to the second variable:

\[
\wedge(x, v) = \sup_{p \in L^2} \{p \cdot v - \Gamma(x, p)\}
\]

It is a function on \( L^2 \times L^2 \) with values in \( \mathbb{R} \cup \{\pm \infty\} \) (note that the value \(-\infty\) is now allowed). For any \( x \in L^2 \), it is a convex l.s.c. function of \( v \) in \( L^2 \); indeed, formula (1) shows that it is the pointwise supremum of a family of l.s.c. functions. For any \( v \in L^2 \), it is a concave function of \( x \) in \( L^2 \), as the following lemma shows (note that it need not be u.s.c.).

**Lemma.** Take \( v \) in \( L^2 \), \( x_1 \) and \( x_2 \) in \( L^2 \) with \( \wedge(x_1, v) < +\infty \) and \( \wedge(x_2, v) < +\infty \), \( \alpha_1 \) and \( \alpha_2 \) non-negative with \( \alpha_1 + \alpha_2 = 1 \). Then:

\[
\wedge(\alpha_1 x_1 + \alpha_2 x_2, v) \geq \alpha_1 \wedge(x_1, v) + \alpha_2 \wedge(x_2, v).
\]

**Proof.** Clear if either \( \wedge(x_1, v) \) or \( \wedge(x_2, v) \) are equal to \(-\infty\). If both are finite, take \( \epsilon > 0 \), and pick \( p_1 \) and \( p_2 \) in \( L^2 \) such that:

\[
p_1 \cdot v - H(x_1, p_1) \geq \wedge(x_1, v) - \epsilon
\]

\[
p_2 \cdot v - H(x_2, p_2) \geq \wedge(x_2, v) - \epsilon
\]

Multiplying these inequalities by \( \alpha_1 \) and \( \alpha_2 \), and adding them, we get (set \( \alpha_1 p_1 + \alpha_2 p_2 = p \)):

\[
p \cdot v - [\alpha_1 H(x_1, p_1) + \alpha_2 H(x_2, p_2)] \geq \alpha_1 \wedge(x_1, v) + \alpha_2 \wedge(x_2, v) - \epsilon
\]

It follows from the definition of \( \wedge \) that:

\[
\wedge(\alpha_1 x_1 + \alpha_2 x_2, v) \geq p \cdot v - H(\alpha_1 x_1 + \alpha_2 x_2, \alpha_1 p_1 + \alpha_2 p_2).
\]

Since \( H \) is convex, we have:

\[
-H(\alpha_1 x_1 + \alpha_2 x_2, \alpha_1 p_1 + \alpha_2 p_2) \geq -[\alpha_1 H(x_1, p_1) + \alpha_2 H(x_2, p_2)].
\]

Comparing the last three inequalities, and using the fact that \( \epsilon \) is arbitrarily small, we get the desired result./

As usual, we denote by \( \partial \Gamma(x, p) \) the subgradient of \( \Gamma \) at \( (x, p) \), i.e. the (closed convex) set of all \( (x', p') \in L^2 \times L^2 \) such that:

\[
(y - x) \cdot x' + (q - p) \cdot p' \leq \Gamma(y, q) - \Gamma(x, p), \quad \forall (y, q) \in L^2 \times L^2.
\]

We shall also denote by \( \partial_v \wedge(x, w) \) the subgradient at \( w \) of the convex l.s.c. function \( v \rightarrow \wedge(x, v) \), and by \( \partial_x(-\wedge)(y, v) \) the subgradient at \( y \) of the convex function \( x \rightarrow -\wedge(x, v) \). Note that all these subgradients can very well be empty. Note also that we can get \( \Gamma \) from \( \wedge \):

\[
\Gamma(x, p) = \sup_{v \in L^2} \{p \cdot v - \wedge(x, v)\}.
\]
Our interest lies in studying the Hamiltonian equation \((-p, x) \in \partial \Gamma(x, p)\). We must first define some boundary conditions; as is usual in convex analysis, we shall do so by using two l.s.c. convex proper functions \(\phi_0\) and \(\phi_1\) from \(H\) to \(\mathbb{R} \cup \{+\infty\}\). We can now state in two equivalent forms the equations we wish to study:

**Proposition 1.** The pair \((x, p) \in H^1 \times H^1\) satisfies the Euler–Lagrange equations:

\[
\begin{align*}
    p &\in \partial_v \wedge (x, \dot{x}) \\
    \frac{dp}{dt} + \partial_x (-\wedge)(x, \dot{x}) &
\end{align*}
\]

\((\mathcal{E})\)

\[
\begin{align*}
    p(0) + \partial \phi_0(x(0)) &\geq 0 \\
    p(T) - \partial \phi_1(x(T)) &\geq 0
\end{align*}
\]

if and only if it satisfies Hamilton’s equations:

\[
\begin{align*}
    \left( -\frac{dp}{dt}, \frac{dx}{dt} \right) &\in \partial \Gamma(x, p) \\
\end{align*}
\]

\((\mathcal{H})\)

\[
\begin{align*}
    p(0) + \partial \phi_0(x(0)) &\geq 0 \\
    p(T) - \partial \phi_1(x(T)) &\geq 0
\end{align*}
\]

**Proof.** The boundary conditions are the same. From the definition of \(\wedge\), it follows that:

\[
(3) \quad p \in \partial_v \wedge (x, \dot{x}) \iff \dot{x} \in \partial_p \Gamma(x, p)
\]

There only remains to show the equivalence of relations \(-\dot{p} \in \partial_x (-\wedge)(x, \dot{x})\) and \(-\dot{p} \in \partial_x \Gamma(x, p)\). The first one means that:

\[
(-\dot{p}) \cdot (y - x) \geq \wedge (x, \dot{x}) - \wedge (y, \dot{x}) \quad \forall y \in L^2
\]

Since relations (3) hold, we have:

\[
\wedge (x, \dot{x}) = p \cdot \dot{x} - \Gamma(x, p).
\]

Writing that into the preceding inequality, we get:

\[
(-\dot{p}) \cdot (y - x) \geq p \cdot \dot{x} - \Gamma(x, p) - \wedge (y, \dot{x})
\]

Using formula (1) yields:

\[
(-\dot{p}) \cdot (y - x) \geq -\Gamma(x, p) + \Gamma(y, p)
\]

which means precisely that \(-\dot{p} \in \partial_x \Gamma(x, p)\). We can retrace our steps, and get the first relation from the second one. The equivalence is thus proved./
For instance, if the Lagrangian happens to be differentiable, the Euler–Lagrange equations can be written in classical form:

\[
\frac{d}{dt} \frac{\partial}{\partial x} (x, \dot{x}) - \frac{\partial}{\partial x} (x, \dot{x}) = 0.
\]

As another example, take \( \Gamma(x, p) = \frac{1}{2} \|p\|^2_2 + V(x) \). Then \( \wedge (x, v) = \frac{1}{2} \|v\|^2_2 - V(x) \); the Lagrangian and Hamiltonian systems are:

\[
\begin{align*}
\dot{x} & \in -\partial V(x), \quad x(0) \in -\partial \phi_0(x(0)), \quad x(T) \in \partial \phi_1(x(T)) \\
-\dot{p} & \in \partial V(x), \quad \dot{x} = p, \quad p(0) \in -\partial \phi_0(x(0)), \quad p(T) \in \partial \phi_1(x(T))
\end{align*}
\]

We now shall formulate equations (\( \mathcal{E} \)) and (\( \mathcal{H} \)) as variational problems. Briefly, there is a non-negative functional \( K(x, p) \) on \( H^1 \times H^1 \) which the solutions of (\( \mathcal{E} \)) and (\( \mathcal{H} \)) minimize; moreover, this minimum has to be zero.

**Proposition 3.** Solutions of problems (\( \mathcal{E} \)) and/or (\( \mathcal{H} \)) are exactly the pairs \((x, p)\) which satisfy:

\[
0 = K(x, p) \leq K(y, q) \quad \forall y \in H^1 \quad \forall q \in H^1
\]

with \( K(y, q) = \Gamma(y, q) + \Gamma^*(-\dot{q}, \dot{y}) - 2q \cdot \dot{y} + \phi_0(y(0)) + \phi^*_0(-q(0)) + \phi_1(y(T)) + \phi^*_1(q(T)) \).

**Proof.** By Fenchel’s inequality:

\[
\begin{align*}
\Gamma(y, q) & + \Gamma^*(-\dot{q}, \dot{y}) \geq q \cdot \dot{y} - y \cdot \dot{q} \\
\phi_0(y(0)) + \phi^*_0(-q(0)) & \geq - y(0)q(0) \\
\phi_1(y(T)) + \phi^*_1(q(T)) & \geq y(T)q(T)
\end{align*}
\]

equality holding if and only if \((-\dot{q}, \dot{y}) \in \partial \Gamma(y, q), \quad -q(0) \in \partial \phi_0(y(0)), \quad q(T) \in \partial \phi_1(y(T)), \quad i.e. \ if \ (y, q) \ solves \ problem \ (\mathcal{H}).\) Adding up these inequalities:

\[
K(y, q) \geq - \int_0^T (y(t)\dot{q}(t) + q(t)\dot{y}(t)) \, dt - y(0)q(0) + y(T)q(T)
\]

Integrating by part, we get \( K(y, q) \geq 0 \). Equality will hold for solutions of problem (\( \mathcal{H} \)) only, which is the desired result./

We can reformulate problem (\( \mathcal{P} \)) as a variational inequality:

**Proposition 4.** Solutions of problem (\( \mathcal{E} \)) and/or (\( \mathcal{H} \)) are exactly the pairs \((x, p)\in H^1 \times H^1 \) which satisfy:

\[
A(x, p; y, q) = 0 \quad y \in H^1, \quad q \in H^1
\]

with \( A(x, p; y, q) = \Gamma^*(-\dot{p}, \dot{x}) - \Gamma^*(-\dot{q}, \dot{y}) + \phi_0(x(0)) - \phi_0(y(0)) + \phi^*_1(p(T)) - \phi^*_1(q(T)) - p \cdot (\dot{x} - \dot{y}) + x \cdot (\dot{p} - \dot{q}) + p(0)(x(0) - y(0)) - x(T)(p(T) - q(T)) \).
Proof. By Proposition 3, the solutions of problems $(\mathcal{E})$ and $(\mathcal{H})$ are the pairs $(x, p) \in H^1 \times H^1$ which satisfy:

$$K(x, p) \leq 0$$

By the definition of Fenchel conjugates:

$$
\Gamma(x, p) = \sup_{a, \bar{y}} \{-\dot{q} \cdot x + \dot{y} \cdot p - \Gamma^*(-\dot{q}, \bar{y})\}
$$

$$
\phi_0^*(-p(0)) = \sup_{y(0)} \{-p(0)y(0) - \phi_0(y(0))\}
$$

$$
\phi_1(x(T)) = \sup_{q(T)} \{q(T)x(T) - \phi_1^*(q(T))\}
$$

Writing these formulas into $K(x, p)$, and noting that $\dot{q}, \dot{y}, y(0)$ and $q(T)$ can be specified independently, we get:

$$K(x, p) = \sup_{y, q} A(x, p; y, q) - [p \cdot x + x \cdot p + p(0)x(0) - p(T)x(T)]$$

Integrating by parts, we see that the bracket is identically zero. Clearly, $K(x, p) \leq 0$ if and only if the pair $(x, p)$ solves the variational inequality $(\mathcal{E})$. We will now give some examples.

Example 1. Newton’s equation, Neumann boundary conditions.

Let $f$ be a convex l.s.c. proper function on $H$. Let $p_0$ and $p_1$ be two points in $H$. Consider the problem:

$$
\begin{align*}
\ddot{x}(t) &\in -\partial f(x, (t)) \quad \text{a.e.} \\
\dot{x}(0) &= p_0, \quad \dot{x}(T) = p_1
\end{align*}
$$

These are equations $(\mathcal{E})$, for:

$$
\Gamma(x, p) = \frac{1}{2} \int_0^T |p(t)|^2 \, dt + \int_0^T f(x(t)) \, dt
$$

$$
\phi_0(\xi) = -p_0\xi, \quad \phi_1(\xi) = p_1\xi
$$

Indeed, it is a standard result in convex analysis (see [2], [5]) that the subdifferential at $x \in L^2$ of the map $y \to \int_0^T f(y(t)) \, dt$ is just the set of all $z \in L^2$ such that $z(t) \in \partial f(x(t))$ almost everywhere. Proposition 3 now tells us that $x$ solves equations (5) if and only if it belongs to $H^1(0, T; H)$, and there exists $p \in H^1$ such that:

$$0 = K(x, p) \preceq K(y, q) \quad (y, q) \in H^1 \times H^1$$
with \( K(y, q) = \int_0^T [f(y(t)) + f^*(\dot{q}(T)) + \frac{1}{2} |q(t)|^2 + \frac{1}{2} |\dot{y}(t)|^2 - 2q(t)\dot{y}(t)] \, dt - p_0 y(0) + p_1 y(T) \) if \( q(0) = p_0 \) and \( q(T) = p_1, +\infty \) otherwise. It then follows that \( p = \dot{x} \).

**Example 2.** Newton’s equation, Dirichlet boundary conditions

Consider the problem:

\[
\ddot{p}(t) \in -\partial f(p(t)) \quad \text{a.e.}
\]

\[
p(0) = p_0, \quad p(T) = p_1
\]

These are equations \((\mathcal{E})\) for:

\[
\mathcal{K}(x, p) = \frac{1}{2} \int_0^T |x(t)|^2 + \int_0^T f(p(t)) \, dt
\]

\[
\phi_0(\xi) = -p_0 \xi, \quad \phi_1(\xi) = p_1 \xi.
\]

Indeed, the relation \((-\dot{p}, \dot{x}) \in \partial \mathcal{K}(x, p)\) decomposes as \(-\dot{p} = x\) and \(\dot{x}(t) \in \partial f(p(t))\) a.e., which is equivalent to the differential equation \((7)\). It follows that \(p\) solves problem \((7)\) if and only if it belongs to \(H^1(0, T; \mathbb{R})\), satisfies the boundary conditions, and there exists \(x \in H^1\) such that:

\[
0 = K(x, p) \leq K(y, q) \quad \forall (y, q) \in H^1 \times H^1
\]

with \( K(y, q) = \int_0^T [\frac{1}{2} |y(t)|^2 + \frac{1}{2} |\dot{q}(t)| + f(q(t)) + f^*(\dot{q}(t)) - 2q(t)\dot{q}(t)] \, dt - p_0 y(0) + p_1 y(T) \) if \( q(0) = p_0 \) and \( q(T) = p_1, +\infty \) otherwise. It then follows that \(x = -\dot{p}\).

**Example 3.** The wave equation, prescribed initial and final state.

Let \( \Omega \) be an open bounded domain in \( \mathbb{R}^n \). We set \( H = L^2(\Omega) \), and we define a convex l.s.c. proper function \( f \) on \( H \) by:

\[
f(\xi) = \frac{1}{2} \int_\Omega |\nabla \xi(w)|^2 \, dw \quad \text{if} \quad \xi \in H^1_0(\Omega)
\]

\[
f(\xi) = +\infty \quad \text{otherwise}.
\]

We then consider the Hamiltonian:

\[
\mathcal{H}(x, p) = \frac{1}{2} \int_0^T |x(t)|^2 + \int_0^T f(p(t)) \, dt
\]

and we prescribe boundary conditions in time by \(\phi_0(\xi) = -p_0 \xi\), and \(\phi_1(\xi) = p_1 \xi\), with \(p_0\) and \(p_1\) given in \(L^2(\Omega)\). Equations \((\mathcal{H})\) then become: (see [2])

\[
p(t) \in H^1_0(\Omega) \cap H^2(\Omega) \quad \text{a.e.}
\]

\[
\frac{d^2 p}{dt^2}(t) = \Delta p(t) \quad \text{a.e.}
\]

\[
p(0) = p_0, \quad p(T) = p_1
\]
which is the wave equation, with homogeneous Dirichlet conditions in the space variables \( p(t)|_{\partial \Omega} = 0 \). By Proposition 3, \( p \) will solve problem (9) if and only if it belongs to \( H^1(0, T; L^2(\Omega)) \), and there exists \( x \in H^1(0, T; L^2(\Omega)) \) such that:

\[
0 = K(x, p) \leq K(y, q)
\]

with

\[
K(y, q) = \int_0^T \int_{\Omega} \left[ \frac{1}{2} \dot{y}(t, w)^2 + \frac{1}{2} \frac{\partial^2 y(t, w)}{\partial t^2} + \frac{1}{2} \sum_{i=1}^n \frac{\partial^2 y(t, w)}{\partial w_i^2} \right] dt dw - p_0 y(0) + p_1 y(T)
\]

if \( q(0) = p_0, q(T) = p_1 \) and \( q(t) \in H_0^1(\Omega) \) a.e. Here the function \( z(t) \) is defined as the solution of the homogeneous Laplace equation:

\[\forall t, \dot{y}(t) = -\Delta z(t), \quad z(t) \in H_0^1(\Omega)./\]

Finally, we will show how to treat initial-value problems, such as:

\[
(\mathcal{H}')
\]

\[
(-\dot{p}, \dot{x}) \in \partial \Gamma(x, p), \quad x(0) = x_0, \quad p(0) = p_0
\]

**Proposition 5.** Solutions of problem \((\mathcal{H}')\) on the time interval \([0, T]\) are exactly the pairs \((x, p)\) which satisfy:

\[
(\mathcal{P}')
\]

\[0 = K'(x, p) \leq K'(y, q)\]

for \((y, q) \in H^1 \times H^1, \quad y(0) = x_0, \quad q(0) = p_0\). Here:

\[K'(y, q) = \Gamma(y, q) + \Gamma^*(-\dot{q}, \dot{y}) - 2q \cdot \dot{y} + y(T)q(T) - x_0 p_0.\]

**Proof.** By Fenchel's inequality:

\[\Gamma(y, q) + \Gamma^*(-\dot{q}, \dot{y}) \geq q \cdot \dot{y} - y \cdot \dot{q}\]

equality holding if and only if \((-\dot{q}, \dot{y}) \in \partial \Gamma(y, q).\) It follows that:

\[K'(y, q) \geq - \int_0^T (y(t)\dot{q}(t) + q(t)\dot{y}(t)) \, dt - x_0 p_0 + y(T)q(T).\]

Integrating by parts, we get \(K'(y, q) \geq 0.\) Equality holds for solutions of problem \((\mathcal{H}')\) only, which is the desired result./

**Proposition 6.** Solutions of problem \((\mathcal{H}')\) on the time interval \([0, T]\) are exactly the pairs \((x, p)\) such that \(x(0) = x_0, \quad p(0) = p_0, \) and

\[
(\mathcal{Z}')
\]

\[A'(x, p; y, q) \leq 0\]
for \((y, q)\in H^1 \times H^1\), \(y(0) = x_0\), \(q(0) = p_0\). Here:

\[A'(x, p; y, q) = \Gamma^*(-\dot{p}, \dot{x}) - \Gamma^*(-\dot{q}, \dot{y}) - p \cdot (x - y) + x \cdot (p - q)\]

**Proof.** By definition of the Fenchel conjugate:

\[\Gamma(x, p) = \sup_{\dot{q}, \dot{y}} \{-\dot{q} \cdot x + \dot{y} \cdot p - \Gamma^*(-\dot{q}, \dot{y})\}\]

Writing this into \(K'(x, p)\), and noting that \(\dot{q}\) and \(\dot{y}\) can be specified arbitrarily, we get:

\[K'(x, p) = \sup A'(x, p; y, q) - [p - x + x \cdot \dot{p} + p_0 x_0 - p(T)x(T)]\]

the supremum being taken over all pairs \((y, q)\in H^1 \times H^1\) and that \(y(0) = x_0\) and \(q(0) = p_0\). The bracket vanishes, so that \(K'(x, p)\leq 0\) if and only if \((\dot{q}')\) is satisfied./

Of course, we can apply the same trick to \(\Gamma^*(-\dot{p}, \dot{x})\) instead of \(\Gamma(x, p)\). In this way, we get statements equivalent to propositions 6 and 4 (we leave the proof to the reader):

**Proposition 6 bis.** Solutions of problem \((\mathcal{H}')\) on the time interval \([0, T]\) are exactly the pairs \((x, p)\) such that \(x(0) = x_0\), \(p(0) = p_0\) and:

\[(\mathcal{H}') \quad B'(x, p; y, q) \leq 0 \quad \forall (y, q) \in H^1 \times H^1, \quad y(0) = y_0, \quad x(0) = x_0.

where \(B'(x, p; y, q) = \Gamma(x, p) - \Gamma(y, q) + \dot{p}(x - y) - \dot{x}(q - y)\).

**Proposition 4 bis.** Solutions of problem \((\mathcal{H})\) are exactly the pairs \((x, p)\in H^1 \times H^1\) which satisfy:

\[(\mathcal{H}) \quad B(x, p; y, q) \leq 0 \quad \forall (y, q) \in H^1 \times H^1

with \(B(x, p; y, q) = \Gamma(x, p) - \Gamma(y, q) + \dot{p}(x - y) - \dot{x}(q - y) + \phi_i(x(0)) - \phi_i(y(0)) + p_0(x(0) - y(0)) + \phi_1(y(T)) - \phi_1(x(T)) + p(T)(x(T) - y(T)).\)

We can now give some more examples:

**Example 4.** Newton’s equation, Cauchy problem.

Let \(f\) be a convex l.s.c. proper function on \(H\), and \(x_0, p_0\) two points of \(H\).

Consider the problem:

\[\ddot{x}(t) = -\delta f(x(t)) \quad \text{a.e.}\]

\[x(0) = x_0, \quad \dot{x}(0) = p_0\]

A function \(x\in H^1(0, T; H)\) solves problem (11) on the time interval \([0, T]\) if and only if \(x(0) = x_0\), and there exists \(p\in H^1\) such that \(p(0) = p_0\) and:

\[A'(x, p; y, q) \leq 0 \quad \forall (y, q) \in H^1 \times H^1, \quad y(0) = x_0, \quad p(0) = p_0.\]
Here $A'(x, p; y, q) = \int_0^T \left[ \frac{1}{2} p(t)^2 + f^*(x(t)) \frac{1}{2} \dot{q}(t)^2 - f^*(\dot{y}(t)) - p(t) \dot{x}(t) - q(t) \right] dt$. It then follows that $p = \dot{x}$.

We can also state this variational inequality as:

$$B'(x, p; y, q) \leq 0 \quad \forall (y, q) \in H^1 \times H^1, \quad y(0) = x_0, \quad p(0) = p_0$$

with

$$B'(x, p; y, q) = \int_0^T \left[ \frac{1}{2} p(t)^2 + f(x(t)) \frac{1}{2} q(t)^2 - f(y(t)) + \dot{p}(t)(x(t) - y(t)) - \dot{x}(t)(q(t) - y(t)) \right] dt.$$ It also follows that $p = \dot{x}$

**Example 5.** The wave equation, Cauchy problem.

Consider, as before, the problem:

$$p(t) \in H^1_0(\Omega) \cap H^2(\Omega) \quad \text{a.e.}$$

$$\frac{d^2 p}{dt^2}(t) = \Delta p(t) \quad \text{a.e.}$$

$$p(0) = p_0, \quad \dot{p}(0) = -x_0$$

The function $p \in H^1(0, T; L^2(\Omega))$ will solve problem (14) on the time interval $[0, T]$ if and only if $p(0) = p_0$, and there exists $x \in H^1(0, T; L^2(\Omega))$ such that

$$A'(x, p; y, q) \leq 0 \quad \forall (y, q) \in H^1 \times H^1, \quad y(0) = x_0,$$

$$p(0) = p_0, \quad q(t) \in H^1_0(\Omega) \quad \text{a.e.}$$

with

$$A'(x, p; y, q) = \int_0^T \int_\Omega \left[ \frac{1}{2} \frac{\partial p}{\partial t}(t, w)^2 - \frac{1}{2} \frac{\partial q}{\partial t}(t, w)^2 + \frac{1}{2} \sum_{i=1}^n \frac{\partial u}{\partial w_i}(t, w)^2 
- \frac{1}{2} \sum_{i=1}^n \frac{\partial z}{\partial w_i}(t, w)^2 - (\frac{\partial x}{\partial t}(t, w) - \frac{\partial y}{\partial t}(t, w)) \right] dt dw$$

Here $u(t)$ and $z(t)$ are the solutions of the Laplace equations:

$$\forall t, \dot{x}(t) = -\Delta u(t), \quad u(t) \in H^1_0(\Omega)$$

$$\forall t, \dot{y}(t) = -\Delta z(t), \quad z(t) \in H^1_0(\Omega)$$

This will also be written as:

$$A'(x, p; y, q) = \int_0^T \left[ \frac{1}{2} \| \dot{p}(t) \|^2 - \frac{1}{2} \| \dot{q}(t) \|^2 + \frac{1}{2} \| \text{grad}(-\Delta)^{-1} \dot{x}(t) \|^2 
- \frac{1}{2} \| \text{grad}(-\Delta)^{-1} \dot{y}(t) \|^2 - p(t)(\dot{x}(t) - \dot{y}(t)) 
+ x(t)(\dot{p}(t) - \dot{q}(t)) \right] dt,$$

all norms to be taken in $H = L^2(\Omega)$.
§2. Existence. The question now is whether these characterizations can actually be used to solve problems (Ⅱ) and/or (Ⅲ). We shall show that, in some cases, they can. Our main tool will be a refined version of Ky Fan’s inequality (see [2]):

**Proposition 1.** Let $\mathcal{H}$ be a closed subspace of $H^1(0, T; H)^2$, and $\mathcal{B}$ its unit ball. Let $\Phi$ be a real function on $\mathcal{H} \times \mathcal{H}$. Assume that, for any $(x, p) \in \mathcal{H}$:

1. the function $(y, q) \mapsto \Phi(x, p; y, q)$ is concave
2. $\Phi(x, p; x, p) = 0$
and that, for any $(y, q) \in \mathcal{H}$, and any $n \in \mathbb{N}$:
3. the function $(x, p) \mapsto \Phi(x, p; y, q)$ is weakly l.s.c. on $n\mathcal{B}$
Assume moreover that:
4. $\exists m \in \mathbb{N}: \{(x, p) \mid \Phi(x, p; y, q) \leq 0 \ \forall (y, q) \in \mathcal{B}\} \subset m\mathcal{B}$.
Then there exists $(\bar{x}, \bar{p}) \in \mathcal{H}$ such that:
5. $\Phi(\bar{x}, \bar{p}; y, q) \leq 0 \ \forall (y, q) \in H^1 \times H^1$.

**Proof.** Let $n \in \mathbb{N}$ be given. By the usual Ky Fan inequality ([6]), applied to $\Phi$ on the set $n\mathcal{B} \times n\mathcal{B}$, there exists $(x_n, p_n)$ in $n\mathcal{B}$ such that:

$$\Phi(x_n, p_n; y, q) \leq 0 \ \forall (y, q) \in n\mathcal{B}.$$ 

Let $n \to \infty$. By assumption (4), the sequence $(x_n, p_n)$ is bounded, and therefore we can extract a subsequence converging weakly to some $(\bar{x}, \bar{p})$. Take any $(y, q) \in H^1 \times H^1$; it belongs to $n\mathcal{B}$ for $n$ large enough, and by assumption (3):

$$\Phi(\bar{x}, \bar{p}; y, q) = \lim_{n \to \infty} \Phi(x_n, p_n; y, q) \leq 0/$$

As a particular case, conclusion (5) will still hold if (4) is replaced by the stronger assumption (see [4]):

6. $\exists m \in \mathbb{N}, (y_0, q_0) \in H^1 \times H^1: \{(x, p) \mid \Phi(x, p; y_0, q_0) \leq 0\} \subset m\mathcal{B}$.

We shall apply these results to the variational inequalities in Propositions 4 and 6; the subspace $\mathcal{H}$ being defined by appropriate boundary conditions. Example 2 and 4 will be taken up again.

**Example 2.** Newton’s equation, Dirichlet boundary condition.
Consider, as before, the problem (with prescribed $T > 0$):

7. \( p(t) \in -\partial f(p(t)) \) a.e.\n
7. \( p(0) = p_0, \quad p(T) = p_1 \)

By Proposition 4, $p$ solves that problem if and only if it belongs to $H^1(0, T; H)$, satisfies the boundary conditions and there exists $x \in H^1$ such
that:

\[ A(Hx, p; y, q) \leq 0 \quad \text{whenever} \quad q(0) = p_0, \quad q(T) = p_1 \]

with

\[ A(x, p; y, q) = \int_0^T \left( \frac{1}{2} |\dot{p}(t)|^2 + f^*(\dot{x}(t)) - \frac{1}{2} |\dot{q}(t)|^2 - f^*(\dot{y}(t)) - p(t)(\dot{x}(t) - \dot{y}(t)) + x(t)(\dot{p}(t) - \dot{q}(t)) \right) dt \]

Note that if a constant is added to \( x \), and another one to \( y \), the value of \( A \) is unchanged. Indeed:

\[
A(x + x_0, p; y + y_0, q) - A(x, p; y, q) = \int_0^T x_0(\dot{p}(t) - \dot{q}(t)) \, dt
\]

\[ = 0 \quad \text{since} \quad p(0) = q(0) \quad \text{and} \quad p(T) = q(T). \]

So we can always assume that \( x(0) = y(0) = 0 \).

**Proposition 2.** Let the function \( f : H \to R \) be convex, continuous, and satisfy the following growth condition:

\[ \exists K > 0, \quad \exists k > 0: \quad f(\xi) \leq K|\xi|^2 + k, \quad \text{all} \quad \xi \in H. \]

Then there exists \( T_K > 0 \) such that problem (7) has at least one solution whenever \( T \in ]0, T_K[ \). Moreover, \( T_K \to \infty \) when \( K \to 0 \).

**Corollary.** Assume the growth of \( f \) is less than quadratic:

\[ f(\xi)/|\xi|^2 \to 0 \quad \text{uniformly as} \quad |x| \to \infty \]

Then problem (7) has a solution for all \( T \).

**Proof.** The function \( A(x, p; y, q) \) satisfies all the assumptions of Proposition 1, where \( \mathcal{H} \) is taken to be the subspace of \( H^1(0, T; H) \) defined by the boundary conditions \( x(0) = 0, \quad p(0) = p_0 \) and \( p(T) = p_1 \). Indeed, (1) and (2) are obvious. As for (3), consider a bounded sequence \( (x_n, p_n) \) in \( \mathcal{H} \) converging weakly to \( (x, p) \). Then \( (x_n, p_n) \) converges weakly in \( L^2(0, T; H) \), which implies that:

\[
x_n(t) = 0 + \int_0^t \dot{x}_n(x) \, ds
\]

\[
p_n(t) = p_0 + \int_0^t \dot{p}_n(s) \, ds
\]

converge to \( x(t) \) and \( p(t) \) for all \( t \). Moreover, \( x_n \) and \( p_n \) are bounded, and hence they converge strongly in \( L^2(0, T; H) \) by Lebesgue's theorem. It follows that \( \dot{x}_n \cdot p_n \) and \( x_n \cdot \dot{p}_n \) converge to \( \dot{x} \cdot p \) and \( x \cdot \dot{p} \). By Fatou's lemma, the function \( x \to \int_0^T f^*(\dot{x}(t)) \, dt \) is strongly l.s.c. on \( H^1(0, T; H) \); since it is convex, it is also weakly l.s.c. Taking everything into account, we see that the function \( (x, p) \to A(x, p; y, q) \) is weakly l.s.c. on bounded sets.
There only remains to prove estimate (4), or (6). We shall use the well-known inequality $ab \leq \frac{1}{2}c a^2 + \frac{1}{2}(b^2/c)$ for all non-negative $a$, $b$, $c$. We have:

$$A(x, p; y, q) = \int_0^T \left[ \frac{1}{2} |\dot{p}(t)|^2 + f^*(\dot{x}(t)) - p(t) \dot{x}(t) + x(t) \dot{p}(t) \right] dt$$

$$+ \int_0^T \left[ p(t) \dot{y}(t) - x(t) \dot{q}(t) \right] dt - \int_0^T \left[ \frac{1}{2} |\dot{q}(t)|^2 + f^*(\dot{y}(t)) \right] dt$$

Once $(y, q)$ is fixed, the last term on the right-hand side is a constant, the second one is a linear function of $(p, q)$, and the first one, by the inequality just mentioned, is greater than or equal to:

$$\int_0^T \left[ \frac{1}{2} |\dot{p}(t)|^2 + f^*(\dot{x}(t)) - \frac{c}{2} |\dot{x}(t)|^2 - \frac{1}{2c} |\dot{p}(t)|^2 - \frac{d}{2} |t(t)|^2 - \frac{1}{2d} |x(t)|^2 \right] dt$$

the constants $c > 0$ and $d > 0$ to be chosen later. Taking into account the initial conditions $x(0) = 0$ and $p(0) = p_0$, we easily get:

$$\|x\|_{L^2} \leq T \|\dot{x}\|_{L^2} \quad \text{and} \quad \|p - p_0\|_{L^2} \leq T \|\dot{p}\|_{L^2}$$

It follows that expression (10) is greater than or equal to:

$$\int_0^T \left[ \frac{1}{2} \left( 1 - d - \frac{T^2}{c} \right) |\dot{p}(t)|^2 \right] dt + \int_0^T \left[ f^*(\dot{x}(t)) - \frac{1}{2} \left( c + \frac{T^2}{d} \right) |\dot{x}(t)| \right] dt$$

$$- \frac{T^{3/2}}{c} |p_0| \|\dot{p}\|_{L^2} - T |p_0|^2$$

Now hypothesis (8) comes into play. Taking the Fenchel conjugate of both sides, we get $f^*(\xi) \geq (1/4K)\xi^2 - k$ for all $\xi \in H$. Taking that into account, as well as the preceding inequalities, we get:

$$A(x, p; y, q) \geq \frac{1}{2} \left( 1 - d - \frac{T}{c} \right) \|\dot{p}\|^2 + \frac{1}{2} \left( \frac{1}{2K} - c - \frac{T}{d} \right) \|\dot{x}\|^2 - kT$$

$$- \frac{T^{3/2}}{c} |p_0| \|\dot{p}\| - T |p_0|^2 - \frac{1}{2} \|\dot{y}\| \|p_0\| + T \|\dot{p}\|$$

$$- T \|\dot{q}\| \|\dot{x}\| - \frac{1}{2} \|\dot{q}\|^2 - \int_0^T f^*(\dot{y}(t)) \, dt$$

Take for instance $d = \frac{1}{2}$ and $c = 1/4K$. Then

$$\frac{1}{2} \left( 1 - d - \frac{T}{c} \right) = \alpha \quad \text{and} \quad \frac{1}{2} \left( \frac{1}{2K} - c - \frac{T}{d} \right) = \beta$$

both are strictly positive whenever $T < 1/8K$. If $y$, $q$, and $T$ are fixed, we have
the inequality:

\[ A(x, p; y, q) \geq \alpha \|p\|^2 + \beta \|\dot{x}\|^2 - \gamma \|p\| - \delta \|\dot{x}\| - \zeta \]

with \(\alpha, \beta, \gamma, \delta, \zeta\) denoting various constants (depending on \(y, q,\) and \(T\)). If \(T < 1/8K\), it is clear that assumption (6) is satisfied, so Proposition 2 is proved with \(T_k = 1/8K\). The corollary immediately follows, since inequality (8) is seen to hold for any \(K\).

The growth condition (8) is natural in this context. For instance, the one-dimensional problem \(\ddot{p} = -p, \ p(0) = p_0, \ p(T) = p_1\), can be solved for all \((p_0, p_1) \in \mathbb{R}^2\) if and only if \(T < 1\), since the solutions have to be 1-periodic.

**Example 4.** Newton's equation, Cauchy problem.

Consider the problem described in the preceding section:

\[ \ddot{x}(t) \in -\partial f(x(t)) \quad \text{a.e.} \]
\[ x(0) = x_0, \quad \dot{x}(0) = p_0 \]

The variational inequality (12) characterizing \((x, p)\) with \(p = \dot{x}\), is exactly the same as the one in the preceding example; only the boundary conditions have changed \((y(0) = x_0, \ p(0) = p_0\) instead of \(y(0) = 0, \ p(0) = p_0, \ p(T) = p_1\)). The same arguments leads us to an analogous result:

**Proposition 3 (Global existence).** Let the function \(f: H \to \mathbb{R}\) be convex, continuous, and satisfy the growth condition (8). Then problem (11) has a solution on the time interval \([-1/8K, 1/8K]\). If growth condition (9) is satisfied, there is a solution for all times \(t \in \mathbb{R}\).

This can easily be transformed into a local existence result:

**Proposition 4 (Local existence).** Let the function \(f: H \to \mathbb{R} \cup \{+\infty\}\) be l.s.c. convex. Let \(x_0 \in H\) be a point of continuity for \(f\). Then, for any \(p_0 \in H\), problem (11) has a solution on some time interval \([-T, T]\), with \(T > 0\).

**Proof.** Since \(f\) is l.s.c. convex and continuous at \(x_0\), it is finite and continuous in some neighbourhood \(U\) of \(x_0\). Moreover:

\[ \exists M: (\eta \in \partial f(\xi), \ \eta \in U) \Rightarrow |\eta| \leq M. \]

We then define a function \(g: H \to \mathbb{R}\) by the formula:

\[ \forall \xi \in H, \quad g(\xi) = \sup\{(\xi - \xi)\eta + f(\xi) \mid \eta \in U, \ \eta \in \partial f(\xi)\} \]

The function \(g\) is easily seen to be convex, finite, and to coincide with \(f\) on
Moreover, it is lipschitzian with constant $K$:
\[
\forall (\xi, \eta) \in H, \quad |g(\xi) - g(\eta)| \leq K |\xi - \eta|
\]
so that it certainly satisfies condition (9).

The initial-value problem:

\[
\begin{align*}
\dot{x}(t) &\in -\partial g(x(t)) \quad \text{a.e.} \\
x(0) &= x_0, \quad \dot{x}(0) = p_0
\end{align*}
\]

has a global solution, by Proposition 3. This solution $x$ is also a solution of (11) as long as $x(t) \in \mathcal{U}$. Hence the result.

For sharper results on the Cauchy problem for Newton’s equation, we refer to [7]. As for the wave equation, we did not succeed in proving existence by our method.

**Bibliography**

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