ERRATUM FOR "CONVEXITY METHODS IN HAMILTONIAN MECHANICS"

p. 2

Formula (7) should be:

$$\log A = (2i\pi)^{-1} \int (zI - A)^{-1} \log z \, dz$$

p. 9

Third line from the bottom: read "Proposition 4" instead of "Proposition 2"

p. 17

Formula (14) should read:

$$R(t_n) x_n = e^{i\theta_n} x_n, \quad (Gx_n, x_n) = 1$$

p. 19-20-21

The proof of Proposition 4 (and probably the proposition itself) is wrong: in formula (25), the third equality does not hold, because $R(t)\xi_k \neq \lambda_k \xi_k$ in general. This was first pointed out to me by John Toland. As a result, pages 19 (starting from the third paragraph), 20 and 21 have to be replaced by the following:

To have a more complete picture, we now turn to Krein-indefinite eigenvalues.

Consider again the system (1), (2). Denote by D the set of $t \geq 0$ such that $R(t)$ has at least one Krein-indefinite eigenvalue λ on the unit circle $\mathcal{U}$.

If $t \in D$, then $R(t)$ must have a G-isotropic λ-eigenvector. Indeed, if λ is not semi-simple, apply Proposition 2.7. If λ is semi-simple, the eigenspace $\ker(R(t) - \lambda I)$ coincides with the invariant subspace $\ker(R(t) - \lambda I)^m$, on which the Hermitian form G is assumed to be indefinite, and which therefore contains an isotropic vector.

Denote by D_m the set of all $t \in D$ such that all Krein-indefinite eigenvalues of $R(t)$ have multiplicity at most m, one of them having exactly multiplicity m. Note that $2 \leq m \leq 2n$, that the D_m partition D:

$$D = \cup D_m, \quad p \neq q \Rightarrow D_p \cap D_q = \emptyset$$

and that the D_m are not closed in general: if $t_i \in D_m$ and $t_i \to t$, then $t \in D_{m'}$ for some $m' \geq m$

Proposition. *D_m is a discrete set: every point in D_m is isolated*

Proof. Assume otherwise. Then there is some point $t \in D_m$ and some sequence $t_k \in D_m$ with $t_k \to t$ and $t_k \neq t$ for every k. By the definition of D_m we find sequences $\lambda_k \in \mathcal{U}$ and $x_k \in \mathbb{C}^{2n}$ such that:

$$\begin{aligned} R(t_k)\, x_k &= \lambda_k x_k \\ ||x_k\| = 1 \qquad &\text{and} \qquad (Gx_k, x_k) = 0 \end{aligned}$$

each λ_k being a root of the characteristic polynomial:

$$P(t_k; X) = \det(R(t_k) - XI)$$

with multiplicity m, all other roots having multiplity $\leq m$. In other words, λ_k is a simple root of the m-th derivative:

$$P^{(m)}(t_k; \lambda_k) = 0$$

By compactness, after extracting a suitable subsequence, we find $\lambda \in \mathcal{U}$ and x with $\|x\| = 1$ and:

$$\begin{aligned} \lambda_k &\to \lambda \\ x_k &\to x \\ (Gx, x) &= 0 \\ R(t)\, x &= \lambda x \\ P^{(m)}(t_.; \lambda) &= 0 \end{aligned}$$

By assumption, $t \in D_m$,so the multiplicity of λ is exactly m, that is, λ is a simple root of $P^{(m)}(t_.; X)$. By the implicit function theorem, there exists an $\epsilon > 0$ and an $\eta > 0$ such that, for $|s - t| < \epsilon$, the polynomial $P^{(m)}(s_.; X)$ has a unique (and simple) root $\varphi(s)$ satisfying $|\varphi(s) - \lambda| < \eta$, the function φ being smooth. Hence:

$$\varphi(t_k) = \lambda_k \quad \text{and} \quad \varphi(t) = \lambda$$

$$\lim_{k\to\infty} \frac{\lambda_k - \lambda}{t_k - t} = \varphi'(t) \in \mathbb{C}$$

We now remember that $R(t)$ is G -unitary. Therefore:

$$\begin{aligned} ((R(t_k) - R(t))\, x_k, Gx) &= (R(t_k)\, x_k, Gx) - (R(t)\, x_k, Gx) \\ &= \lambda_k (x_k, Gx) - \left(x_k, R(t)^\star Gx\right) \\ &= \lambda_k (x_k, Gx) - \left(x_k, GR(t)^{-1} x\right) \\ &= \lambda_k (x_k, Gx) - \left(x_k, \overline{\lambda} Gx\right) \\ &= (\lambda_k - \lambda)(x_k, Gx) \end{aligned}$$

Divide both sides by $(t_k - t)$ and let $k \to \infty$. We get:

$$(JA(t)\, R(t)\, x, Gx) = \varphi'(t)\,(x, Gx)$$

The right-hand side vanishes since x is G-isotropic. As for the left-hand side, replacing G by $-iJ$, we get:

$$\begin{aligned}(JA(t)R(t)x, Gx) &= (JA(t)R(t)x, -iJx)\\ &= i\lambda(A(t)x, x)\end{aligned}$$

which cannot vanish since $A(t)$ is positive definite. This is a contradiction and proves the proposition. □

Corollary. *D is a closed set with empty interior*

Proof. D is the set of $t \geq 0$ such that $R(t)$ has an eigenvalue $\lambda \in \mathcal{U}$ with some g-isotropic λ-eigenvector, so it has to be closed. On the other hand, $D = \cup D_m$. Since D is closed, $D = \cup \overline{D_m}$. By the preceding proposition, each $\overline{D_m}$ has empty interior, and by Baire's Theorem, D itself has empty interior. □

It t_0 is an isolated point in D, we can describe precisely the behaviour of the Krein-, indefinite eigenvalues: they immediately split up into Krein-definite eigenvalues, and eigenvalues which leave the unit circle:

Corollary. *Let t_0 be an isolated point in D, and let $\lambda \in \mathcal{U}$ be an eigenvalue of $R(t_0)$ with Krein type (p_0, q_0) [The rest as in Corollary 5 p. 20]*

Proof. Since t_0 is an isolated point in D, there is some open interval $\mathcal{N}$ around t_0 such that, for $t \in \mathcal{N}$ and $t \neq t_0$, $R(t)$ has only Krein-definite eigenvalues on $\mathcal{U}$. [The rest as in the proof of Corollary 5 p. 20] □

In other words, Krein-positive and Krein-negative eigenvalues leave the unit circle in pairs, each one cancelling the other, while the remaining ones continue their motion on $\mathcal{U}$ in the direction prescribed by their Krein sign, positive for positive ones and negative for the negative ones. A Krein-indefinite eigenvalue is a place wher a Krein-positive eigenvalue collides with a Krein-negative one. We formalize this idea by a definition.

Let t_0 be a (possibly non-isolated) point in D, and let $\lambda \in \mathcal{U}$ be an eigenvalue of $R(t_0)$ with Krein type (p_0, q_0). Choose some neighbourhood $\mathcal{N}$ of λ in $\mathbb{C}$ (not $\mathcal{U}$) and some $\epsilon > 0$ such that, whenever $|t - t_0| < \epsilon$, the $R(t)$ have the same number of eigenvalues in $\mathcal{N}$ (counted with mutliplicity), and they all converge to λ when $t \to t_0$.

By the first corollary, there exists a sequence $t_k \to t_0$, with $t_k < t_0$, such that the eigenvalues of $R(t_k)$ in $\mathcal{N} \cap \mathcal{U}$ are all Krein-definite. Inspecting the negative side of λ in $\mathcal{N} \cap \mathcal{U}$, we find p_k Krein-positive eigenvalues and q_k Krein-negative ones. The number:

$$p_0^- = p_k - q_k$$

is non-negative and independent of k. To see this, use Corollary 3: as s increases from t_k to t_{k+1}, the Krein-negative eigenvalues move away from λ on $\mathcal{N} \cap \mathcal{U}$, and can be forced away from $\mathcal{U}$ and back towards λ only by colliding with Krein-positive eigenvalues. In other words, in $\mathcal{N} \cap \mathcal{U}$, positive and negative eigenvalues are created or annihilated in pairs, so the difference $p_k - q_k$ is constant, and it has to be non-negative, otherwise there would be one negative eigenvalue in excess, which would eventually move away from λ.

Similarly, inspecting the positive side of λ in $\mathcal{N}\cap\mathcal{U}$, we find p'_k Krein-positive eigenvalues and q'_k Krein-negative ones, and we define:

$$q_0^- = q'_k - p'_k$$

which again is non-negative and independent of k.

Using sequences $t_k \to t_0$ with $t_k > t_0$, we define p_0^+ (on the positive side of λ) and q_0^+ (on the negative side). Arguing as in the second corollary, one proves that:

$$p_0^+ - q_0^+ = p_0 - q_0 = p_0^- - q_0^-$$

Definition. Set:

$$\begin{aligned} r_0^- &= p_0 - p_0^- = q_0 - q_0^- \\ r_0^+ &= p_0 - p_0^+ = q_0 - q_0^+ \end{aligned}$$

We refer to $2r_0^-$ as the number of eigenvalues which arrive on the unit circle at λ and to $2r_0^+$ as the number of eigenvalues which leave the unit circle at λ.

The rest as in the book, from p. 21, line 3 from the bottom. Note that Proposition 5.11 holds without changes.

p. 23

Formula (42) should read:

$$\lambda\left(A\left(t_0\right)\xi,\xi\right) = 0$$

p. 25

Formula (18) should read:

$$\begin{aligned} \|\Pi_s u\|^2 &= \sum_{n\neq 0} \frac{s^2}{4n^2\pi^2}|u_n|^2 \\ &\leq \sum_{n\neq 0} \frac{s^2}{4\pi^2}|u_n|^2 = \frac{s^2}{4\pi^2}\|u\|^2 \end{aligned}$$

p. 35

Middle of the page: read "Proposition 4.2" instead of "Proposition 3.2"

p. 41

Formula (58): the second line should be:

$$= \frac{1}{2}\int_0^T \left[\left(J\dot{x}, \int_0^t J\dot{x}\left(s\right)ds\right) + \left(B\left(t\right)J\dot{x}, J\dot{x}\right)\right]dt$$

p.50

(a) Formulas (126) and (127) should read

$$\begin{aligned} j_T\left(e^{i0}\right) &= j_T(1) + \frac{m_0}{2} + n \\ j_T\left(e^{-i0}\right) &= j_T(1) + \frac{m_0}{2} + n \end{aligned}$$

(b) In the proof, read Proposition 13 instead of Proposition 11

p. 56

(a) Corollary 6. In the statement, add the condition $\omega \neq 1$. In the proof, replace Corollary 5.15 by Corollary 5.14.

(b) Insert a new corollary, for which I am indebted to Salem Mathlouti

Corollary. *Denote by $m \geq 2$ the multiplicity of 1 as a Floquet multiplier. Then, for any $\omega_0 \in \mathcal{U}$ wit h $\omega_0 \neq 1$, we have:*

$$j(\omega_0) \geq \frac{m}{2}$$

Proof. We have:

$$j\left(\omega_0 e^{-i0}\right) \leq j(\omega_0) + p_0 \leq j(\omega_0) + m_0$$

On the other hand, denoting by (p_k, q_k) the Krein types of all the Floquet multipliers lying between ω_0 and 1, we have:

$$n \leq j\left(\omega_0 e^{-i0}\right) + \sum p_k - \sum q_k$$

It follows that:

$$\sum p_k - \sum q_k \geq n - j\left(\omega_0 e^{-i0}\right) \geq n - m_0 - j(\omega_0)$$

Counting the eigenvalues, we find:

$$\sum p_k - \sum q_k \leq n - m_0 - \frac{m}{2}$$

and the result follows by comparing the two last inequalities. □

p. 58-59-60

Formulas (31), (32), (33), (37), (38), (40), (47), replace $\sum$ by Σ

p. 62

Middle of the page, replace $\ker(A(t/\theta - I)$ by $\ker(R_\theta(t/\theta) - I)$

p. 63

In formulas (65), (66) and (67), replace $\sum$ by Σ

p.72

In Proposition 6, add two new formulas after (64) and (65):

- for $1<\beta\leq 2$:

$$i_k = i_1 + (k-1)(i_1+n+1) = (i_1+n+1)k - (n+1)$$

- for $\alpha > 2$:

$$i_k = i_1 + (k-1)(i_1+n) = (i_1+n)k - n$$

p.75

Line before formula (5): remove ¨unique¨

p. 82

In formula (8) for F, and in the formula for $F^{\star\star}$ in the middle of the page, replace $\sum_{x^\star\in X^\star}$ by $\sup_{x^\star\in X^\star}$

p. 90

Formula (37) should read:

$$\partial(G\circ A)(x) = A^\star \partial G(Ax) \;\; \forall x\in X$$

p. 105

(a) Proof of Proposition 5, second line, read "Theorem 3.2 and Corollary 3.3"

(b) *Replace formula (62) and the preceding line by*:

We now wish to apply Theorem 2. Introduce the space X of all $x\in L^\alpha$such that $\dot{x}\in L^\beta$and $x(T)=Mx(0)$, and consider the functional Ψ on X defined by:

$$\begin{aligned}\Psi(x) &= \frac{1}{2}\langle Ax,x\rangle + \mathcal{H}^\star(Ax)\\ &= \int_0^T\left[\frac{1}{2}(J\dot{x},x)+H^\star(t,-J\dot{x})\right]dt\end{aligned}$$

p. 106

(a) First line after formula (67): $\overline{q}(t) := (q_1-q_0)t/T + q_0$

(b) Formula (73) should read:

$$\begin{aligned}\Psi(p,q): &= \int_0^T\left[-p\dot{q}+p\overline{q}+H^\star\left(\dot{q}+\frac{d\overline{q}}{dt},-\dot{p}\right)\right]dt\\ &= \int_0^T\left[-p\left(\dot{q}+\frac{d\overline{q}}{dt}\right)+H^\star\left(\dot{q}+\frac{d\overline{q}}{dt},-\dot{p}\right)\right]dt + q_1p(T)-q_0p(0)\end{aligned}$$

p. 112

The action functional on the middle of the page should read:

$$\Phi(x) = \int_0^T \left[\frac{1}{2} (J\dot{x} + A_\infty(t)\, x, x) + N(t, x) \right] dt$$

p.115

Formula (33) should read:

$$x := \Lambda_0^{-1} u + x_0$$

p. 134

After formula (14), insert "with $\overline{q}(t) := tT^{-1}(q_1 - q_0)$

p. 136

Formula (4) should read:

$$\forall x \in X, \quad \Phi(x) \geq \Phi(y) - \epsilon d(x, y)$$

p. 149

Formula (7) should read:

$$\begin{aligned} \dot{x} &= JH'(t, x) \\ x(0) &= x(T) \end{aligned}$$

p. 154

(a) The last term in formula (50) is $c_1 \|w\|_\beta$

(b) There should be $\geq$ instead of $\leq$ in the second line

p. 156

The last term in the unnumbered formula between (69) and (70) is (w_n, ϵ_n)

p. 158

Fourth paragraph, line 4: read $\beta < 2$ instead of $\beta > 2$

p. 174

(a) Formula (28) should read $u_k + he_k \in \mathcal{P}_0$

(b) The two lines between formulas (34) and (35) should be: ¨Since u_k is $kT/2$-periodic, the phase shift does not affect the value of the integral

p. 187

Introduction, line before last: "and refer the"

p. 221

Line after formula (47) should read: "It is *essential* if it i-essential for infinitely many $i \geq 1$.

p. 222

Second line after Lemma 8, replace $c_{(n_1)p}$ with $c_{(n+1)p}$

p. 223

Formula (63) should read:

$$\gamma_\alpha^- (\Sigma) = \liminf_{i \to \infty} \left[(-c_i)^{\frac{2-\alpha}{\alpha}} i \right]^{-1}$$

p. 225

(a) Formula (84) should read:

$$\hat{i}(x) = \lim_{k \to \infty} \frac{1}{k} i_{kT}(x)$$

(b) In the following line, read Theorem I.7.7 instead of Theorem I.7.8

p. 231

(a) The unnumbered formula should read:

$$\hat{i}\left(x^i\right) = \frac{2}{\alpha_i} \sum_j \alpha_j$$

(b) The last sentence should read: "Hence the mean index per unit of action:"