Generations playing a Chichilnisky game[a]

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Abstract

The Chichilnisky criterion of intergenerational equity is time-inconsistent and has no optimal solution in the Ramsey model. Hence, seeking optimality is both irrelevant and futile. This paper investigates Markov-perfect equilibria in the game that generations with Chichilnisky preferences play in the Ramsey model. The time-discounted utilitarian optimum is the unique equilibrium path when the initial stock is small, implying that the weight on the infinite future in the Chichilnisky criterion plays no role. However, this part of the Chichilnisky criterion may lead to more stock conservation than the time-discounted utilitarian optimum if the initial stock is large. Uniqueness is obtained by assuming that each generation coordinates on a(n) (almost) best equilibrium and takes into account that future generations will do as well. This analysis of uniqueness is based on von Neumann-Morgenstern stability.

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1 Introduction

The Ramsey model (Ramsey, 1928) is the workhorse of growth theory, used to analyze both the management of reproducible assets and stocks of environmental and natural resources.

In the Ramsey model, output depends on a one-dimensional stock k and is split between consumption c and stock accumulation $\dot{k}$. One possibility is to interpret the function that turns stock k into output $c + \dot{k}$ as a production function and k as reproducible capital. Then the initial capital stock can be assumed to be small, and the question is to how much capital to accumulate. Another possibility is to interpret the function that turns k into $c + \dot{k}$ as a natural growth function and k as a renewable resource. Then the initial resource stock can be assumed to be large, and the question is how much resource to conserve.

Economists usually apply the time-discounted utilitarian (TDU) criterion, which seeks to maximize

$$\delta \int_0^\infty e^{-\delta t} u(c) dt$$

over all feasible paths. In this criterion, u is a utility function that turns consumption into transformed value ('utility'). When the TDU criterion is applied to the Ramsey model, it leads to capital accumulation in the former interpretation, with a small initial stock of reproducible stock, but does not lead to resource conversation in the latter interpretation, with a large initial stock of a natural resource.

When considering the question of intertemporal distribution as a problem of intergenerational equity, one might argue that the TDU criterion is deficient from an axiomatic perspective – in spite of Koopmans's (1960) axiomatization – as it does not treat generations equally. The TDU criterion leads also to problematic consequences in the Ramsey model, as it does not support the intuition that we should be willing to assist an infinite future if all the future generations are worse off than us.

Alternatives like undiscounted utilitarianism and (lexicographic or ordinary) maximin treat generations equally. They entail that future generations are assisted if they are worse off than us, but these alternatives provide extremely different answered to question of our responsibility to save for the benefit of future generations than are better than us (see Asheim, 2010): According to undiscounted utilitarianism, the responsibility to save for the benefit of future generations than are better than us is essentially unlimited, while there is no such responsibility when maximin is applied. Hence, compared to the TDU criterion, undiscounted utilitarianism and maximin might be claimed to lead to more desirable consequences when the Ramsey model is interpreted as a model of resource conservation, but these criteria lead to extreme and perhaps undesirable consequences when the Ramsey model is interpreted as a model of capital accumulation.

Chichilnisky (1996) points out that the TDU criterion is a dictatorship of the present, in the sense that consumption beyond some finite future time does not play any role if two different consumption paths are strictly ranked. She suggests *sustainable preference* as an alternative. A sustainable preference is a numerically representable social welfare function satisfying the axiom of Strong Pareto (in the sense of being sensitive to the interests of each generation), and being neither a dictatorship of the present (also generations beyond any given T play a role) or a dictatorship of the future (not only generations beyond any given T play a role).

The requirement that a sustainable preference not be a dictatorship of the present rules out the TDU criterion. The requirement that a sustainable preference be numerically representable rules out undiscounted utilitarianism and lexicographic maximin. The requirement that a sustainable preference satisfy Strong Pareto rules out ordinary maximin.

The following is a particular version of a sustainable preference that we will apply in the continuous time framework considered in this paper:

$$(1-\alpha)\left(\delta \int_0^\infty e^{-\delta t} u(c) dt\right) + \alpha \lim_{t\to\infty} u(c)\,, \text{ with } 0 < \alpha < 1.$$

In this version of a sustainable preference, the criterion is simply to maximize a convex combination of the TDU value and limit of utility. This particular kind of a sustainable preference can be used to rank paths for which consumption converges when time goes to infinity, and on this domain it is within the class of representations considered in Chichilnisky's (1996) Theorems 1 and 2.

However, even in this simplified form, it is problematic to use Chichilnisky's criterion when applied on the set of converging consumption paths in the Ramsey model. These problems are discussed extensively by Heal (1998). The problems of applying the Chichilnisky criterion is two-fold:

- There is a generic problem of non-existence, as the value of the first TDU part of the criterion is increased by the delaying the response to the second part of the criterion, which captures the concern for the infinite future. This problem of non-existence is present also when applied to the Ramsey model.

- The criterion is time-inconsistent, as the weight on any absolute time t of the first TDU part increases when the time of evaluation is advanced, while the weight on the infinite future through the second part does not change.

The problems of non-existence and time-inconsistency imply that searching for a optimal paths in the Ramsey model when applying the Chichilnisky criterion in the Ramsey model is both futile and irrelevant. Consequently, this paper investigates Markov-perfect equilibria in the game that generations with Chichilnisky preferences play in the Ramsey model. We show that the equilibrium path always coincides with the TDU optimum when the initial stock is small, implying that the weight on the infinite future in the Chichilnisky criterion plays no role. However, we also show that this part of the Chichilnisky criterion may lead to more stock conservation than the time-discounted optimum if the initial stock is large.

These consequences of the Chichilnisky criterion might be considered attractive as it supports the intuition that we should seek to assist future generations if they are worse off than us, but not to save as much for their benefit if they turn out to be better off.

The paper is organized as follows. In Section 2 we introduce the Ramsey model, and investigate TDU optimal paths when the stock is constrained to remain within restricted subintervals. This will serve as a building block for the analysis of Markov-perfect equilibria when, in Section 3–6, applying the Chichilnisky criterion to the Ramsey model. In Section 3 we first establish that there does not exist and optimal path for the Chichilnisky criterion in the Ramsey model. In Section 4 we then consider continuous Markov-perfect equilibria on subintervals, and establish that the Chichilnisky criterion is outcome-equivalent to the TDU criterion.

In Section 5, we then show that this conclusion is changed when we consider discontinuous Markov-perfect equilibrium. To be precise: The Chichilnisky criterion is still outcome-equivalent to the TDU optimum when the Ramsey model is interpreted as a model of capital accumulation with a small initial stock. However, when the Ramsey model is interpreted as a model of resource conversation with a large initial stock, the weight on the infinite future in the Chichilnisky criterion may induce generations to conserve the resource stock to a larger extent than under the TDU criterion.

In the penultimate Section 6 we address the problem of coordination: What if the first generation coordinates on the discontinuous Markov-perfect equilibrium that leads to an outcome maximizing the value of the Chichilnisky criterion? What if the first generation takes into account that future generations will do so in turn? Our analysis indicates that uniqueness is obtained by assuming that each generation coordinates on a(n) (almost) best equilibrium and takes into account that future generations will do as well. This uniqueness result allows us to perform comparative statics with respect to the discount rate δ and the weight α on the infinite future. In the final Section 7 we offer concluding remarks by comparing our results in the context of the Chichilnisky criterion with other criteria that also support the intuition that we should seek to assist future generations if they are worse off than us, but not to save as much for their benefit if they turn out to be better off.

2 TDU optimum in the Ramsey model

Denote by k the stock of an augmentable good. In the *Ramsey model*, instantaneous output $f(k)$ is split between flow of consumption c and stock accumulation $\dot{k}$. One possibility is to interpret f as a net production function and k as a stock of reproducible capital. Then the initial capital stock k_0 can be assumed to be small, and the question is to how much capital to accumulate. Another possibility is to interpret f as a natural growth function and k as a stock of a renewable resource. Then the initial resource stock can be assumed to be large, and the question is how much resource to conserve.

To allow for both these interpretations, let $f : (0, K) \to \mathbb{R}_{++}$ be a strictly concave and continuously differentiable function, with K finite or infinite, satisfying $\lim_{k \to 0^+} f(k) = 0$, $\lim_{k \to K^-} f(k) = 0$ if $K < \infty$, $\lim_{k \to 0^+} f'(k) = \infty$, and $\lim_{k \to K^-} f'(k) = 0$ if $K = \infty$. This includes the case where f is an increasing function satisfying the Inada conditions, but allows also for f having an interior maximum, due to depreciation (in the former interpretation) or reduced natural regeneration for stocks exceeding the maximum sustainable yield (in the latter interpretation). In any case, the technology of the economy is described by the system:

$$\dot{k} = f(k) - c, \quad k(0) = k_0 \in (0, K), \tag{1}$$

$$c(t) > 0, \quad k(t) > 0. \tag{2}$$

Note that if $c \in L^1(0, T)$, then k is absolutely continuous on $(0, T)$, and conversely. We denote by $L^1_{\text{loc}}(0, \infty)$ the space:

$$L^1_{\text{loc}}(0, \infty) = \bigcap\nolimits_{T>0} L^1(0, T) .$$

Given $k_0 \in (0, K)$, any pair (c, k) satisfying (1) and (2), with $c \in L^1_{\text{loc}}(0, \infty)$, will be called *feasible*. The set of all feasible pairs (c, k) will be denoted by $A(k_0)$.

Let $u : \mathbb{R}_{++} \to \mathbb{R}$ be a twice differentiable strictly increasing and strictly concave utility function satisfying $\lim_{c \to 0^+} u'(c) = \infty$.

Define $\mathbf{k}_\infty : \mathbb{R}_{++} \to (0, K)$ by $f'(\mathbf{k}_\infty(\delta)) = \delta$. It follows from the assumptions on f that $\mathbf{k}_\infty$ is well-defined continuous and strictly decreasing function.

The following result is classical:

Proposition 1 *The unrestricted TDU problem:*

$$\sup_{(c,k) \in A(k_0)} \int_0^\infty e^{-\delta t} u(c) dt$$

has a unique solution $(c^*(t), k^*(t))$ *for every initial stock* $k_0 \in (0, K)$. *Both* k^* *and* c^* *are monotonic in* t, *and:*

$$\lim_{t \to \infty} k^*(t) = \mathbf{k}_\infty(\delta), \ \lim_{t \to \infty} c^*(t) = f(\mathbf{k}_\infty(\delta)) \ .$$

To study the implications of the Chichilnisky criterion in the Ramsey problem, we need to understand first the *restricted* TDU problem. Fix $k_0 \in (0, K)$, and let $I \subseteq (0, K)$ be a nonempty interval with $I \ni k_0$. The interval I may be unrestricted and coincide with $(0, K)$ or be an open, half-closed or closed subinterval of $(0, K)$ within which the stock k is constrained to remain. Given $k_0 \in (0, K)$, any pair (c, k) satisfying (1) and (2), with $k(t) \in I$ for all $t \in [0, \infty)$ and $c \in L^1_{\text{loc}}(0, \infty)$, will be called *$I$-feasible*. The set of all I-feasible pairs (c, k) will be denoted by $A(k_0, I)$. Say that the pair (k^*, c^*) is *I-optimal* if, for any other I-feasible pair (k, c), we have:

$$\liminf_{T \to \infty} \int_0^T e^{-\delta t} \big(u(c^*(t)) - u(c(t)) \big) dt \ \geq \ 0 \, .$$

Lemma 1 *If the I-optimum exists, then it is unique.*

Proof. It is easily checked that the set of all I-feasible consumption paths is convex. Since u is strictly concave, the TDU criterion is strictly concave, and the maximum, if it exists, is unique. ■

With any I-feasible pair (k, c) we associate the path of present-value consumption prices $p : [0, \infty) \to \mathbb{R}_{++}$ defined by:

$$p(t) = e^{-\delta t} u'(c(t)) \, .$$

We say that (k^*, c^*) is *I-competitive* if c^* is absolutely continuous, so that the associated present-value price path p^* is differentiable almost everywhere, and, for almost all $t \in [0, \infty)$, $k^*(t)$ satisfies profit-maximization:[1]

$$k^*(t) = \arg\max_{k \in I} \{p^*(t) f(k) + \dot{p}^*(t) k\}\,. \tag{3}$$

Say that (k^*, c^*) satisfies the *capital value transversality* (CVT) condition if

$$\lim_{T \to \infty} p^*(T) k^*(T) = 0\,.$$

We are now in a position to state a sufficient condition for optimality.

Lemma 2 *If an I-feasible pair (k^*, c^*) is I-competitive, satisfies the CVT condition, and $\int_0^\infty e^{-\delta t} u\,(c^*(t))\,dt < \infty$, then (k^*, c^*) is I-optimal.*

Proof. Assume that the I-feasible pair (k^*, c^*) is I-competitive, satisfies the CVT condition, and $\int_0^\infty e^{-\delta t} u\,(c^*(t))\,dt < \infty$. For any other I-feasible pair (k, c) we have, using the definition of p^* and the concavity of u:

$$\int_0^T e^{-\delta t}\,(u(c^*(t)) - u(c(t)))\,dt \geq \int_0^T p^*(t)\,(c^*(t) - c(t))\,dt\,.$$

Hence, by equation (1) and the property that (3) holds for almost all t:

$$\int_0^T e^{-\delta t}\,(u(c^*(t)) - u(c(t)))\,dt \geq \int_0^{T'} p^*(t)\big(\dot{k}(t) - \dot{k}^*(t)\big) + \dot{p}^*(t)\,(k(t) - k^*(t))\,\,.$$

Integrating by parts the right-hand side, and using the fact that $k\,(0) = k_0 = k^*\,(0)$:

$$\int_0^T e^{-\delta t}\,(u(c^*(t)) - u(c(t)))\,dt \geq p^*\,(T)\,(k\,(T) - k^*\,(T))\,\,.$$

Letting $T \to \infty$ and using the CVT condition:

$$\lim\inf_{T \to \infty} \int_0^T e^{-\delta t}\,(u(c^*(t)) - u(c(t)))\,dt \geq 0\,.$$

Hence, (k^*, c^*) is I-optimal. ■

This provides us with three cases where the I-optimal consumption paths exists and is unique. Note that uniqueness follows immediately from the convexity of the feasible set and the concavity of the utility function u. Write $\underline{k} := \mathbf{k}_\infty(\delta)$.

[1] The first term, $pf(k)$, is the value of net production, while the negative of the second term, $-\dot{p}k = (-\dot{p}/p)pk$, is the cost of holding capital, with $-\dot{p}/p$ being the consumption interest rate. Hence, $pf(k) + \dot{p}k$ can be interpreted as profit. Note that (3) cannot be defined at time at which c and thus p is not differentiable.

(a) If $\underline{k} \in I$, $\underline{k} = \sup I \notin I$ or $\underline{k} = \inf I \notin I$, then there exists a unique I-competitive pair (k, c) satisfying the CVT condition. If $k_0 = \underline{k}$, we have $k(t) = \underline{k}$ for all t. If $k_0 \neq \underline{k}$, then $k(t)$ belongs to the interior of I for all $t > 0$, and converges to, but never reaches, $\underline{k}$. Condition (3) is satisfied for all $t \in [0, \infty)$, implying that net marginal productivity $f'(k)$ equals the consumption interest rate $-\dot{p}/p$. This leads to the Euler equation,

$$u''(c)\dot{c} = (\delta - f'(k))u'(c)\,,$$

and the Keynes-Ramsey rule,

$$f'(k) = \delta + \eta\tfrac{\dot{c}}{c}\,,$$

where $\eta = -u''(c)c/u'(c)$.

(b) If $\underline{k} > \sup I \in I$, then there exists a unique I-competitive pair (k, c) satisfying the CVT condition. The stock path reaches $\max I$ in finite time T with $\dot{k}(T) = 0$, and stays at $\max I$, with $c(t) = f(\max I)$, for $t \geq T$. We have that $f'(k) > -\dot{p}/p$ for $t > T$, implying that the Euler equation is satisfied only in the interior phase, for $0 < t < T$. Note that (3) is not satisfied for $t = T$, as c and thus p are not differentiable at this point in time.

(c) If $\underline{k} < \inf I \in I$, then there exists a unique I-competitive pair (k, c) satisfying the CVT condition. The stock path reaches $\min I$ in finite time T with $\dot{k}(T) = 0$, and stays at $\min I$, with $c(t) = f(\min I)$, for $t \geq T$. We have that $f'(k) < -\dot{p}/p$ for $t > T$, implying that the Euler equation is satisfied only in the interior phase, for $0 < t < T$. Note that (3) is not satisfied for $t = T$, as c and thus p are not differentiable at this point in time.

We claim that in the remaining cases, that is, $\underline{k} > \sup I \notin I$ or $\underline{k} < \inf I \notin I$, there is no I-optimal pair. Indeed, suppose for instance $\underline{k} > \sup I \notin I$. Set $J = I \cup \{\sup I\}$. Clearly, the set of I-feasible pairs $A(k_0, I)$ is a subset of the set of J-feasible pairs $A(k_0, J)$, so the maximum of the TDU criterion over all J-feasible pairs is at least as large as the supremum over all I-feasible pairs:

$$\max_{A(k_0,J)} \int_0^\infty e^{-\delta t} u\left(c\left(t\right)\right) dt \geq \sup_{A(k_0,I)} \int_0^\infty e^{-\delta t} u\left(c\left(t\right)\right) dt$$

We know from case (b) above that the J-maximum is unique, and is achieved by a pair (k^*, c^*) where k^* stays in I for $0 \leq t < T$ and is equal to $\sup I$ for $t \geq T$. We approximate (k^*, c^*) by a sequence of I-feasible pairs (k_n, c_n) as follows. Denote by T_n the time when $k^*(t) = \sup I - \frac{1}{n}$, and set:

$$k_n(t) = \begin{cases} k^*(t) & \text{for } 0 \leq t \leq T_n \\ \max I - \frac{1}{n} & \text{for } T_n \leq t \end{cases}$$

with c_n being the associated consumption path. Clearly:

$$\sup_{A(k_0,I)} \geq \int_0^\infty e^{-\delta t} u(c_n(t))\, dt \to \int_0^\infty e^{-\delta t} u(c^*(t))\, dt = \max_{A(k_0,J)}$$

so $\sup_{A(k_0,I)} = \max_{A(k_0,J)}$. On the other hand, the maximum on the right-hand side is achieved only at (k^*, c^*), which does not belong to $A(k_0, I)$. So the supremum is not achieved.

3 Chichilnisky criterion in the Ramsey model

Fix $k_0 \in (0, K)$, and consider the class $\mathcal{K}$ of $(0, K)$-feasible paths for which the corresponding consumption path is converging. For any stock path $k \in \mathcal{K}$ with associated converging consumption path c, write $c_\infty = \lim_{t\to\infty} c(t)$.

Say that the stock path $k^* \in \mathcal{K}$ with associated converging consumption path c^* is *Chichilnisky-optimal* if

$$(1-\alpha)\delta \int_0^\infty e^{-\delta t} \left(u(c^*(t)) - u(c(t))\right) dt + \alpha \left(c^*_\infty - c_\infty\right) \;\geq\; 0 \tag{4}$$

for any stock path $k \in \mathcal{K}$ with associated converging consumption path c.

The Chichilnisky criterion, as given by (4), consists of a TDU part and a part that depends on the limit of the path. This particular form is a special case of what Chichilnisky (1996) calls a *sustainable preference*. It is clearly time-inconsistent [INCLUDE A PROOF, BUT NOT A PRIORITY AT THIS POINT IN TIME], as the weight on the stream in TDU part is increased when the time of evaluation is advanced, while the weight on the limit is not affected with the time of evaluation is advanced. Furthermore, the following result can be established.

Proposition 2 *There does not exist an optimal path for the Chichilnisky criterion, as given by (4), when applied in the Ramsey model*

Sketch of proof. [THIS PROOF CAN EASILY BE MADE STRINGENT, BASED ON YOUR NOTES, BUT THIS SHOULD BE PRIORITY AT THIS POINT

INT TIME.] *Step 1: A stock path $k \in \mathcal{K}$ with associated consumption path c satisfying $c_\infty \le f(\underline{k})$ is not Chichilnisky-optimal.* Since $f'(\underline{k}) = \delta > 0$ and f is continuously differentiable, there is $k' > \underline{k}$ with $f(k') > f(\underline{k})$. By following the $(0, K)$-optimal stock path converging to $\underline{k}$ sufficiently long before the deviating to a stock path converging to some k' with associated consumption path converging to $f(k')$, any stock path with associated consumption path converging to $c_\infty \le f(\underline{k})$ can be improved.

Step 2: A stock path $k \in \mathcal{K}$ with associated consumption path c satisfying $c_\infty > f(\underline{k})$ is not Chichilnisky-optimal. Assume that there is a Chichilnisky-optimal stock path $k \in \mathcal{K}$ with associated consumption path c satisfying $c_\infty > f(\underline{k})$. There is k_∞ with $f(k_\infty) = c_\infty$. By following the $(0, K)$-optimal stock path converging to $\underline{k}$ sufficiently long before the deviating to a stock path converging to k_∞ with associated consumption path converging to $f(k_\infty) = c_\infty$, the stock path k can be improved, leading to a contradiction. ■

The fact that the Chichilnisky criterion is time-inconsistent and does not have an optimal path in the Ramsey model, implies that seeking an optimal path is both irrelevant and futile. For the rest of the paper, we therefore investigate Markov-perfect equilibria in the game that generations with Chichilnisky preferences play in the Ramsey model.

4 Continuous equilibrium strategies

We first consider continuous Markov-strategies on subintervals in this section in order to prepare the ground for the analysis of discontinuous Markov-strategies in the subsequent sections.

Definition 1 *A continuous Markov-strategy is a pair (I, σ), where $I \subseteq (0, K)$ is a nonempty interval and $\sigma : I \to \mathbb{R}_{++}$ is continuously differentiable on the interior of I, such that, for any initial $k_0 \in I$, there is a unique solution $k^* : [0, \infty) \to I$ to $k(0) = k_0$ and $\dot{k} = f(k) - \sigma(k)$ for $t \in [0, \infty)$ which satisfies that:*

- *the stock path k^* is I-feasible*
- *there is some $k_\infty \in I$, not depending on k_0, such that $\lim_{t\to\infty} k^* = k_\infty$.*

There are two possible situations. Either k_∞ belongs to the interior of I, in which case σ is continuously differentiable at k_∞, which is an attracting fixed point for the one-dimensional dynamical system $\dot{k} = f(k) - \sigma(k)$. Or k_∞ is a boundary point of I. In either case, since $k(t) \to k_\infty$ when $t \to \infty$, we must have $f(k(t)) - \sigma(k(t)) \to 0$, so $\sigma(k(t)) \to f(k_\infty)$ and since σ is continuous, we find $\sigma(k_\infty) = f(k_\infty)$. Hence, the solution k^* is in $\mathcal{K}$, so we can compute the associated value of the C-criterion:

$$J(I, \sigma, k_0) := (1-\alpha)\Big(\delta \int_0^\infty e^{-\delta t} u(c^*(t))dt\Big) + \alpha u(f(k_\infty))$$

where $c^*(t)$ is the consumption path generated by (I, σ) and $k_0 \in I$

We now introduce the concept of continuous Markov-perfect equilibrium strategy, henceforth denoted by CES: it is a strategy such that deviating on any bounded time interval is not profitable, provided that the stock k remains in I. To formalize this, let (I, σ) be a strategy with solution converging to some k_∞, take the initial stock $k_0 \in I$ and two times T_1 and T_2, with $0 \leq T_1 \leq T_2 < \infty$, and any control $c(t) > 0$, $T_1 \leq t \leq T_2$. Extend it to a control $c_{k_0,T_1,T_2}(t)$ on $t \geq 0$ by:

$$c_{k_0,T_1,T_2}(t) = \begin{cases} \sigma(k) & \text{for } 0 \leq t \leq T_1 \\ c(t) & \text{for } T_1 \leq t \leq T_2 \\ \sigma(k) & \text{for } T_2 \leq t \end{cases} \tag{5}$$

We shall say that c_{k_0,T_1,T_2} is I-*admissible* if there is a unique solution $k : [0, \infty) \to I$ to $k(0) = k_0$ and $\dot{k} = f(k) - c_{k_0,T_1,T_2}(t)$ for $t \in [0, \infty)$ which satisfies that k remains in I for all $t \in [0, \infty)$ and there is some $k_\infty \in I$ such that $\lim_{t\to\infty} k = k_\infty$.

Definition 2 A continuous Markov-strategy (I, σ) is a continuous equilibrium strategy (CES) if, for every $k_0 \in I$ and every I-admissible choice of c_{k_0,T_1,T_2},

$$J(I, \sigma, k_0) \geq (1-\alpha)\Big(\delta \int_0^\infty e^{-\delta t} u(c_{k_0,T_1,T_2}(t))dt\Big) + \alpha u(f(k_\infty)) .$$

Proposition 3 *The pair (I, σ) is a CES if and only if, for every $k_0 \in I$, the solution $k^* : [0, \infty) \to I$ to $k(0) = k_0$ and $\dot{k} = f(k) - \sigma(k)$ for $t \in [0, \infty)$ is I-optimal.*

Proof. Assume that the solution $k^* : [0, \infty) \to I$ to $k(0) = k_0$ and $\dot{k} = f(k) - \sigma(k)$ for $t \in [0, \infty)$ is I-optimal. Then deviating from k^* on any bounded time interval cannot improve the TDU part of $J(I, \sigma, k_0)$ and cannot influence the part that depends on the limit of the path. Therefore, the pair (I, σ) is a CES.

Suppose that the solution $k^* : [0, \infty) \to I$ to $k(0) = k_0$ and $\dot{k} = f(k) - \sigma(k)$ for $t \in [0, \infty)$ is not I-optimal. By following the I-optimal stock path sufficiently long before the deviating to a stock path converging to $k_\infty^* = \lim_{t\to\infty} k^*(t)$, the TDU part of $J(I, \sigma, k_0)$ can be improved, while not influencing the part that depends on the limit of the path. Therefore, the pair (I, σ) is not a CES. ■

Proposition 3 implies that, in the Ramsey model, the Markov-perfect equilibrium strategy of the Chichilnisky criterion yields the same outcome as the TDU-optimal path, if the Markov-perfect equilibrium strategy is required to be continuous. For later use, denote by $((0, K), \sigma^*)$ the CES on $(0, K)$.

As we will see in the following section, considering discontinuous strategies yields additional possibilities.

5 Discontinuous equilibrium strategies

A discontinuous Markov-strategy combines a finite number of continuous Markov-strategies so that the flow of consumption is determined for all positive stock levels.

Definition 3 *A discontinuous Markov-strategy is a collection*

$$\sigma = \{(I_1, \sigma_1), \ldots, (I_n, \sigma_n)\}$$

where $\{I_1, \ldots, I_n\}$ *is a partition of the set of possible stock sizes* $(0, K)$ *and, for every* $i = 1, \ldots, n$, (I_i, σ_i) *is a continuous Markov-strategy.*

If, for every $i = 1, \ldots, n$, (I_i, σ_i) is a CES, we can define the associated *value function* $V_\sigma : (0, K) \to \mathbb{R}$ as follows: For every $k \in (0, K)$,

$$V_\sigma(k) = J(I, \sigma_j, k)\,,$$

where j satisfies that $k \in I_j$. Since $J(I, \sigma_i, \cdot)$ is a continuous function on I_i for every $i = 1, \ldots, n$, it follows that V_σ is a piecewise continuous function. Furthermore, since $J(I, \sigma_i, \cdot)$ is a continuously differentiable function on the interior of I_i for every $i = 1, \ldots, n$, it follows that V_σ is a piecewise continuously differentiable function.

By Propositions **??** and 3 that we have three cases to consider for each I_i:

(a) If $\underline{k} \in I_i$, $\underline{k} = \sup I_i \notin I_i$ or $\underline{k} = \inf I_i \notin I_i$, then, for any $k_0 \in I_i \backslash \{\underline{k}\}$, the solution k^* converges to, but never reaches, $\underline{k}$. (If $k_0 = \underline{k}$, then the stock equals $\underline{k}$ for all t.)

(b) If $\underline{k} > \sup I_i$, then $\sup I_i \in I_i$ and, for any $k_0 \in I_i$, the solution k^* reaches $\sup I$ in finite time.

(c) If $\underline{k} < \inf I_i$, then $\inf I_i \in I_i$ and, for any $k_0 \in I$, the solution k^* reaches $\inf I_i$ in finite time.

It follows that V_σ is continuous from the left for $k < \underline{k}$ and continuous from the right for $k > \underline{k}$. Furthermore, since $\underline{k}$ cannot be a point of discontinuity, points of discontinuity are reached in finite time.

If, at points of discontinuity, the value function V_σ jumps upwards, then a deviation would increase value, and therefore be profitable. If, on the other hand, at points of discontinuity, the value function V_σ jumps downwards, then a deviation would decrease value, and therefore not be profitable. [ADD FIGURE] Hence, only in this latter case, where it is possible to remain at the top of "cliffs", is discontinuity compatible with equilibrium. [EXPLAIN BETTER] Mathematically, this corresponds to the value function V_σ being upper semi-continuous, thereby motivating the following definition.

Definition 4 *A discontinuous Markov-strategy $\sigma = \{(I_1, \sigma_1), \dots, (I_n, \sigma_n)\}$ is a discontinuous equilibrium strategy (DES) if*

(i) for every $i = 1, \dots, n$, (I_i, σ_i) is a CES, and

(ii) the value function $V_\sigma : (0, K) \to \mathbb{R}$ is upper semi-continuous.

Lemma 3 *Let $\sigma = \{(I_1, \sigma_1), \dots, (I_n, \sigma_n)\}$ be a DES. If k is a point of discontinuity of $V_\sigma : (0, K) \to \mathbb{R}$, then $k > \underline{k}$.*

Sketch of proof. For all $k' \in (0, K)$, there is $\sigma = \{(I_1, \sigma_1), \dots, (I_n, \sigma_n)\}$ where, for every $i = 1, \dots, n$, (I_i, σ_i) is a CES, satisfying that, for some j, $k' \in I_j$ and $\sigma_j(k') = f(k')$. Hence, the solution $k_j^* : [0, \infty) \to I$ to $k = k'$ and $\dot{k} = f(k) - \sigma_j(k)$ for $t \in [0, \infty)$ leads to a constant consumption path $c_j^* : [0, \infty) \to \mathbb{R}_{++}$ with $c^*(t) = f(k')$ for all $t \geq 0$ and $V_\sigma(k') = u(f(k'))$. We also have that $V_\sigma(\underline{k}) = u(f(\underline{k}))$ since $\underline{k}$ is an attracting fixed point in I_ℓ for the one-dimensional dynamical system $\dot{k} = f(k) - \sigma_\ell(k)$, where ℓ satisfies that $\underline{k} \in I_\ell$ and σ_ℓ is the restriction of σ^* to I_ℓ.

It can be shown [WHAT IS THE BEST WAY TO SHOW THIS RESULT?] that, for $k \in (0, K)$ such that V_σ is differentiable,

$$V_\sigma'(k) = (1 - \alpha)\delta u'(\sigma(k)) .$$

Because, at a point of discontinuity $k' \in (0, K)$, we have that $V_\sigma(k') = u(f(k'))$, and V_σ is continuous from the left for $k \leq \underline{k}$, it follows that $V_\sigma(k) \geq u(f(k))$ for all $k \in (0, \underline{k}]$. Hence, a point of discontinuity $k' \in (0, \underline{k})$ would contradict that V_σ is upper semi-continuous. The result follows since V_σ cannot be discontinuous at $k' = \underline{k}$. ■

It follows from Lemma 3 that any point of discontinuity must exceed $\underline{k}$. Hence, for any DES $\sigma = \{(I_1, \sigma_1), \ldots, (I_n, \sigma_n)\}$, there is j and $k > \underline{k}$, such that $I_\ell = (0, k)$ and $\underline{k}$ is an attracting fixed point in I_ℓ for the one-dimensional dynamical system $\dot{k} = f(k) - \sigma_\ell(k)$, where σ_ℓ is the restriction of σ^* to I_ℓ. Therefore, for any initial stock $k_0 \leq \underline{k}$, the stock will converge towards $\underline{k}$. However, if the initial stock k_0 exceeds $\underline{k}$, then there might be points of discontinuity $k' > \underline{k}$ such that the stock will converge towards k'. This is summarized in the following proposition.

Proposition 4 *Let $\sigma = \{(I_1, \sigma_1), \ldots, (I_n, \sigma_n)\}$ be a DES. For every $k_0 \in (0, K)$, write $k^*_\infty := \lim_{t\to\infty} k^*_j(t)$, where j satisfies that $k_0 \in I_j$ and $k^*_j : [0, \infty) \to I$ is the solution to $k(0) = k_0$ and $\dot{k} = f(k) - \sigma_j(k)$.*

(i) If $k_0 \in (0, \underline{k})$, then $k^*_\infty = \underline{k}$.

(ii) If $k_0 \in [\underline{k}, K)$, then $k^*_\infty \geq \underline{k}$.

The former case corresponds to the interpretation of f as a net production function and k as a stock of reproducible capital. Then the initial capital stock k_0 can be assumed to be small, and the question is to how much capital to accumulate. Proposition 4 implies that in a DES, capital is accumulated as in the TDU-optimal path. Hence, the Chichilnisky criterion leads to the same behavior as the TDU criterion.

The latter case corresponds to the interpretation of f as a natural growth function and k as a stock of a renewable resource. Then the initial resource stock can be assumed to be large, and the question is how much resource to conserve. Proposition 4 implies that in a DES, more resource might be conserved than in the TDU-optimal path. Hence, the Chichilnisky criterion leads to more conservation than the TDU criterion does.

6 Coordinating on a DES

To study the scope of resource conservation for the Ramsey model under the Chichilnisky criterion, it is sufficient to consider the class of DESs of the form

$$\sigma^\kappa = \{(I_1, \sigma_1), (I_2, \sigma_2)\}$$

where $(0, K)$ is partitioned into two elements $I_1 = (0, \kappa)$ and $I_2 = [\kappa, K)$ with $\kappa > \underline{k}$. In particular, σ_1 is the restriction of σ^* to I_1.

To determine what values of κ are compatible with σ^κ being a DES, consider the CES $((0, K), \sigma^*)$. Define the associated value function $V_{\sigma^*} : (0, K) \to \mathbb{R}$ by:

$$V_{\sigma^*}(k) = J\left((0, K), \sigma^*, k\right) .$$

Since $\sigma^*(\underline{k}) = f(\underline{k})$, it follows that $V_{\sigma^*}(\underline{k}) = u(f((\underline{k}))$. Moreover:

$$V'_{\sigma^*}(\underline{k}) = (1 - \alpha)\delta u'(\sigma^*(\underline{k})) = (1 - \alpha)f'(\underline{k})u'(f(\underline{k})) < u'(f(\underline{k}))f'(\underline{k}) ,$$

since $f'(\underline{k}) = \delta$ and $\alpha > 0$. From the assumptions on f and the property that V_{σ^*} is continuously differentiable on $(0, K)$, it follows that there is κ^* such that

$$V_{\sigma^*}(k) \begin{cases} > u(f(k)) & \text{for } 0 < k < \underline{k} \\ = u(f(k)) & \text{for } k = \underline{k} \\ < u(f(k)) & \text{for } \underline{k} < k < \kappa^* \\ = u(f(k)) & \text{for } k = \kappa^* \\ > u(f(k)) & \text{for } \kappa^* < k < K . \end{cases} \tag{6}$$

Since $\sigma^*(\kappa) = f(\kappa)$, we have that the value function corresponding to σ^κ, V_{σ^κ}, has the property that $V_{\sigma^\kappa}(\kappa) = u(f(\kappa))$. Furthermore, because V_{σ^κ} coincides with V_{σ^*} on $(0, \kappa)$, it follows that σ^κ is upper semi-continuous if and only if $\underline{k} < \kappa \le \kappa^*$.

This analysis shows that κ^* is the maximum stock that can be conserved if the initial stock k_0 exceeds κ^*. If the initial stock k_0 satisfies $\underline{k} \le k_0 \le \kappa^*$, then k_0 is the maximum stock that can be conserved, as stock accumulation is not possible in a DES if the stock k exceeds $\underline{k}$. If the initial stock is smaller than $\underline{k}$, then in any DES the stock accumulates to $\underline{k}$.

Assume now that the generation at time 0 is endowed with the stock k_0 and seeks to coordinate on a DES that leads to an outcome maximizing the value of the Chichilnisky criterion. Of central interest for the analysis of this question is the stock $\bar{k} := \mathbf{k}_\infty\left((1 - \alpha)\delta\right)$. Hence, $\bar{k}$ is defined by

$$f'(\bar{k}) = (1 - \alpha)\delta .$$

The stock $\bar{k}$ has the property that

$$\lim_{k \to \bar{k}^+} V'_{\bar{k}}(k) = u'(f(\bar{k}))f'(\bar{k}) .$$

This entails that $\bar{k}$ is the maximum stock at which the generation at time 0 would want to conserve the initial stock when coordinating on a DES that leads to an outcome maximizing the value of the Chichilnisky criterion.

If the initial stock is larger than $\bar{k}$, then the increase of TDU part of the Chichilnisky criterion achieved by running down the stock – and thus temporarily increasing consumption – more than compensates for the reduced value of infinite part of the criterion that such a run-down of the stock leads to. However, if the initial stock is smaller than $\bar{k}$, then it pays to conserve the stock at the initial stock, given that a DES does not allow the stock to be accumulated beyond $\underline{k}$. Finally, if the initial stock does not exceed $\underline{k}$, then any DES leads to accumulation of the stock to $\underline{k}$.

These results can be summarized as follows: Assume that the generation at time 0 has the stock k_0 and coordinates a DES σ designed to maximize the value of the Chichilnisky criterion.

(a) If $0 < k_0 \leq \underline{k}$: All DESs induce the same behavior as the TDU-optimum, accumulating the stock to $\underline{k}$. It is not possible to accumulate beyond $\underline{k}$.

(b) If $\underline{\leq} k_0 \leq \bar{k}$: Stay put, e.g. by choosing $\sigma^{\bar{k}}$. It is not possible to accumulate, and not worthwhile to decrease the stock, given the trade-off between the two parts of the Chichilnisky criterion.

(c) If $\bar{k} < k_0 < K$: It is not worthwhile to stay put, as the increase of TDU part of the Chichilnisky criterion achieved by running down the stock exceeds the cost in terms of a reduced value of the part depending on the infinite future. The path will converge to some k_∞ satisfying $\bar{k} \leq k_\infty \leq \kappa^*$.

In case (c), convergence to $k_\infty > \bar{k}$ is not consistent with taking into account that future generations will coordinate in turn. However, due to the time-inconsistency of the Chichilnisky criterion, it might indeed be the case that initially the value of the Chichilnisky criterion is maximized by choosing $k_\infty > \bar{k}$.

To handle this kind of time-inconsistency in the coordination of DES, we present a modeling that is inspired by the analysis of renegotiation-proofness in repeated games. In particular, our formulation is based on Asheim (1997), but where we maintain the restriction to Markov-perfect strategies. This analysis of uniqueness is based on von Neumann-Morgenstern stability.

Let $\Sigma = \{\sigma = \{(I_1, \sigma_1), \ldots, (I_n, \sigma_n)\} \mid \sigma \text{ is a DES}\}$ denote the class of DES. Consider a partition $(G^\varepsilon, B^\varepsilon)$ of Σ and define $V^\varepsilon : (0, K) \to \mathbb{R}$ by

$$V^\varepsilon(k) = \sup_{\sigma \in G^\varepsilon} V_\sigma(k) \text{ for every } k > 0\,.$$

G^ε is *ε-internally stable* if, for all $\sigma \in G^\varepsilon$, and every $k > 0$,

$$V_\sigma(k) \geq V^\varepsilon(k) - \varepsilon\,.$$

G^ε is *ε-externally stable* if, for all $\sigma \in B^\varepsilon$, there is $k > 0$ s.t.

$$V_\sigma(k) < V^\varepsilon(k) - \varepsilon\,.$$

G^ε is *ε-stable* if it is ε-internally stable and ε-externally stable.

The following lemma paves the way for the proposition below.

Lemma 4 *For every $\varepsilon > 0$, there is a unique ε-stable G^ε.*

Proposition 5 *For all $\delta > 0$ and $k \in (\underline{k}, \bar{k})$, there exists $\varepsilon > 0$ such that if $\sigma \in G^\varepsilon$, then σ has a discontinuity in $(k - \delta, k + \delta)$.*

For all $\delta > 0$, there exists $\varepsilon > 0$ such that if $\sigma \in G^\varepsilon$, then σ has no discontinuity in $[\bar{k} + \delta, K)$.

The interpretation is that, in the limit, when $\varepsilon \to 0$:

- If $0 < k < \underline{k}$, then $\sigma(k) = \sigma^*(k)$.
- If $\underline{k} \leq k \leq \bar{k}$, then $\sigma(k) = \sigma^k(k)$.
- If $\bar{k} < k < K$, then $\sigma(k) = \sigma^{\bar{k}}(k)$.

This is essentially a uniqueness result, although the limiting strategy is not a DES. Rather, as $\varepsilon \to 0$, the points of discontinuity appear closer and closer, so that the outcome from any initial k_0 satisfying $\underline{k} \leq k \leq \bar{k}$ approaches the path where the stock remains constant at k_0.

The uniqueness result allows for comparative statics.

- As $\delta \to 0$ for fixed $\alpha \in (0, 1)$, the outcome for any $k_0 \in (0, K)$ becomes identical with the TDU-optimal path, which in turn approaches the undiscounted utilitarian optimum (if it exists). Hence, the weight on the infinite future in the Chichilnisky criterion plays no role.

- As $\alpha \to 1$ for fixed $\delta > 0$, the outcome is the TDU optimum for $k_0 \in (0, \underline{k})$, while k_0 is conserved if $\underline{k} \leq k < \hat{k}$, where $\hat{k}$ satisfies $f'(\hat{k}) = 0$ if f reaches a maximum or $\hat{k} = \infty$ otherwise. Hence, increasing the weight on the infinite future in the Chichilnisky criterion does not change the behavior for small k_0, but ensures that resource conservation is the outcome any initial k_0 unless conserving the stock at k_0 is inefficient.

7 Concluding remarks

We have shown that the Markov-perfect equilibria, when the Chichilnisky criterion is applied in the Ramsey model, support the intuition that we should seek to assist future generations if they are worse off than us, but not to save as much for their benefit if they turn out to be better off.

This reinforces the results obtained by Asheim and Mitra (2010) and Zuber and Asheim (2012) for the criteria of *sustainable discounted utilitarianism* (SDU) and *rank discounted utilitarianism* (RDU) respectively. These criteria are also numerically representable, and they are neither a dictatorship of the present (also generations beyond any given T play a role) nor a dictatorship of the future (not only generations beyond any given T play a role). However, they do not satisfy the axiom of Strong Pareto, and are thus not examples of a sustainable preference.

When applied to the Ramsey model, both SDU and RDU lead to capital accumulation (leading to outcomes that are identical to the TDU optimal path) when $(0 < k < \underline{k})$, while k_0 is conserved if $\underline{k} \leq k \leq \hat{k}$. Moreover, these optimal paths are time-consistent so that a game-theoretic analysis is not called for.

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