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AGGREGATION AND MARKET DEMAND: AN EXTERIOR DIFFERENTIAL CALCULUS VIEWPOINT

BY P. A. CHIAPPORI AND I. EKELAND[1]

We analyze under which conditions a given vector field can be disaggregated as a linear combination of gradients. This problem is typical of aggregation theory, as illustrated by the literature on the characterization of aggregate market demand and excess demand. We argue that exterior differential calculus provides very useful tools to address these problems. In particular, we show, using these techniques, that any analytic mapping in $\mathbb{R}^n$ satisfying Walras Law can be locally decomposed as the sum of n individual, utility-maximizing market demand functions. In addition, we show that the result holds for arbitrary (price-dependent) income distributions, and that the decomposition can be chosen such that it varies continuously with the mapping. Finally, when income distribution can be freely chosen, then decomposition requires only $n/2$ agents.

KEYWORDS: Microeconomics, consumer theory, aggregation, market demand.

1. INTRODUCTION: AGGREGATION AND GRADIENT STRUCTURES

IN MANY SITUATIONS, economists are interested in the behavior of aggregates formed by adding several elementary demand or supply functions. In turn, each of these elementary components results from some maximizing decision process at the "individual" level. A standard illustration is the characterization of aggregate market demand or excess demand in an exchange economy, a problem initially raised by Sonnenschein (1973a, b). A number of authors have addressed this problem, starting with Mantel (1974) and Debreu (1974), and including McFadden et al. (1974), Mantel (1976, 1977), Diewert (1977), and Geanakopolos and Polemarchakis (1980). Here, agents maximize utility subject to a budget constraint, and individual demands add up to an aggregate demand or excess demand function. Recently this research has been extended to incomplete markets by Bottazzi and Hens (forthcoming) and Gottardi and Hens (1995). A different but related example is provided by Browning and Chiappori (1998), who consider the demand function of a two-person household, where each member is characterized by a specific utility function and decisions are only assumed to be Pareto-efficient.

These models share a common feature: they lead to the same type of mathematical problem. In all cases, the economic context has the following translation: some given vector field $\mathbf{X}(\mathbf{p})$, mapping $\mathbb{R}^n_+$ to $\mathbb{R}^n$, must be decom-

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posed as a linear combination of k gradients. Here, k is the number of individuals, $\mathbf{X}(\mathbf{p})$ is the original (aggregate) function, and gradients are the natural mathematical translation of the underlying optimization problem. Formally, one seeks to write $\mathbf{X}(\mathbf{p})$ as

$$\mathbf{X}(\mathbf{p}) = \lambda_1(\mathbf{p})\, D_{\mathbf{p}} V^1(\mathbf{p}) + \cdots + \lambda_k(\mathbf{p})\, D_{\mathbf{p}} V^k(\mathbf{p}) \tag{1.1}$$

where the $\lambda_i(\mathbf{p})$ and the $V^i(\mathbf{p})$ are scalar functions (V^i being in general interpreted as an indirect utility function), and where $D_{\mathbf{p}} V^i(\mathbf{p})$ is the gradient of $V^i(\mathbf{p})$ at $\mathbf{p}$:

$$D_{\mathbf{p}} V^i(\mathbf{p}) = \left(\frac{\partial V^i}{\partial p_1}, \ldots, \frac{\partial V^i}{\partial p_n} \right)'.$$

Note that, depending on the context, these functions may have to fulfill specific, additional conditions, such as positiveness, monotonicity, (quasi)-convexity, and budget constraints.

The main purpose of this paper is to investigate what conditions on $\mathbf{X}$ make such a decomposition possible.

From a mathematical point of view, the structure of (1.1) is highly specific. In the first half of this century, Elie Cartan (1945) developed a set of concepts, usually referred to as exterior differential calculus (from now on EDC), that proved especially convenient to deal with problems of this type. Surprisingly enough, however, these tools have hardly ever been used in the field of economic theory. One obvious exception is the pioneering paper by Russell and Farris (1993), which shows that Gorman's rank theorem is a consequence of well-known results on Lie groups. More recently, Russell (1994) proposes a measure of "quasi-rationality" directly based upon EDC. However, these works only consider individual behavior.[2]

In this paper, we apply the tools of EDC to standard economic aggregation problems. Specifically, we describe in some detail how a very powerful theorem of EDC, due to Cartan and Kähler, can be used to address a range of issues related to the aggregation problem. To our knowledge, this result has not been used in economics thus far, although its scope potentially includes many important issues. To illustrate the latter claim, we consider the classical problem, initially raised by Sonnenschein (1993b), of the characterization of aggregate demand.[3] Two versions have been considered in the literature. The first version considers the excess demand of an exchange economy. Given some arbitrary, continuous function Z that satisfies homogeneity and Walras Law, is it possible to construct an economy (i.e., a set of preferences and initial endowments) for which Z is the aggregate excess demand? This problem was solved by Mantel

[2] A referee rightly points out that EDC is mentioned by Hurwicz, in the context of integration of consumer demand.

[3] Another application to "collective" household behavior a la Browning and Chiappori is provided in a companion paper (Chiappori and Ekeland (1998a)).

(1974) and Debreu (1974), and additional results were derived by Geanakoplos and Polemarchakis (1980). The second problem is similar, except that it considers market demand (instead of excess demands). Even in the local version put forth by Shafer and Sonnenschein (1982), this problem was still unresolved, except for the case of finite data sets (Andreu (1982)).

In what follows, we provide a formal proof of the general conjecture already formulated in Sonnenschein's paper. Namely, we establish that when the number of agents is at least equal to the number of goods, then any (smooth enough) function satisfying Walras Law can be locally seen as the aggregate market demand of some economy, even when the distribution of income is imposed a priori. Our approach is general—although, for the sake of brevity, we only consider the market demand case in the paper.

In addition, we provide three other original results. The first result is general, while the second two are specific to market demand.

- The first result states that the decomposition can be chosen such that it varies continuously with the initial, aggregate demand function. In other words, to any new demand that is "close" to the initial one, one can associate a decomposition "close" to the initial one. Although somewhat natural, this result turned out to be very difficult to prove using the Debreu-Mantel technique, whereas it is a simple consequence of our approach.
- For the second result, assume that the initial distribution of income, instead of being price-independent as in the original setting, is some *given* function of prices. Then, again, any smooth enough function can be decomposed as an aggregate demand, provided that the number of agents is at least equal to the number of commodities.
- Finally, for the third result, assume the income distribution is some *arbitrary* and a priori unspecified function of prices. When is it possible to decompose an arbitrary, smooth function as the aggregate demand of an economy where the (price-dependent) income distribution can be freely chosen? In this new setting, the number of agents required is considerably reduced. Technically, decomposition is possible even when the number of agents is half the number of goods.

The structure of the paper is as follows: In the next section, we quickly review the structure of the problems under consideration, and describe their formulation in terms of exterior differential calculus. Section 3 presents the main results, while the extensions are discussed in Section 4.

2. CHARACTERIZATION OF AGGREGATE EXCESS AND MARKET DEMAND

2.1. *A Statement of the Problem*

The problem of characterizing the structure of aggregate excess and market demand was initially raised in Sonnenschein's seminal paper (1993b). The excess demand problem can be stated as follows. Take some continuous mapping $\mathbf{Z}(\mathbf{p})$:

$\mathbb{R}^\ell \to \mathbb{R}^\ell$, such that (Walras Law)

$$\mathbf{p}\cdot\mathbf{Z}(\mathbf{p}) = 0. \tag{2.1}$$

Can we find n individual demand functions $\mathbf{z}^1(\mathbf{p}),\dots,\mathbf{z}^n(\mathbf{p})$ such that

$$\mathbf{X}(\mathbf{p}) = \mathbf{z}^1(\mathbf{p}) + \cdots + \mathbf{z}^n(\mathbf{p}) \tag{2.2}$$

where $\mathbf{z}^i(\mathbf{p}) = \mathbf{x}^i(\mathbf{p}) - \boldsymbol{\omega}^i$ and $\mathbf{x}^i(\mathbf{p})$ is a solution of

$$\begin{aligned} &V^i(\mathbf{p}) = \max U^i(\mathbf{x}^i) \\ &\mathbf{p}\cdot\mathbf{x}^i = \mathbf{p}\cdot\boldsymbol{\omega}^i, \\ &\mathbf{x}^i \geq 0, \end{aligned} \tag{2.3}$$

for some well-behaved utility function U^i and some positive initial endowments $\boldsymbol{\omega}^1,\dots,\boldsymbol{\omega}^n$?

The market demand problem is similar, except that the initial mapping $\mathbf{X}(\mathbf{p})$ should satisfy a different version of Walras Law, namely

$$\mathbf{p}\cdot\mathbf{X}(\mathbf{p}) = n \tag{2.4}$$

and (2.2) and (2.3) should respectively be replaced by

$$\mathbf{X}(\mathbf{p}) = \mathbf{x}^1(\mathbf{p}) + \cdots + \mathbf{x}^n(\mathbf{p}) \tag{2.5}$$

and

$$\begin{aligned} &V^i(\mathbf{p}) = \max U^i(\mathbf{x}^i), \\ &\mathbf{p}\cdot\mathbf{x}^i = 1, \\ &\mathbf{x}^i \geq 0. \end{aligned} \tag{2.6}$$

As is well known, the former question—the characterization of aggregate excess demand—has been solved by Mantel (1974) and Debreu (1974), whereas the so-called "market demand" problem is still unresolved. As a matter of fact, the techniques we shall now describe apply to both problems in basically the same way. This is partly due to the local nature of our approach. It has been known since Sonnenschein's original paper that, in contrast to the excess demand problem, when characterizing market demand one must account for complex nonnegativity restrictions. In particular, Sonnenschein exhibits a counterexample of a function $\mathbf{X}$ that cannot be globally decomposed as above because of these constraints. Recently, Brown and Matzkin (1996) have provided a precise characterization of the restrictions arising from nonnegativity constraints. However, the local version of the question remains open: is it possible, for any given $\mathbf{p} \gg 0$, to find individual demand functions $\mathbf{x}^1(\mathbf{p}),\dots,\mathbf{x}^n(\mathbf{p})$, defined in some neighborhood of $\mathbf{p}$, such that (2.2) and (2.6) are fulfilled in this neighborhood?

A result initially demonstrated by Sonnenschein (1973b) and then generalized by Diewert (1977) and Mantel (1977), states that for $n \geq \ell$ any continuous function satisfying Walras Law, when considered *at some given point* $\bar{\mathbf{p}}$ "looks like" aggregate market demand, in the following sense: It is possible to find individual demand functions $\mathbf{x}^1(\mathbf{p}),\dots,\mathbf{x}^n(\mathbf{p})$ such that

$$\mathbf{X}(\bar{\mathbf{p}}) = \sum_i \mathbf{x}^i(\bar{\mathbf{p}}), \quad \text{and} \quad D_{\mathbf{p}}\mathbf{X}(\bar{\mathbf{p}}) = \sum_i D_{\mathbf{p}}\mathbf{x}^i(\bar{\mathbf{p}}).$$

In their 1982 survey, Shafter and Sonnenschein ask whether it is possible to go beyond this result, and find the $\mathbf{x}^i(\mathbf{p})$ such that $\mathbf{X}(\mathbf{p})$ coincides with $\Sigma_i \mathbf{x}^i(\mathbf{p})$ in an open neighborhood of $\bar{\mathbf{p}}$. While Andreu (1982) has demonstrated this property for finite sets of price-income bundles, the continuous version had not yet been established. In what follows, we show that the answer to the question is yes, at least if we assume that the function $\mathbf{X}$ is analytic in such an open neighborhood of $\bar{\mathbf{p}}$ (which implies, in particular, that it is infinitely differentiable).

2.2. *The Basic Partial Differential Equations*

Excess demand: Both problems can actually be stated as partial differential equations. We start with excess demand. If V^i denotes consumer i's indirect utility, utility maximization implies $D_{\mathbf{p}}V^i(\mathbf{p}) = -\alpha_i \cdot \mathbf{z}^i(\mathbf{p})$, where α_i is the Lagrange multiplier. It follows that

$$(2.7) \qquad \mathbf{Z}(\mathbf{p}) = -\frac{1}{\alpha_1(\mathbf{p})} D_{\mathbf{p}}V^1(\mathbf{p}) - \cdots - \frac{1}{\alpha_n(\mathbf{p})} D_{\mathbf{p}}V^n(\mathbf{p})$$
$$= \lambda_1(\mathbf{p})\, D_{\mathbf{p}}V^1(\mathbf{p}) + \cdots + \lambda_n(\mathbf{p})\, D_{\mathbf{p}}V^n(\mathbf{p})$$

and $\mathbf{Z}(\mathbf{p})$ must be a linear combination of n gradients. In addition:

- the V^i are (quasi) convex;
- the λ_i are negative;
- furthermore, the budget constraint implies

$$(2.8) \qquad \mathbf{p} \cdot D_{\mathbf{p}}V^i(\mathbf{p}) = 0 \qquad \forall i.$$

The problem is thus to find, in a neighborhood of some given $\bar{\mathbf{p}}$, functions $\lambda_1,\dots,\lambda_n$ and $V^1,\dots,V^n$ satisfying (2.7) and the set of conditions (2.8).

Market demand: The statement of the market demand problem is similar. First, if V^i denotes consumer i's indirect utility, we know that utility maximization implies $D_{\mathbf{p}}V^i(\mathbf{p}) = -\alpha_i \cdot \mathbf{x}^i(\mathbf{p})$, where α_i is the Lagrange multiplier. It follows that

$$(2.9) \qquad \mathbf{X}(\mathbf{p}) = -\frac{1}{\alpha_1(\mathbf{p})} D_{\mathbf{p}}V^1(\mathbf{p}) - \cdots - \frac{1}{\alpha_n(\mathbf{p})} D_{\mathbf{p}}V^n(\mathbf{p})$$
$$= \lambda_1(\mathbf{p})\, D_{\mathbf{p}}V^1(\mathbf{p}) + \cdots + \lambda_n(\mathbf{p})\, D_{\mathbf{p}}V^n(\mathbf{p})$$

and $X(\mathbf{p})$ must be a linear combination of n gradients. In addition:

- the V^i are (quasi) convex and decreasing;
- the λ_i are negative;
- furthermore, the budget constraint implies

$$(2.10)\qquad \mathbf{p}\cdot D_{\mathbf{p}}V^i(\mathbf{p}) = 1/\lambda_i \qquad \forall i.$$

The problem is thus to find, in a neighborhood of some given $\bar{\mathbf{p}}$, functions $\lambda_1,\dots,\lambda_n$ and $V^1,\dots,V^n$ satisfying (2.9) and the set of conditions (2.10).

It should be noted that the two problems above are basically similar; in both cases, we have to solve a partial differential equation in a set of functions that satisfies specific constraints. The only difference lies in the fact that the constraints ((2.8) in one case, (2.10) in the other), although similar, are not identical. Surprisingly, this minor difference results in important discrepancies in the solution process. As we shall see, the market demand problem is much more difficult to solve than the excess demand one, even when the power of the EDC techniques is fully exploited. This may explain why the former remained unsolved for twenty five years whereas a solution to the latter was found within a few months. It must be emphasized that these differences are totally independent of the nonnegativity restrictions; they are related to the mathematical nature of the (local) problem.

2.3. *Mathematical Resolution: The Basic Strategy*

Let us now describe the general strategy used throughout the proof.

One approach might be to consider the basic partial differential equations on $\mathbb{R}^{\ell}$ directly, and try to solve them using some standard technique. However, this strategy does not work here. For example, in the case of market demand, the problem is to find (quasi convex) functions $V^1,\dots,V^n$ that solve

$$(2.11)\qquad \mathbf{X}(\mathbf{p}) = \sum_{i=1}^{n} \frac{D_{\mathbf{p}}V^i(\mathbf{p})}{\mathbf{p}\cdot D_{\mathbf{p}}V^i(\mathbf{p})}.$$

But this PDE does not belong to any usual class, and we cannot apply standard existence results. Thus, the problem must be reformulated.

The basic idea of the following proof is to enlarge the space under consideration, and to adopt a geometric viewpoint. Let us consider the space $E = \{\mathbf{p}, \lambda_1,\dots,\lambda_n, \boldsymbol{\Delta}^1,\dots,\boldsymbol{\Delta}^n\} = \mathbb{R}^{\ell+n+\ell n}$, where the vector $\boldsymbol{\Delta}^i$ will later be interpreted as the gradient $D_{\mathbf{p}}V^i$. Assume that the original problem is solved—say, $(\lambda_i(\mathbf{p}), \boldsymbol{\Delta}^i(\mathbf{p}))$. Then the equations $\lambda_i = \lambda_i(\mathbf{p})$ and $\boldsymbol{\Delta}^i = \boldsymbol{\Delta}^i(\mathbf{p})$ define a (ℓ-dimensional) manifold $\mathscr{S}$ in E (which is the graph of the map $\mathbf{p} \to (\lambda_i, \boldsymbol{\Delta}^i)$). It is clear from (2.9) and (2.10) that $\mathscr{S}$ is contained in the ℓn-dimensional manifold $\mathscr{M}$

defined by

$$\mathbf{X}(\mathbf{p}) = \sum_i \lambda_i \boldsymbol{\Delta}^i, \qquad (2.12)$$
$$\mathbf{p} \cdot \boldsymbol{\Delta}^i = 1/\lambda_i \qquad \forall i.$$

In addition, since each function $\boldsymbol{\Delta}^i(\mathbf{p})$ is the gradient of a quasi-convex function, it satisfies the cross-derivative restrictions:

$$(2.13) \qquad \forall i, j \qquad \frac{\partial \boldsymbol{\Delta}^i}{\partial p_j} = \frac{\partial \boldsymbol{\Delta}^j}{\partial p_i}$$

plus some positivity conditions.[4]

In summary:

• if $(\lambda_i(\mathbf{p}), \boldsymbol{\Delta}^i(\mathbf{p}))$ solves the original problem, then the manifold $\mathscr{S}$ it defines satisfies (2.12) and (2.13), plus the positivity constraints;

• conversely, if we can find functions $\lambda_i(\mathbf{p})$ and $\boldsymbol{\Delta}^i(\mathbf{p})$ satisfying (2.12) and (2.13) plus the positivity constraints, then each $\boldsymbol{\Delta}^i(\mathbf{p})$ is the gradient of some function V^i, and the $(V^1, \ldots, V^n)$ solve the original problem.

In other words, the mathematical problem can be stated as follows: *solve the system of partial differential equations* (2.13) *on the manifold $\mathscr{M}$ defined by* (2.12).

3. THE MAIN RESULT

3.1. *Mathematical Preliminaries*

As it turns out, one of the most important applications of EDC is precisely to provide existence theorems for partial differential equations on manifolds. Although we do not attempt to present EDC in detail,[5] we briefly indicate the main intuitions underlying the approach through a few examples.

Cauchy Theorem: Let us start from the simplest version of our problem, namely the Cauchy Theorem for ordinary differential equations. It states that, given a point $\bar{x} \in \mathbb{R}$ and a C^1 function f, defined from some neighborhood $\mathscr{U}$ of $(0, \bar{x})$ into $\mathbb{R}$, there exists some $\epsilon > 0$ and a C^1 function φ: $]-\epsilon, \epsilon[\to \mathscr{U}$ that solves

$$(3.1) \qquad \frac{d\varphi}{dt} = f(t, \varphi(t)) \qquad \forall t \in]-\epsilon, \epsilon[$$

with the initial condition

$$\varphi(0) = \bar{x}.$$

In particular, $d\varphi(0)/dt = f(0, \bar{x})$. If $f(0, \bar{x}) = 0$, the solution is trivial ($\varphi(t) = \bar{x}$ for all t); so we assume that $f(0, \bar{x})$ does not vanish.

[4] The latter will turn out to be manageable in our local approach, since if they hold at some point, they will hold in the neighborhood as well.

[5] See Bryant et al. (1991), Arnold (1978), Griffiths and Jensen (1987), or Chiappori and Ekeland (1996) for a pedagogical summary.

Clearly, a differential equation on $\mathbb{R}$ is the simplest case of a system of PDE on a manifold. Now, how do we solve this system locally around 0? Intuition suggests considering the *linearized* version of the differential equation. We thus look for some linear function $\tilde{\varphi} = \alpha t + \beta$, which solves the system obtained by replacing in the right-hand side the function $f(t, \varphi(t))$ with its value at $t = 0$, namely $f(0, \bar{x})$. Such a solution obviously exists, and is given by

$$\alpha = f(0, \bar{x}), \qquad \beta = \bar{x}.$$

Now, the question is whether the existence of a solution for the linearized version of the problem implies the existence of a solution for the initial, nonlinear equation. The Cauchy theorem essentially states that the answer is yes, under mild regularity conditions on f. The intuition is clear. Take some $\nu << \epsilon$, and construct a linear approximation to the solution as follows. First, take $\tilde{\varphi}(t) = f(0, \bar{x}) \cdot t + \bar{x}$ on $[0, \nu[$; then $\tilde{\varphi}(\nu) = f(0, \bar{x}) \cdot \nu + \bar{x}$ and from (3.1)

$$\frac{d\varphi}{dt}(\nu) = f(\nu, f(0, \bar{x}) \cdot \nu + \bar{x}).$$

Next, take $\tilde{\varphi}(t) = \tilde{\varphi}(\nu) + f(\nu, f(0, \bar{x}) \cdot \nu + \bar{x})(t - \nu)$ on $[\nu, 2\nu[$ (that is, a linear continuation with slope $d\varphi(\nu)/dt$). Finally, continue until the neighborhood consists of only $[\epsilon - \nu, \epsilon[$.

This piecewise linear function can be made arbitrarily close to the "true" solution when ν becomes arbitrarily small (this is where the mild regularity conditions are needed). This indicates why one may expect a link between existence problems in the linear and the nonlinear versions.

Can this result be generalized? I.e., can we expect that, for *any* system of PDE, the existence of a solution for its linearization guarantees the existence of a (local) solution to the general system? The Cartan-Kähler theorem essentially states that this is the case, but only under specific conditions. Indeed, the above statement *would certainly not be true in full generality*. To see why, consider the following two counterexamples.

Counterexample 1: We modify the previous example as follows: Take two functions f and g from $\mathbb{R}^2$ into $\mathbb{R}$, with $f(0, \bar{x}) = g(0, \bar{x}) = y \neq 0$ and consider the differential equation:

$$\frac{d\varphi}{dt} = f(t, \varphi(t)) \qquad \forall t \in]-\epsilon, \epsilon[,$$

$$\frac{d\varphi}{dt} = g(t, \varphi(t)) \qquad \forall t \in]-\epsilon, \epsilon[,$$

with the initial condition

$$\varphi(0) = \bar{x}.$$

Clearly, a solution does not exist in general (unless f and g coincide in some *open* neighborhood of $(0, \bar{x})$). However, if we linearize around $(0, \bar{x})$, the linear version does have a solution, namely:

$$\alpha = f(0, \bar{x}) = g(0, \bar{x}) = y, \qquad \beta = \bar{x}.$$

Hence the linearized problem has a solution, but the solution does not locally extend. The reason is that the equality $f(t,x)=g(t,x)$—which is necessary for the existence of a solution—holds at $(0,\bar{x})$, but not in the neighborhood; the technical translation in Cartan's language is that $(0,\bar{x})$ is not an *ordinary point*. What we need is a regularity condition that will exclude such pathological situations. Technically, this condition will state that the relevant equality holds at ordinary points. This requires a general definition of the concept of an ordinary point, which is quite easy in the case above, but may be more complex in general. The Cartan-Kähler theorem provides such a definition.

Counterexample 2: The first counterexample introduced a system of differential equations. We now introduce several variables. Take two functions f and g from $\mathbb{R}^2$ into $\mathbb{R}$, and consider the system:

$$\frac{\partial\varphi}{\partial t_1}=f(t_1,t_2),\qquad \frac{\partial\varphi}{\partial t_2}=g(t_1,t_2), \tag{3.2}$$

with some initial condition (say, $\varphi(0,0)=0$). As is well known, a solution cannot exist unless f and g satisfy

$$\frac{\partial f}{\partial t_2}=\frac{\partial g}{\partial t_1}.$$

However, the linear version (linearized around (0,0)) does have a solution, namely $\tilde{\varphi}=\alpha_1 t_1+\alpha_2 t_2$, with

$$\alpha_1=f(0,0),\qquad \alpha_2=g(0,0).$$

We still are in a case where the linearized problem has a solution, but the latter does not locally extend. The issue, here, is not whether the point is ordinary or not; all points are ordinary in this context. Rather, the problem comes from the fact that the system (3.2) is not "complete," in the following sense. If these two equations are satisfied for all (t_1,t_2) in some open neighborhood, then we can differentiate them; in particular, it must be the case that

$$\frac{\partial^2\varphi}{\partial t_1\,\partial t_2}=\frac{\partial f(t_1,t_2)}{\partial t_2},\qquad \frac{\partial^2\varphi}{\partial t_1\,\partial t_2}=\frac{\partial g(t_1,t_2)}{\partial t_1}. \tag{3.3}$$

These two equations are immediate *analytic* consequences of the previous ones. But, *algebraically*, they are independent. In particular, the linearization of the system consisting of (3.2) *and* (3.3) is different from that of (3.2) alone.

The system (3.2) is incomplete (or, in the language of EDC, "not closed") in the sense that it does not include some equations (such as (3.3)) that are algebraically different from, but analytically implied by the initial ones. A second condition for applying Cartan-Kähler is that the system must be closed in the previous sense.

3.2. *The Argument*

We now come to the resolution of the market demand problem. An important remark, first, is that the system is closed, in the sense of the previous section. Hence, the second condition needed to apply Cartan-Kähler is automatically fulfilled.

Following the approach alluded to above, the proof is in two steps.

• *Step One*: Solve the *linearized* problem (at some given point $\bar{\mathbf{p}}$). Specifically, choose the values of λ_i and $\boldsymbol{\Delta}^i = D_{\mathbf{p}} V^i$—say, $\bar{\lambda}_i$ and $\bar{\boldsymbol{\Delta}}^i$— at $\bar{\mathbf{p}}$ arbitrarily. In particular, choose $\bar{\lambda}_i < 0$, $\bar{\boldsymbol{\Delta}}^i << 0$ and $\bar{\boldsymbol{\Delta}} = (\bar{\boldsymbol{\Delta}}^1, \dots, \bar{\boldsymbol{\Delta}}^n)$ invertible; if these properties hold at $\bar{\mathbf{p}}$, they will hold by continuity in a neighborhood as well. Also, these values must satisfy the relations:

$$\sum_i \bar{\lambda}_i \boldsymbol{\Delta}^i = \mathbf{X}(\bar{\mathbf{p}})$$

and

$$\bar{\mathbf{p}} \cdot \bar{\boldsymbol{\Delta}}^i = 1/\bar{\lambda}_i \qquad \forall i.$$

Now, linearize λ_i and $\boldsymbol{\Delta}^i$ (as functions of $\mathbf{p}$) around $\bar{\mathbf{p}}$:

$$\frac{\partial \lambda_i}{\partial p_j} = N_i^j,$$

$$\frac{\partial \boldsymbol{\Delta}_k^i}{\partial p_j} = M_k^{i,j}.$$

Solving the linearized problem is equivalent to finding vectors $N^i = (N_i^j)$ and matrices $M^i = (M_k^{i,j})$ that satisfy the integration equations, i.e., (2.13), plus the equations expressing that λ_i and Δ^i remain on the manifold $\mathscr{M}$ (the latter are obtained by differentiating (2.12)). In addition, we want the V^i to be convex.

Formally, we need the following conditions:

—$\boldsymbol{\Delta}^i$ is the gradient of a convex function; this implies that

$$M^i \text{ is symmetric positive} \qquad (i = 1, \dots, n);$$

—"the point remains on the manifold," which leads to

$$D_{\mathbf{p}} \mathbf{X}(\bar{\mathbf{p}}) = \sum_i \left(\boldsymbol{\Delta}^i D_{\mathbf{p}}\, \lambda_i' + \lambda_i\, D_{\mathbf{p}}\, \boldsymbol{\Delta}^i \right) = \sum_i \left(\boldsymbol{\Delta}^i N_i' + \lambda_i M^i \right), \tag{3.4}$$

$$M^i \mathbf{p} + \boldsymbol{\Delta}^i = -\frac{1}{\lambda_i^2} N_i \Leftrightarrow N_i' = -\lambda_i^2 (\mathbf{p}' M^i + \boldsymbol{\Delta}^{i\prime}). \tag{3.5}$$

This is the set of linear equations that have to be solved in M^i and N_i.

• *Step Two*: The second, and more tricky step is to show that all points in $\mathscr{M}$ are *ordinary* in the sense of Cartan. This step is crucial in order to go from a solution to the linearized version at each point to a solution to the general, nonlinear problem. This move may not be possible otherwise, as illustrated by the counterexamples in the previous section. Formally, this requirement means that the co-dimension of the space of solutions has to be computed in two different ways and that the final results must agree.

Is it possible to find vectors N^i and matrices M^i that satisfy the previous conditions? The answer is yes. Three remarks are relevant at this point:

• The technique used in this proof applies not only to the aggregate demand problem, but also to other problems of the same type. In particular, the case of incomplete markets is considered in Chiappori and Ekeland (1998b).

• In the case of market demand, the existence of a solution to the linearized problem (Step One above) is in fact a consequence of known results in the literature, due to Sonnenschein (1973b), Diewert (1977), and Mantel (1977). These results, however, are not sufficient for the present purpose, because they do not allow us to check the codimension properties of Step Two. The proof we provide allows us to compute the required dimensions.

• It is important to understand that finding *one* particular solution of the linearized problem is not enough. What we need is a characterization of *all* possible solutions to the linearized problem, since we shall have to compute the dimensions of the corresponding spaces. This is of course more difficult than finding one solution. But it is a problem for which the whole apparatus of linear algebra can be used.

• Finally, and for the sake of completeness, it can be noted that in the language of EDC, the system (2.13) can be rewritten simply as

$$\sum_j d\boldsymbol{\Delta}^{ij} \wedge dp_j = 0 \qquad \forall i \tag{3.6}$$

(see Chiappori-Ekeland (1996) for a detailed explanation). We are looking for an n-dimensional integral manifold of the exterior differential system (3.6) on the manifold $\mathscr{M}$.

The proof then relies upon the Cartan–Kähler theorem.

THEOREM 3.1: *Consider some open set $\mathscr{U}$ in $\mathbb{R}^{\ell}\backslash\{0\}$ and some analytic mapping $\mathbf{X}$: $\mathscr{U} \to \mathbb{R}^n$ such that $\mathbf{p}\cdot\mathbf{X}(\mathbf{p}) = n$. For all $\bar{\mathbf{p}} \in \mathscr{U}$ and for all $(\bar{\mathbf{x}}_1,\dots,\bar{\mathbf{x}}_n) \in \mathbb{R}^{n^2}$ and $(\bar{\lambda}_1,\dots,\bar{\lambda}_n) \in \mathbb{R}^n$ that satisfy*

$$\bar{\mathbf{x}}_1 + \cdots + \bar{\mathbf{x}}_n = \mathbf{X}(\bar{\mathbf{p}}),$$

$$\forall i, \qquad \bar{\lambda} > 0,$$

there exist n functions $U^1,\dots,U^n$, where each U_i is defined in some convex neighborhood $\mathscr{U}_i$ of $\bar{\mathbf{x}}_i$ and is analytic and strictly concave, n mappings $(\mathbf{x}_1,\dots,\mathbf{x}_n)$

and n functions $(\lambda_1, \dots, \lambda_n)$, *all defined in some neighborhood* $\mathscr{V}$ *of* $\bar{\mathbf{p}}$ *and analytic in* $\mathscr{V}$, *such that, for all* $\mathbf{p} \in \mathscr{V}$:

$$\mathbf{p} \cdot \mathbf{x}_i(\mathbf{p}) = 1 \qquad (i = 1, \dots, n),$$

$$U_i(\mathbf{x}_i(\mathbf{p})) = \max\{U_i(\mathbf{x}) | \mathbf{x} \in \mathscr{U}_i, \mathbf{p} \cdot \mathbf{x} \leq 1\} \qquad (i = 1, \dots, n),$$

$$\frac{\partial U_i}{\partial x^j}(\mathbf{x}_i(\mathbf{p})) = \lambda_i(\mathbf{p}) p_j \qquad (i = 1, \dots, n;\ j = 1, \dots, n),$$

$$\sum_{i=1}^{n} \mathbf{x}_i(\mathbf{p}) = \mathbf{X}(\mathbf{p}),$$

$$\mathbf{x}_i(\bar{\mathbf{p}}) = \bar{\mathbf{x}}_i \qquad (i = 1, \dots, n),$$

$$\lambda_i(\bar{\mathbf{p}}) = \bar{\lambda}_i \qquad (i = 1, \dots, n).$$

PROOF: See Appendix A.

Note that both the individual demands and the Lagrange multipliers (i.e., each agent's marginal utility of income) can be freely chosen at $\bar{\mathbf{p}}$. In particular, nonnegativity constraints can be ignored, since one can choose individual demands to be strictly positive at $\bar{\mathbf{p}}$, and they will remain positive in a neighborhood.

3.3. *Continuity of the Decomposition*

A nice property of our approach relates to what can be called the "continuity" of the decomposition with respect to small perturbations of the initial function.

Let $\mathbf{X}(\mathbf{p})$ be some given, analytic function. From the previous theorem, we know that, in a neighborhood $\mathscr{U}$ of some given point $\bar{\mathbf{p}}$, $\mathbf{X}(\mathbf{p})$ can be decomposed as the sum of n individual market demands:

$$\mathbf{X}(\mathbf{p}) = \sum_{i=1}^{n} \mathbf{x}_i(\mathbf{p}).$$

Let us fix a particular decomposition of this kind, and let ε be a positive scalar. Is it possible to find some positive ε' with the following property: for any analytic function $\mathbf{Y}(\mathbf{p})$ such that

$$\| \mathbf{X}(\bar{\mathbf{p}}) - \mathbf{Y}(\bar{\mathbf{p}}) \| < \varepsilon',$$

it is possible to find, in some neighborhood $\mathscr{U} \subset \mathscr{U}$ of $\bar{\mathbf{p}}$, a decomposition of $\mathbf{Y}(\mathbf{p})$ as

$$\mathbf{Y}(\mathbf{p}) = \sum_{i=1}^{n} \mathbf{y}_i(\mathbf{p})$$

such that

- for all i, $\mathbf{y}_i(\mathbf{p})$ is an individual demand function, with $\mathbf{p} \cdot \mathbf{y}_i(\mathbf{p}) = 1$;
- $\|\mathbf{x}_i(\mathbf{p}) - \mathbf{y}_i(\mathbf{p})\| < \varepsilon$ for all $\mathbf{p} \in \mathscr{U}$?

In words, the decomposition described in the main theorem must be "robust" to small perturbations of the initial function, in the sense that any function "close to" the initial one can be decomposed into individual demands that are "close to" the initial ones.

The answer is yes, and is in fact an immediate consequence of the technique we adopt here. Remember that the value, at $\bar{\mathbf{p}}$, of individual demands can be chosen arbitrarily (provided they add up to $\mathbf{X}(\bar{\mathbf{p}})$). So we can choose the $\mathbf{y}_i(\bar{\mathbf{p}})$ such that

$$\mathbf{y}_i(\bar{\mathbf{p}}) = \mathbf{x}_i(\bar{\mathbf{p}}) - \frac{\mathbf{X}(\bar{\mathbf{p}}) - \mathbf{Y}(\bar{\mathbf{p}})}{n},$$

which implies that

$$\|\mathbf{y}_i(\bar{\mathbf{p}}) - \mathbf{x}_i(\bar{\mathbf{p}})\| < \frac{\varepsilon'}{n}. \tag{3.7}$$

Also, both the $\mathbf{x}_i(\mathbf{p})$ and the $\mathbf{y}_i(\mathbf{p})$ are analytic on $\mathscr{U}$ so that for any ε, one can always find some ε' such that (3.7) implies

$$\|\mathbf{x}_i(\mathbf{p}) - \mathbf{y}_i(\mathbf{p})\| < \varepsilon \qquad \text{for all } \mathbf{p} \in \mathscr{U}.$$

In fact, one can even do (slightly) better. Assume that, at $\bar{\mathbf{p}}$, $\mathbf{Y}$ is close to $\mathbf{X}$ in the C^1 sense; i.e., assume that

$$\left\| D_{\mathbf{p}}\mathbf{X}(\bar{\mathbf{p}}) - D_{\mathbf{p}}\mathbf{Y}(\bar{\mathbf{p}}) \right\| < \varepsilon'.$$

Now, consider the basic equations of the linearized version, i.e., (3.4) and (3.5). Replacing $\mathbf{X}$ by $\mathbf{Y}$ does not change the latter, while in the former the left-hand side is modified only by ε'. We know, then, that the solution chosen to construct the M^i matrices (i.e., the partials of $\mathbf{x}_i(\mathbf{p})$) can then be approximated to construct the partials of $\mathbf{y}_i(\mathbf{p})$. It follows that the $\mathbf{y}_i(\mathbf{p})$ can be chosen close to the $\mathbf{x}_i(\mathbf{p})$ in the C^1 sense as well; i.e., such that

$$\left\| D_{\mathbf{p}}\mathbf{x}_i(\mathbf{p}) - \mathbf{D}_{\mathbf{p}}\mathbf{y}_i(\mathbf{p}) \right\| < \varepsilon \qquad \text{for all } \mathbf{p} \in \mathscr{U}.$$

4. TWO EXTENSIONS

We now consider two extensions of our main result.

4.1. *Arbitrary Income Distributions*

In the previous section, Theorem 3.1 has been established under the assumption that each member's income was constant, and equal to 1:

$$\mathbf{p} \cdot \mathbf{x}_i(\mathbf{p}) = 1 \qquad (i = 1, \ldots, n).$$

A natural question to ask is whether it extends to more general income distributions. Hence, we consider the following, general version of the problem.

Take some arbitrary functions $\mu_1(\mathbf{p}),\dots,\mu_n(\mathbf{p})$ that satisfy $\mu_i(\bar{\mathbf{p}})>0$, $i=1,\dots,n$, and $\sum \mu_i(\mathbf{p})=n$ for all $\mathbf{p}$. Here, $\mu_i(\mathbf{p})$ is interpreted as i's nominal income, which is allowed to depend on $\mathbf{p}$ in an arbitrary (but given) way. Now, can we locally (around $\bar{\mathbf{p}}$) decompose some arbitrary, analytic function $\mathbf{X}(\mathbf{p})$ as the sum of n individual demands $\mathbf{x}_i(\mathbf{p})$, such that each $\mathbf{x}_i(\mathbf{p})$ is a solution to the program

$$\max U^i(\mathbf{x}^i),$$
$$\mathbf{p}\cdot\mathbf{x}=\mu_i(\mathbf{p}),$$

for some well-chosen utility functions?

Again, the answer is yes, at least when the distributions $\mu_i(\mathbf{p})$ are analytic, as stated in the following theorem:

THEOREM 4.1: *Consider some open set $\mathscr{U}$ in $\mathbb{R}^n-\{0\}$, some analytic functions $\mu_1(\mathbf{p}),\dots,\mu_n(\mathbf{p})$ that satisfy $\mu_i(\bar{\mathbf{p}})>0$, $i=1,\dots,n$, and $\sum\mu_i(\mathbf{p})=n$ for all $\mathbf{p}$, and some analytic mapping $\mathbf{X}$: $\mathscr{U}\to\mathbb{R}^n$ such that $\mathbf{p}\cdot\mathbf{X}(\mathbf{p})=1$. For all $\bar{\mathbf{p}}\in\mathscr{U}$ and for all $(\bar{\mathbf{x}}_1,\dots,\bar{\mathbf{x}}_n)\in\mathbb{R}^{n^2}$ and $(\bar{\lambda}_1,\dots,\bar{\lambda}_n)\in\mathbb{R}^n$ that satisfy*

$$\bar{\mathbf{x}}_1+\cdots+\bar{\mathbf{x}}_n=\mathbf{X}(\bar{\mathbf{p}}),$$
$$\forall i,\qquad \lambda_i>0,$$

there exist n functions $U^1,\dots,U^n$, where each U_i is defined in some convex neighborhood $\mathscr{U}_i$ of $\bar{\mathbf{x}}_i$ where it is analytic and strictly concave, n mappings $(\mathbf{x}_1,\dots,\mathbf{x}_n)$ and n functions $(\lambda_1,\dots,\lambda_n)$, all defined in some neighborhood $\mathscr{V}$ of $\bar{\mathbf{p}}$ and analytic in $\mathscr{V}$, such that, for all $\mathbf{p}\in\mathscr{V}$:

$$\mathbf{p}\cdot\mathbf{x}_i(\mathbf{p})=\mu_i(\mathbf{p})\qquad (i=1,\dots,n),$$
$$U_i(\mathbf{x}_i(\mathbf{p}))=\max\{U_i(\mathbf{x})|\mathbf{x}\in\mathscr{U}_i,\ \mathbf{p}\cdot\mathbf{x}\le\mu_i(\mathbf{p})\}\qquad (i=1,\dots,n),$$
$$\frac{\partial U_i}{\partial x^j}(\mathbf{x}_i(\mathbf{p}))=\lambda_i(\mathbf{p})p_j\qquad (i=1,\dots,n;\ j=1,\dots,n),$$
$$\sum_{i=1}^n\mathbf{x}_i(\mathbf{p})=\mathbf{X}(\mathbf{p}),$$
$$\mathbf{x}_i(\bar{\mathbf{p}})=\bar{\mathbf{x}}_i\qquad (i=1,\dots,n),$$
$$\lambda_i(\bar{\mathbf{p}})=\bar{\lambda}_i\qquad (i=1,\dots,n).$$

The proof is in Appendix B; in fact, it is exactly identical to the previous one, since the $\mu_i(\mathbf{p})$ only introduce minor changes into the basic argument. Note that this result applies to the case where $\mathbf{p}\cdot\mathbf{x}_i(\mathbf{p})=\mu_i$ for some fixed income distribution $\mu_1,\dots,\mu_n$ that satisfies $\sum\mu_i=n$; this is the way the problem is often stated in the literature.

4.2. *Unknown Income Distributions*

Finally, we consider a different version of the problem. In the previous section, we asked the following question: Given functions $(\mathbf{X}(\mathbf{p}), \mu_i(\mathbf{p}))$ representing market demand and individual incomes (as function of prices), do there exist individual demand functions $\mathbf{x}_i(\mathbf{p})$ such that

$$\mathbf{X}(\mathbf{p}) = \Sigma \mathbf{x}_i(\mathbf{p}),$$

$$\mathbf{p} \cdot \mathbf{x}_i(\mathbf{p}) = \mu_i(\mathbf{p}) \qquad \forall i, \mathbf{p}.$$

Now we ask: Given $\mathbf{X}(\mathbf{p})$, do there exist functions $\mathbf{x}_i(\mathbf{p})$ *and* $\mu_i(\mathbf{p})$ such that

$$\mathbf{X}(\mathbf{p}) = \Sigma \mathbf{x}_i(\mathbf{p}),$$

$$\mathbf{p} \cdot \mathbf{x}_i(\mathbf{p}) = \mu_i(\mathbf{p}) \qquad \forall i, \mathbf{p}.$$

The difference is that instead of being initially given, the $\mu_i(\mathbf{p})$ can be arbitrarily chosen to "match" the initial function $\mathbf{X}$. Economically, the interpretation is that there exists some price-dependent income distribution, about which we have no information.

How many individual agents are necessary to decompose the market demand $\mathbf{X}$ in this new setting? The answer to this question is a direct consequence of a recent result (Chiappori and Ekeland (1997)). It can be shown that, under this new formulation, *only* $(n-1)/2$ *consumers are necessary* to decompose an arbitrary, smooth aggregate market demand.

This result can in turn be related to another problem, namely the characterization of household demand (see Browning and Chiappori (1998), Chiappori and Ekeland (1998a). In this context, the emphasis is put on the properties of aggregate demand in "small" economies, where the internal distribution of income is not observed. The main result is that in an economy (a household) with k members, the Slutsky matrix must be the sum of a symmetric, negative matrix and a matrix of rank at most $(k-1)$. This condition does not generate additional restrictions upon the aggregate demand function unless $n \geq 2k+1$. Note, however, that the latter result is more general, since the model allows for public goods, externalities, etc. (see Chiappori and Ekeland (1998a) for a detailed presentation).

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APPENDIX

A. PROOF OF THEOREM 3.1

A.1. *The Linearized Problem*

We look for matrices M^i and vectors N_i such that

$$M^i \text{ is symmetric positive definite} \qquad (i=1,\dots,n),$$

and

$$D_{\mathbf{p}}\mathbf{X}(\bar{\mathbf{p}}) = \sum_i (\boldsymbol{\Delta}^i D_{\mathbf{p}}\lambda_i' + \lambda_i D_{\mathbf{p}}\boldsymbol{\Delta}^i) = \sum_i (\boldsymbol{\Delta}^i N_i' + \lambda_i M^i), \tag{A.1}$$

$$M^i\mathbf{p} + \boldsymbol{\Delta}^i = -\frac{1}{\lambda_i^2}N_i \Leftrightarrow N_i' = -\lambda_i^2(\mathbf{p}'M^i + \boldsymbol{\Delta}^{i\prime}) \tag{A.2}$$

(the last two equations reflecting the fact that "the point remains on the manifold").

- Substituting (A.2) into (A.1) gives

$$S + \sum_i \lambda_i^2\,\boldsymbol{\Delta}^i\boldsymbol{\Delta}^{i\prime} = \sum_i \bar{\lambda}_i M^i - \sum_i \lambda_i^2\,\boldsymbol{\Delta}^i\mathbf{p}'\,M^i \tag{A.3}$$

where $S = D_{\mathbf{p}}\mathbf{X}(\bar{\mathbf{p}})$ is given and is such that

$$\mathbf{p}'S = -\mathbf{X}',$$

which implies that

$$\mathbf{p}'\left(S + \sum_i \lambda_i^2\,\boldsymbol{\Delta}^i\boldsymbol{\Delta}^{i\prime}\right) = -\mathbf{X}' + \sum_i \lambda_i\boldsymbol{\Delta}^{i\prime} = 0.$$

We now concentrate upon the set of all solutions to (A.3). In our case, (A.3) should be considered as an equation on the set of symmetric, positive definite matrices. However, for reasons that will become clear in the computation of Cartan characters that follows, we shall also consider (A.3) as an equation on the set of all $(n\times n)$ matrices.

As a consequence, let us first study the two operators Φ and Φ_S defined respectively by:

$$\begin{aligned}&(\mathbb{R}^{n^2})^n = \{(M^1,\dots,M^n) \mid M^i \in \mathbb{R}^{n^2}\} \to \mathbb{A},\\ &\Phi(M^1,\dots,M^n) = \sum_i \lambda_i M^i - \sum_i \lambda_i^2\,\boldsymbol{\Delta}^i\mathbf{p}'\,M^i,\end{aligned} \tag{A.4}$$

and

$$\begin{aligned}&\mathbb{S}^n = \{(M^1,\dots,M^n) \mid M^i \in \mathbb{S}\} \to \mathbb{A},\\ &\Phi_S(M^1,\dots,M^n) = \sum_i \lambda_i M^i - \sum_i \lambda_i^2\,\boldsymbol{\Delta}^i\mathbf{p}'\,M^i\end{aligned} \tag{A.5}$$

where $\mathbb{S}$ is the set of $(n\times n)$ symmetric matrices, and $\mathbb{A}$ is the set of $(n\times n)$ matrices such that $\mathbf{p}'A=0$; note that Φ_S is simply the restriction of Φ to the space $\mathbb{S}^n$, which is obviously a subspace of $(\mathbb{R}^{n^2})^n$.

A.1.1. *Kernel of Φ and Φ_S*

We first characterize the kernels of Φ and Φ_S. This is done in the following Lemma:

LEMMA 1: *$(M^1,\dots,M^n)$ belong to $\ker\Phi$ (resp. to $\ker\Phi_S$) if and only if there exist a symmetric, $(n\times n)$ matrix $\beta = ((\beta_{k,s}))$ such that*

$$M^k\mathbf{p} = \sum_s \lambda_s^2\beta_{k,s}\boldsymbol{\Delta}^s, \tag{A.6}$$

$$\sum_k \lambda_k M^k = \sum_{k,s}\lambda_s^2\lambda_k^2\,\beta_{k,s}\boldsymbol{\Delta}^k(\boldsymbol{\Delta}^s)'. \tag{A.7}$$

PROOF: We prove the result for Φ_S (the proof for Φ is identical). The kernel is defined as

$$\ker \Phi_S = \left\{ (M^1,\dots,M^n) \in \mathbb{S}^n \,\middle|\, \sum_i \lambda_i M^i - \sum_i \lambda_i^2 \, \boldsymbol{\Delta}^i \mathbf{p}' M^i = 0 \right\}. \tag{A.8}$$

Transposing,

$$\sum_i \lambda_i M^i - \sum_i \lambda_i^2 M^i \mathbf{p} \, \boldsymbol{\Delta}^{i\prime} = 0.$$

Adding up,

$$\sum_i \lambda_i^2 [\, \boldsymbol{\Delta}^i \mathbf{p}' M^i - M^i \mathbf{p} \, \boldsymbol{\Delta}^{i\prime}] = 0.$$

Define $\mathbf{y}_i = \lambda_i^2 \boldsymbol{\Delta}^i$ and $\mathbf{z}^i = M^i \mathbf{p}$; then the previous relation becomes

$$\sum_i (\mathbf{y}_i \mathbf{z}^{i\prime} - \mathbf{z}^i \mathbf{y}_i') = 0.$$

By a celebrated Lemma due to Elie Cartan (see Bryant and al. (1991)), this is equivalent to the existence of some symmetric matrix β such that

$$\mathbf{z}^i = M^i \mathbf{p} = \sum_s \beta^{i,s} \mathbf{y}_s = \sum_s \lambda_s^2 \, \boldsymbol{\Delta}^s \beta^{i,s}.$$

Substituting in (A.8) leads to

$$\sum_k \lambda_k M^k = \sum_{k,s} \lambda_s^2 \lambda_k^2 \, \beta^{k,s} \boldsymbol{\Delta}^k (\boldsymbol{\Delta}^s)'.$$

These two relations fully characterize the kernel of Φ_S.

We must check that these two equations are compatible. We have that

$$\sum_k \lambda_k M^k \mathbf{p} = \sum_s \lambda_k \lambda_s^2 \, \beta^{k,s} \boldsymbol{\Delta}^s$$

from (A.6), and

$$\sum_k \lambda_k M^k \mathbf{p} = \sum_{k,s} \lambda_s^2 \lambda_k^2 \, \beta^{k,s} \boldsymbol{\Delta}^k (\boldsymbol{\Delta}^s)' \mathbf{p}$$

from (A.7). But since $(\boldsymbol{\Delta}^s)' \mathbf{p} = 1/\lambda_s$, and β is symmetric, these relations are equivalent.

A.1.2. *Dimension of* $\ker \Phi$ *and* $\ker \Phi_S$

Let L_1 denote the subspace generated by the right-hand sides of (A.6) and (A.7):

$$L_1 = \left\{ (\mathbf{w}_1,\dots,\mathbf{w}_n, W) \in \mathbb{R}^{n^2} \times \mathbb{S} \,\middle|\, \sum_s \lambda_s^2 \, \boldsymbol{\Delta}^s \beta_{i,s} = \mathbf{w}_i \text{ and } \sum_{k,s} \lambda_s^2 \lambda_k^2 \, \beta^{k,s} \boldsymbol{\Delta}^k (\boldsymbol{\Delta}^s)' = W \right\}.$$

It can be readily checked that the operator $\beta \to (\mathbf{w}_1,\dots,\mathbf{w}_n, W)$ is injective; it follows that

$$\dim L_1 = \frac{n(n+1)}{2}.$$

Then

$$\ker \Phi_S = \left\{ (M^1,\dots,M^n) \in \mathbb{S}^n \,\middle|\, \left(M^1 \mathbf{p},\dots, M^n \mathbf{p}, \sum_k \lambda_k M^k \right) \in L_1 \right\}.$$

Consider the operator G: $(M^1,\dots,M^n) \to (M^1\mathbf{p},\dots,M^n\mathbf{p},\Sigma_k \lambda_k M^k)$. Its image belongs to the subspace L_2 defined by

$$L_2 = \left\{(\mathbf{w}_1,\dots,\mathbf{w}_n, W) \in \mathbb{R}^{n^2} \times \mathbb{S} | W\mathbf{p} = \sum_k \lambda_k \mathbf{w}_k \right\}.$$

By the compatibility condition, just verified,

$$L_1 \cap L_2 = L_1.$$

Then

$$\begin{aligned} co \dim \ker \Phi_S &= \dim L_2 - \dim L_1 \\ &= \frac{n(n+1)}{2} + n^2 - n - \frac{n(n+1)}{2} \\ &= n^2 - n; \end{aligned}$$

hence

$$\dim \ker \Phi_S = \tfrac{1}{2} n^2 (n+1) - n(n-1)$$

and

$$\dim \operatorname{Im} \Phi_S = n(n-1).$$

But Φ_S maps $\mathbb{S}^n$ to $\mathbb{A}$, with $\dim \mathbb{A} = n^2 - n = n(n-1)$. Lemma 2 follows:

LEMMA 2: *Φ_S, considered as a linear mapping from $\mathbb{S}^n$ to $\mathbb{A}$, is onto.*

All these results can be summarized as follows:

LEMMA 3: *For any matrix S such that $\mathbf{p}'S = -\mathbf{X}'$, there exist n symmetric matrices $(M^1,\dots,M^n)$ such that*

$$S + \sum_i \lambda_i^2 \, \boldsymbol{\Delta}^i \boldsymbol{\Delta}^{i\prime} = \sum_i \bar{\lambda}_i M^i - \sum_i \lambda_i^2 \, \boldsymbol{\Delta}^i \mathbf{p}' M^i. \tag{A.9}$$

Moreover, the set of all such n-uples is an affine space of dimension $\frac{1}{2}n^2(n+1) - n(n-1)$.

The case for Φ is exactly similar, except for the initial space, which is $(\mathbb{R}^{n^2})^n$ (and hence of dimension n^3) instead of $\mathbb{S}^n$ (which is of dimension $\frac{1}{2}n^2(n+1)$). The corresponding result is as follows.

LEMMA 4: *For any matrix S such that $\mathbf{p}'S = -\mathbf{X}'$, there exist n matrices $(M^1,\dots,M^n)$ such that*

$$S + \sum_i \lambda_i^2 \, \boldsymbol{\Delta}^i \boldsymbol{\Delta}^{i\prime} = \sum_i \bar{\lambda}_i M^i - \sum_i \lambda_i^2 \, \boldsymbol{\Delta}^i \mathbf{p}' M^i. \tag{A.10}$$

Moreover, the set of all such n-uples is an affine space of dimension $n^3 - n(n-1)$.

These two results, and particularly the dimensions of the kernels, will be used in the computation of Cartan characters in the next subsection.

A.1.3. *Looking for a particular, positive solution*

The final step is to show that there exists a solution to the previous equation such that all matrices $(M^1, \dots, M^n)$ are *positive* definite. The key idea is to show the following Lemma:

LEMMA 5: *There exist n symmetric, positive definite matrices $Q^1, \dots, Q^n$ such that*

$$(Q^1, \dots, Q^n) \in \ker \Phi.$$

Assume this Lemma holds true, and let $(M^1, \dots, M^n)$ be any n-uple of symmetric matrices such that (A.10) is satisfied. Then for any positive scalar k, the n-uple $(M^1 + kQ^1, \dots, M^n + kQ^n)$ satisfies (A.10); moreover, for k large enough, these matrices are positive definite, which would complete the proof.

We now prove Lemma 5. We know from Lemma 1 that, to any n-uple of symmetric matrices $(Q^1, \dots, Q^n)$ in the kernel, one can associate a symmetric matrix β satisfying the relations (A.6) and (A.7). For the sake of simplicity, define P^k and $\gamma_{k,s}$ by

$$P^k = \lambda_k Q^k, \qquad \gamma_{k,s} = \lambda_s^2 \lambda_k^2 \beta^{k,s}.$$

The P^i must be negative (remember $\lambda_k < 0$), and satisfy the equations

$$P^k \mathbf{p} = \sum_s \frac{\gamma_{k,s}}{\lambda_k} \boldsymbol{\Delta}^s = \Delta \Gamma \frac{\mathbf{e}_k}{\lambda_k},$$

$$\sum_k P^k = \sum_{k,s} \gamma_{k,s} \boldsymbol{\Delta}^k (\boldsymbol{\Delta}^s)' = \Delta \Gamma \Delta',$$

where $\mathbf{e}'_k = (0, \dots, 1, \dots, 0)$ is the kth vector of the canonical basis, Γ denotes the matrix $(\gamma_{k,s})$, and Δ denotes the matrix $(\boldsymbol{\Delta}^1, \dots, \boldsymbol{\Delta}^n)$.

Using the fact that the matrix Δ is invertible, we may define Γ^k by:

$$P^k = \Delta \Gamma^k \Delta'.$$

Note that there is a one-to-one correspondence between the P^k and the Γ^k and that, in addition, P^k is symmetric (resp. negative definite) if and only if Γ^k is symmetric (resp. negative definite). Also, we have that $\mathbf{p} \cdot \boldsymbol{\Delta}^k = 1/\lambda_k$ for all k, which implies that $\Delta' \mathbf{p} = \Sigma_i (\mathbf{e}_i / \lambda_i)$. So we are looking for n matrices Γ^k that are symmetric, negative definite, and such that

$$\Gamma^k \left(\sum_i \frac{\mathbf{e}_i}{\lambda_i} \right) = \left(\sum_i \Gamma^i \right) \frac{\mathbf{e}_k}{\lambda_k}$$

(then $\Gamma = \Sigma_i \Gamma^i$).

We may suppose $\lambda_k = 1$ $\forall k$. Indeed, assume that the Γ^k are solutions of the previous problem with $\lambda_k = 1$; define $\bar{\Gamma}^k$ by $\bar{\gamma}^k_{i,j} = \lambda_i \lambda_j \gamma^k_{i,j}$; then the $\bar{\Gamma}^k$ are solutions of the initial problem. We are thus faced with the problem

$$\text{(A.11)} \qquad \Gamma^k \left(\sum_i e_i \right) = \left(\sum_i \Gamma^i \right) e_k$$

to be solved by symmetric, negative definite matrices Γ^k.

We can now exhibit a particular solution $\Gamma^{(1)} = (\Gamma^{(1)1}, \dots, \Gamma^{(1)n})$ by

$$\Gamma^{(1)1} = \begin{pmatrix} \gamma_1^1 & \gamma_2^1 & \gamma_3^1 & \cdots & \gamma_n^1 \\ \gamma_2^1 & \gamma_2^1 & \gamma_3^1 & \cdots & \gamma_n^1 \\ \gamma_3^1 & \gamma_3^1 & \gamma_3^1 & \cdots & \gamma_n^1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \gamma_n^1 & \gamma_n^1 & \gamma_n^1 & \cdots & \gamma_n^{11} \end{pmatrix},$$

$$\Gamma^{(1)2} = \begin{pmatrix} \gamma_2^1 & \gamma_2^1 & \gamma_3^1 & \cdots & \gamma_n^1 \\ \gamma_2^1 & \gamma_2^1 & \gamma_3^1 & \cdots & \gamma_n^1 \\ \gamma_3^1 & \gamma_3^1 & \gamma_3^1 & \cdots & \gamma_n^1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \gamma_n^1 & \gamma_n^1 & \gamma_n^1 & \cdots & \gamma_n^2 \end{pmatrix},$$

$$\cdots$$

$$\Gamma^{(1)n} = \begin{pmatrix} \gamma_n^1 & \gamma_n^1 & \gamma_n^1 & \cdots & \gamma_n^1 \\ \gamma_n^1 & \gamma_n^1 & \gamma_n^1 & \cdots & \gamma_n^1 \\ \gamma_n^1 & \gamma_n^1 & \gamma_n^1 & \cdots & \gamma_n^1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \gamma_n^1 & \gamma_n^1 & \gamma_n^1 & \cdots & \gamma_n^1 \end{pmatrix}.$$

These matrices are symmetric and satisfy (A.11); moreover, one can easily fix the coefficients such that they are all negative. The only problem left is that, among them, only $\Gamma^{(1)1}$ is definite. Thus the final step is to define other solutions using permutations of rows and columns; i.e., we define $\Gamma^{(i)} = (\Gamma^{(i)1}, \dots, \Gamma^{(i)n})$ by

$$\Gamma^{(i)1} = \begin{pmatrix} \gamma_1^i & \cdots & \gamma_1^i & \gamma_{i+1}^i & \cdots & \gamma_n^i \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \gamma_1^i & \cdots & \gamma_1^i & \gamma_{i+1}^i & \cdots & \gamma_n^i \\ \gamma_{i+1}^i & \cdots & \gamma_{i+1}^i & \gamma_{i+1}^i & \cdots & \gamma_n^i \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \gamma_n^i & \cdots & \gamma_n^i & \gamma_n^i & \cdots & \gamma_n^i \end{pmatrix},$$

$$\Gamma^{(i)i} = \begin{pmatrix} \gamma_i^i & \gamma_i^i & \cdots & \gamma_i^i & \gamma_{i+1}^i & \cdots & \gamma_n^i \\ \gamma_i^i & \gamma_{i-1}^i & \cdots & \gamma_{i-1}^i & \gamma_{i+1}^i & \cdots & \gamma_n^i \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \gamma_i^i & \gamma_{i-1}^i & \cdots & \gamma_1^i & \gamma_{i+1}^i & \cdots & \gamma_n^i \\ \gamma_{i+1}^i & \gamma_{i+1}^i & \cdots & \gamma_{i+1}^i & \gamma_{i+1}^i & \cdots & \gamma_n^i \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \gamma_n^i & \gamma_n^i & \cdots & \gamma_n^i & \gamma_n^i & \cdots & \gamma_n^i \end{pmatrix},$$

$$\Gamma^{(i)n} = \begin{pmatrix} \gamma_n^i & \gamma_n^i & \gamma_n^i & \cdots & \gamma_n^i \\ \gamma_n^i & \gamma_n^i & \gamma_n^i & \cdots & \gamma_n^i \\ \gamma_n^i & \gamma_n^i & \gamma_n^i & \cdots & \gamma_n^i \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \gamma_n^i & \gamma_n^i & \gamma_n^i & \cdots & \gamma_n^i \end{pmatrix}.$$

These matrices have the same properties as the $\Gamma^{(1)k}$, except that only $\Gamma^{(i)i}$ is definite here. Finally, take the Γ^k defined by

$$\Gamma^k = \sum_i a_i \Gamma^{(i)k}, \qquad a_i > 0.$$

These matrices are symmetric, negative definite, and solve the equation above. *Q.E.D.*

A.2. *Computing the Cartan Characters*

We now proceed to the second step of our proof. Remember that we are looking for an n-dimensional integral manifold of the exterior differential system:

$$\sum_j d\boldsymbol{\Delta}^{ij} \wedge dp_j = 0 \qquad \forall i \leq n,$$

satisfying

$$dp_1 \wedge \cdots \wedge dp_n \neq 0 \tag{A.12}$$

in the n^2-dimensional manifold M.

Our first task is to compute the codimension of G^n (see subsection 4.5). To do this, take $x = (\mathbf{p}, \boldsymbol{\lambda}, \boldsymbol{\Delta}) \in M$. Any n-dimensional subspace E of the tangent space $T_x M$ satisfying (A.12) is defined by the set of equations:

$$d\lambda_i = \sum_j N_i^j \, dp_j,$$

$$d\Delta_k^i = \sum_j M_k^{ij} \, dp_j,$$

where the N_i^j and the M_k^{ij} satisfy (A.1) and (A.2). The mapping $E \to (N_i^j, M_k^{ij})$ is well-defined and one-to-one, so that the (N_i^j, M_k^{ij}) provide a local coordinate system for the set of n-dimensional subspaces of $T_x M$. Also, Lemma 4 above shows that the set of (N_i^j, M_k^{ij}) that satisfy (A.1) and (A.2) is a subspace of dimension $n^3 - n(n-1)$. This is precisely the dimension of $\mathbb{R}^n \times \mathbb{P}^n(\mathbb{R}^n)$, as expected.

We now have to find the codimension of G^n in $\mathbb{R}^n \times \mathbb{P}^n(\mathbb{R}^n)$. In the (N_i^j, M_k^{ij}) coordinate system, G^n is defined by (A.1) and (A.2) plus the additional equations $M_k^{ij} = M_k^{ji}$. Lemma 3 above shows that the set of solutions to these has dimension $\frac{1}{2}n^2(n+1) - n(n-1)$. The codimension of G^n is thus

$$\begin{aligned} c &= [n^3 - n(n-1)] - [\tfrac{1}{2}n^2(n+1) - n(n-1)] \\ &= \tfrac{1}{2}n^2(n-1). \end{aligned}$$

Now, fix $\bar{x} = (\bar{\mathbf{p}}, \bar{\boldsymbol{\lambda}}, \boldsymbol{\Delta}) \in M$, with $\bar{\boldsymbol{\lambda}} \ll \mathbf{0}$. From the previous subsection, we know that we can find N_i^j and M_k^{ij} such that the matrices $\overline{M}_k$ are symmetric and positive definite. Let $\overline{E}$ be the corresponding integral element; we claim that it is ordinary. Indeed, consider the 1-forms given by

$$\pi_k^i = d\Delta_k^i - \sum_j \overline{M}_k^{ij} \, dp_j,$$

$$\pi^i = \sum_j \overline{N}_i^j \, dp_j,$$

so that $\overline{E}$ is defined by $\pi_k^i = 0$, $\pi^i = 0$. We have

$$\begin{aligned} \sum_i d\Delta_k^i \wedge dp_i &= \sum_i \left(\pi_k^i - \sum_j \overline{M}_j^{ij} \, dp_j\right) \wedge dp_i \\ &= \sum_i \pi_k^i \end{aligned}$$

since $\overline{M}_k$ is symmetric. By the criterion described in Section 4.5, we have that, for $0 \leq \lambda \leq n$,

$$H_\ell^* = Span\{\pi_k^i | k \leq n,\ i \leq \ell\}.$$

Therefore $c_\ell = \ell n$ for $\ell \leq n$, and

$$\begin{aligned} c_0 + \cdots + c_{n-1} &= n(0 + 1 + \cdots + n - 1) \\ &= n^2 \frac{n-1}{2}. \end{aligned}$$

This coincides with the codimension of G^n. So $(\bar{x}, \overline{E})$ is ordinary and we can apply the Cartan–Kähler Theorem.

B. PROOF OF THEOREM 4.1

This proof obtains by slightly modifying the previous proof. Indeed, the problem at $\bar{\mathbf{p}}$ becomes

$$\mathbf{X}(\bar{\mathbf{p}}) = \sum_i \lambda_i \boldsymbol{\Delta}^i,$$

$$\bar{\mathbf{p}}' \cdot \boldsymbol{\Delta}^i = \frac{\mu_i(\bar{\mathbf{p}})}{\lambda_i} \qquad \forall i;$$

hence equation (A.1) is unchanged, while (A.2) becomes

$$\text{(B.1)} \qquad M^i \mathbf{p} + \boldsymbol{\Delta}^i = -\frac{\mu_i(\bar{\mathbf{p}})}{\lambda_i^2} N_i + \frac{D_{\mathbf{p}} \mu_i}{\lambda_i} \Leftrightarrow N_i' = -\frac{\lambda_i^2}{\mu_i(\bar{\mathbf{p}})} (\mathbf{p}' M^i + \boldsymbol{\Delta}^{i\prime}) + \frac{\lambda_i}{\mu_i(\bar{\mathbf{p}})} (D_{\mathbf{p}} \mu_i)'.$$

Substituting (B.1) into (A.1) yields

$$\text{(B.2)} \qquad S + \sum_i \frac{\lambda_i^2}{\mu_i(\bar{\mathbf{p}})} \boldsymbol{\Delta}^i \boldsymbol{\Delta}^{i\prime} - \sum_i \frac{\lambda_i}{\mu_i(\bar{\mathbf{p}})} \boldsymbol{\Delta}^i (D_{\mathbf{p}} \mu_i)' = \sum_i \bar{\lambda}_i M^i - \sum_i \frac{\lambda_i^2}{\mu_i(\bar{\mathbf{p}})} \boldsymbol{\Delta}^i \mathbf{p}' M^i.$$

Note that

$$\mathbf{p}' \sum_i \frac{\lambda_i}{\mu_i(\bar{\mathbf{p}})} \boldsymbol{\Delta}^i (D_{\mathbf{p}} \mu_i)' = \sum_i (D_{\mathbf{p}} \mu_i)' = 0 \qquad \text{since} \quad \sum_i \mu_i = n,$$

so that we still have

$$\mathbf{p}' \left(S + \sum_i \lambda_i^2 \boldsymbol{\Delta}^i \boldsymbol{\Delta}^{i\prime} - \sum_i \frac{\lambda_i}{\mu_i(\bar{\mathbf{p}})} \boldsymbol{\Delta}^i (D_{\mathbf{p}} \mu_i)' \right) = 0.$$

Now, the proof does not depend on the left-hand side of (B.2). As for the right-hand side, we may define:

$$\tilde{\boldsymbol{\Delta}}^i = \frac{\boldsymbol{\Delta}^i}{\mu_i(\bar{\mathbf{p}})}$$

so that

$$\bar{\mathbf{p}}' \cdot \tilde{\boldsymbol{\Delta}}^i = \frac{1}{\lambda_i} \qquad \forall i.$$

Then the right-hand side of (B.2) becomes

$$\sum_i \bar{\lambda}_i M^i - \sum_i \lambda_i^2 \tilde{\boldsymbol{\Delta}}^i \mathbf{p}' M^i$$

and the previous proof applies exactly.

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