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The golden rule when preferences are time inconsistent

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Abstract We investigate the classical Ramsey problem of economic growth when the planner uses non-constant discounting. It is well-known that this leads to time inconsistency, so that optimal strategies are no longer implementable. We then define equilibrium strategies to be such that unilateral deviations occurring during a small time interval are penalized. Non-equilibrium strategies are not implementable, so only equilibrium strategies should be considered by a rational planner. We show that there exists such strategies which are (a) smooth, and (b) lead to stationary growth, as in the classical Ramsey model. Finally, we prove an existence and multiplicity result: for logarithmic utility and quasi-exponential discount, there is an interval I such that, for every k in I, there is an equilibrium strategy converging to k. We conclude by giving an example where the planner is led to non-constant discount rates by considerations of intergenerational equity.

Keywords Time inconsistency · Markov strategies · Ramsey models · Nash equilibria · Intergenerational equity · Implicit differential equation

JEL Classification E43 · O44

1 Introduction

Forty years ago, Phelps and Pollak [14] noted that the classical Ramsey model of economic growth [15] relies on a *stationarity postulate*: discounting at a constant rate, even if this rate is 0 as Ramsey suggests, means that "the present generation's preference ordering of consumption streams is invariant to changes in their timing". This mathematical postulate then

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implies another, deeper, economic postulate, which they call *perfect altruism*: "each generation's preference for their own consumption relative to the next generation's is no different from their preferences for any future generation's consumption relative to the succeeding generation's". In other words, a constant discount rate requires each generation to save as if it would still be around when the time comes to reap the benefits of its investments; they act as if, paraphrasing Keynes' famous words, in the long run, we'll all be there.

Phelps and Pollak then represented the preferences of the present generation by the utility function:

$$u(c_0) + \delta \sum_{n=1}^{\infty} \alpha^n u(c_n) \tag{1}$$

where c_n represents the consumption of generation n, α reflects time preference, and δ is the degree to which the present generation values other people's consumption relative to their own: as δ increases from 0 to ∞, we go from extreme selfishness ("après moi le déluge") to extreme altruism, $\delta = 1$ being the standard case where the discount rate is constant. They then proceed to show that the first-best strategy (straightforward optimization of the criterion 1) is not implementable, unless the present generation can commit future ones. In the absence of such a commitment mechanism, one has to resort to a second-best strategy, in fact a Stackelberg equilibrium of the associated leader-follower game, which they compute explicitly for CRRA utilities, $u'(c) = c^{-p},\ p > 0$. They find that this second-best strategy is not Pareto optimal. A few years later, Dasgupta [2,3] also revisited the Ramsey problem, seeking to define and test various criteria of distributive justice between generations, and he reached the same conclusions.

The line of research opened up by Phelps and Pollak has been active ever since. We refer to [6] for a critical review of the literature up to 2002, including other shortcomings of the constant discounting model and other attempts to find an alternative model. In this paper, we will focus on one of the many issues that have been raised about the non-constant discounting model, namely the non-uniqueness of equilibria. In [13], Phelps revisited his earlier work with Pollak, and put it in a continuous-time framework, akin (but not equivalent) to the original Ramsey model: the generation alive at time t equates the marginal utility of its consumption, $u'(c(t))$, to the marginal utility of bequeathed capital, $V'(k(t))$:

$$V(k) := \int_0^{\infty} \left(u\left(c\left(k(t)\right)\right) - u\left(c\left(\bar{k}\right)\right)\right) dt, \quad k(0) = k$$

Here $c(k)$ is the candidate equilibrium strategy, and it is assumed that $k(t) \to \bar{k}$ when $t \to \infty$, fast enough so that the right-hand side converges. Phelps then pointed out that there is no way of determining $\bar{k}$, beyond putting upper and lower bounds on possible values. In other words, if all generations agreed what sustainable level of capital $\bar{k}$ they were aiming for, their equilibrium strategy would be determined, but he could see no way of determining $\bar{k}$ from standard rationality arguments. He concluded that this indeterminacy may leave room for an "ethic", that specifies some obligations that each generation is expected to meet: "by telling each generation what to expect of other generations, morals may make determinate the altruistic behavior of each generation"

This line of thought was followed by Krusell and Smith [12] in the discrete-time framework, and, more interestingly for our purposes, by Karp in the continuous-time framework. In a series of remarkable papers [8–11], he puts the whole analysis back into the framework of the Ramsey model, which, as is well-known, drives the economy to some steady state $\bar{k}$. Karp then concludes, as Phelps does, that there is not enough information to identify the steady state. He does, however, provide upper and lower bounds that any candidate steady state must satisfy, and he shows that there is a natural ordering between them.

It would seem that by now, there is not much else to be said. There is, however, still one piece missing in the puzzle: to our knowledge, no one has ever proved the existence, much less the multiplicity, of equilibrium strategies in time-inconsistent versions of the Ramsey model. What has been done is to identify necessary conditions that $\bar{k}$ has to satisfy if there is to be an equilibrium strategy $c(k)$ such that $k(t) \to \bar{k}$, and then to show that there is a continuum of such $\bar{k}$: Karp calls them "candidate steady states". But he does not show that any $\bar{k}$ in this continuum corresponds to an equilibrium strategy, i.e. that, corresponding to a multiplicity of candidates, there is really a multiplicity of strategies; the reason, as he points out in [9], is that he "cannot appeal to any of the standard sufficiency conditions", so that his analysis "relies entirely on necessary conditions". Krusell and Smith [12] proved such a result in the discrete-time framework, but the strategies they consider are discontinuous (step functions with countably many jumps).

In the following, we will provide some non-standard sufficiency conditions, and prove that, in the case of logarithmic utility and quasi-exponential discount, there is a multiplicity of continuous equilibrium strategies: an open interval $I \subset R$ of candidate steady states is identified, as in Karp, but in addition it is shown that for each k in that interval, there exists an equilibrium strategy which is C^2 and converges to that steady state. We believe this result is robust, in the sense that it would hold for more general preferences and discount rates, but we have chosen this framework in other to keep the computations down. The basic intuition of Phelps is thus laid on firm mathematical ground.

The model is given in Sect. 2. In Sect. 3, we define equilibrium strategies directly in the continuous framework, and in Sect. 4 we characterize them by a non-local version of the Hamilton-Jacobi-Bellman (henceforth HJB) equation. Similar results are known in the discrete-time case (see for instance the hyperbolic Euler relation of Harris and Laibson [7]). In the continuous-time case, Karp obtained a dynamic programming equation, namely Eq. 5 of [9]. In this paper, we carry the computations further than he does, so that we obtain the necessary conditions in two equivalent forms, both of which we believe to be more transparent than Karp's formulation, and in addition we prove that they are also sufficient. In other words, we prove a verification theorem. In this way, we obtain a theory of strategic equilibrium which closely parallels the classical HJB theory of optimal control, and which we feel is interesting for its own sake.

In Sect. 5, we explore the case when the discount factor is quasi-exponential:

$$R(t) = \lambda \exp(-\delta t) + (1 - \lambda) \exp(-\rho t)$$

with $\delta < \rho$ and $\lambda > 0$. This case was already considered by Karp in the case $0 < \lambda < 1$, but we also consider the case $\lambda > 1$, which corresponds to altruistic time preferences. As mentioned earlier, we show that there is an open (non-empty) interval $I \subset R$ such that all steady states belong to the closure $\bar{I}$ and every $k \in I$ is the steady state of some C^2 equilibrium strategy, defined on a neighbourhood of k and converging to k, and we define a natural ordering between them. Finally, in Sect. 6, we show that a very simple model of intergenerational equity leads to quasi-exponential discount. Previous versions of this paper have been made available in [4,5].

2 The model

We consider the classical Ramsey model with non-constant discount rate. There is only one good in the economy, which can either be consumed or used for production. The decision-maker at time t discounts the utility of future consumption with a non-constant discount rate. Specifically, the utility he derives from a consumption schedule $c(s)$, with $t \leq s_1 \leq s \leq s_2$, is

$$U_t\,(c\,(\cdot)) = \int_{s_1}^{s_2} R\,(s-t)\,u\,(c(s))\,ds \tag{2}$$

for some utility function $u : [0, \infty) \to R$ and some discount function R. We assume that u is strictly increasing, twice continuously differentiable and strictly concave. The discount function R is continuously differentiable and positive, with

$$R(0) = 1, \quad R(t) \geq 0, \quad \int_0^\infty R(s)ds < \infty \tag{3}$$

The decision maker at time t strives to maximize the objective:

$$\int_t^\infty R\,(s-t)\,u\,(c(s))\,ds \tag{4}$$

under the resource constraint:

$$\frac{dk(s)}{ds} = f\,(k(s)) - c(s), \quad k(t) = k_t, \tag{5}$$

where $k(s)$ is the amount of capital available at time s, and k_t is the initial amount available at time t. We assume that the production function f is strictly increasing, concave and twice continuously differentiable, with $f(0) = 0$. As usual, consumption c and capital k are non-negative:

$$c(t) \geq 0, \quad k(t) \geq 0 \tag{6}$$

It is by now well-known that using non-exponential discount rates creates time inconsistency, in the sense that the relative ordering of two consumptions schedules $c_1(s)$ and $c_2(s)$, both defined on $t_1 \leq s \leq t_2$, can be reversed by the mere passage of time. To see this, take $t_1 < t_2 \leq s_1$, and suppose $U_{t_1}\,(c_1\,(\cdot)) < U_{t_1}\,(c_2\,(\cdot))$, so that, at time t_1, the decision-maker prefers c_2 to c_1. If $R(t) = \exp(-rt)$, then $U_{t_2}\,(c\,(\cdot)) = \exp\,(r\,(t_2 - t_1))\,U_{t_1}\,(c\,(\cdot))$, so we have $U_{t_2}\,(c_1\,(\cdot)) < U_{t_2}\,(c_2\,(\cdot))$ and the relative ordering persists. In the non-exponential case, however, $U_{t_2}\,(c\,(\cdot))$ is no longer proportional to $U_{t_1}\,(c\,(\cdot))$, and we may have $U_{t_2}\,(c_1\,(\cdot)) > U_{t_2}\,(c_2\,(\cdot))$, so that the ordering is reversed: at time t_2 schedule $c_2\,(\cdot)$ is preferred to $c_1\,(\cdot)$. As a consequence, the successive decision-makers cannot agree on a common optimal policy: each of them will have his own. Optimizing the intertemporal welfare (4) at $t = t_1$ under the constraint (5) will lead to a t_1-optimal policy c_1 defined on $s \geq t_1$. At any subsequent time $t_2 > 0$, the decision-maker will revisit the problem, setting $t = t_2$ in (4) and (5). He will then find an optimal schedule $c_2\,(\cdot)$ on $s \geq t_2$ that is different from $c\,(\cdot)$.

In other words, for general discount functions, there are a plethora of *temporary* optimal policies: each of them will be optimal when evaluated from one particular point in time, but will cease to be so when time moves forward. In the absence of a commitment technology, there is no way for the decision-maker at time t to implement his optimal (first-best) solution. He will then consider the problem as a leader-follower game between successive players, and seek a Stackelberg equilibrium.

3 Equilibrium strategies: construction and definition

Whereas in the time-consistent case we are looking for optimal *controls* , in the time-inconsistent case we will be looking for equilibrium *strategies*. We restrict our analysis to stationary Markov strategies, in the sense that the policy depends only on the current capital stock and not on past history, current time or some extraneous factors.

Definition 2 A *stationary Markov strategy* is a pair I, σ, where $I \subset [0, \infty)$ is an interval and $\sigma : I \to [0, \infty)$ is a continuously differentiable function such, if $k_0 \in I$, then the solution $k(t)$ of the differential equation:

$$\frac{dk}{dt} = f(k(t)) - \sigma(k(t)), \quad k(0) = k_0 \tag{7}$$

remains in I for all $t > 0$.

For each $k \in I$, the strategy prescribes a consumption $c = \sigma(k)$. Such a strategy is given by $c = \sigma(k)$. Equation 7 then describes the dynamics of capital accumulation from $t = 0$.

Definition 3 A stationary Markov strategy (I, σ) *converges to* $\bar{k}$ if $\bar{k} \in I$ and for every $k_0 \in I$, the solution $k(t)$ of (7) converges to $\bar{k}$ when $t \to \infty$.

We will say that $\bar{k}$ is a *steady state* of σ. Note that we must have

$$f\left(\bar{k}\right) = \sigma\left(\bar{k}\right)$$

In order to keep notations simple, we will denote stationary Markov strategies as σ instead of (I, σ), unless the underlying interval I needs to be considered. We shall say that σ is convergent if it converges to some $\bar{k}$. We will consider only convergent strategies, so that the integral (4) is obviously convergent, and its successive derivatives can be computed by differentiating under the integral.

We will define equilibrium strategies by going to the limit from the discrete case. Cut the time interval $[0, \infty[$ into subintervals of length $\varepsilon > 0$. Let t be the left endpoint of such an interval, and assume the decision-maker at time t can decide the consumption c on all of the interval $[t, t + \varepsilon[$. We will seek a leader-follower equilibrium of this game, and let $\varepsilon \longrightarrow 0$. One interpretation of that is that the decision-maker holds power during his lifetime, but that the latter is small compared to the planning horizon.

A convergent Markov strategy $c = \sigma(k)$ has been announced and is public knowledge. The decision maker begins at time $t = 0$ with capital stock k. If all future decision-makers apply the strategy σ, the resulting path $k_0(t)$ obeys

$$\frac{dk_0}{dt} = f(k_0(t)) - \sigma(k_0(t)), \quad t \geq 0 \tag{8}$$

$$k_0(0) = k. \tag{9}$$

The decision-maker at time 0 is free to fix consumption on $[0, \varepsilon]$ at any level c. She expects all future decision-makers to apply the strategy σ, and she asks herself if it is in her own interest to apply the same strategy, that is, to consume $\sigma(k)$. If she commits to another bundle, c say, the immediate utility flow during $[0, \varepsilon]$ is $u(c)\varepsilon$. At time ε, the resulting capital will be $k + (f(k) - c)\,\varepsilon$, and from then on, the strategy σ will be applied which results in a capital stock k_c satisfying

$$\frac{dk_c}{dt} = f(k_c(t)) - \sigma(k_c(t)), \quad t \geq \varepsilon \tag{10}$$

$$k_c(\varepsilon) = k + (f(k) - c)\,\varepsilon. \tag{11}$$

The capital stock k_c can be written as $k_c(t) = k_0(t) + k_1(t)\varepsilon$ where [1]

$$\frac{dk_1}{dt} = \left(f'(k_0(t)) - \sigma'(k_0(t))\right) k_1(t), \quad t \geq \varepsilon \tag{12}$$

$$k_1(\varepsilon) = \sigma(k) - c \tag{13}$$

where f' and σ' stand for the derivatives of f and σ. Summing up, we find that the total gain for the decision-maker at time 0 from consuming bundle c during the interval of length ε when she can commit, is

$$u(c)\varepsilon + \int_{\varepsilon}^{\infty} R(s)u\left(\sigma\left(k_0(t) + \varepsilon k_1(t)\right)\right) dt,$$

and in the limit, when $\varepsilon \to 0$, and the commitment span of the decision-maker vanishes, expanding this expression to the first order leaves us with two terms

$$\int_0^{\infty} R(t)u\left(\sigma\left(k_0(t)\right)\right) dt + \varepsilon\left[u(c) - u\left(\sigma(k)\right) + \int_0^{\infty} R(t)u'\left(\sigma\left(k_0(t)\right)\right)\sigma'\left(k_0(t)\right) k_1(t)dt\right]. \tag{14}$$

where k_1 solves the linear equation

$$\frac{dk_1}{dt} = \left(f'(k_0(t)) - \sigma'(k_0(t))\right) k_1(t), \quad t \geq 0 \tag{15}$$

$$k_1(0) = \sigma(k) - c. \tag{16}$$

Note that the first term of (14) does not depend on the decision taken at time 0, but the second one does. This is the one that the decision-maker at time 0 will try to maximize. In other words, given that a strategy σ has been announced and that the current state is k, the decision-maker at time 0 faces the optimization problem:

$$\max_{c \geq 0} P_1(k, \sigma, c) \tag{17}$$

where

$$P_1(k, \sigma, c) = u(c) - u\left(\sigma(k)\right) + \int_0^{\infty} R(t)u'\left(\sigma\left(k_0(t)\right)\right)\sigma'\left(k_0(t)\right) k_1(t)dt. \tag{18}$$

In the above expression, $k_0(t)$ solves the Cauchy problem (8), (9) and $k_1(t)$ solves the linear Eqs. 15 and 16.

The same analysis will apply to decision-making at any time $t > 0$. We are led to the following:

Definition 4 A convergent Markov strategy (I, σ) is an *equilibrium strategy* for the intertemporal decision model (4) under the constraints (5) and (6) if, for every $k \in R$, the maximum in problem (17) is attained for $c = \sigma(k)$:

[1] To see this, plug $k_c(t) = k_0(t) + k_1(t)\varepsilon$ into (10) for $t \geq \varepsilon$, keeping only the terms of first order in ε, and get

$$\frac{dk_c}{dt} = f\left(k_0(t)\right) + \varepsilon f'\left(k_0(t)\right) k_1(t) - \sigma\left(k_0(t)\right) - \varepsilon\sigma'\left(k_0(t)\right) k_1(t).$$

Comparing this with (8) gives (12). Equation 13 is obtained by substituting the expansion $k_0(\varepsilon) = k + \varepsilon \frac{dk_0}{ds}(0) = k + \varepsilon\left(f(k) - \sigma(k)\right)$ into (11).

$$\sigma(k) = \arg\max_{c\geq 0} P_1(k, \sigma, c) \tag{19}$$

The intuition behind this definition is simple. Each decision-maker can commit only for a small time ε, so she can only hope to exert a very small influence on the final outcome. In fact, if the decision-maker at time 0 plays c when she is called to bat, while all the others are applying the strategy σ, the end payoff for her will be of the form

$$P_0(k, \sigma) + \varepsilon P_1(k, \sigma, c)$$

where the first term of the right hand side does not depend on c. In the absence of commitment, the decision-maker at time 0 will choose whichever c maximizes the second term $\varepsilon P_1(k, \sigma, c)$. Saying that σ is an equilibrium strategy means that she will choose $c = \sigma(k)$. Given the stationarity of the problem, if the strategy $c = \sigma(k)$ is chosen at time 0, it will be chosen at any future time t and as a result, the strategy σ can be implemented in the absence of commitment. Conversely, if a strategy σ for the intertemporal decision model (2), (5), and (6) is not an equilibrium strategy, then it cannot be implemented unless the decision-maker at time 0 has some way to commit his successors. Typically, an optimal strategy will not be an equilibrium strategy.

Note that this is the minimal requirement for rationality: if a strategy $\sigma(k)$ is not an equilibrium strategy, then, by definition, at some time t the decision-maker will have an incentive to deviate unilaterally and consume some $c(t) \neq \sigma(k(t))$ during some brief period. In other words, the strategy $\sigma(k)$ will not be implemented past time t. No rational planner would consider such a strategy at time 0, because she knows that the plan simply would not be executed.

4 Characterization of the equilibrium strategies

The main result of this section is that the equilibrium strategy can be fully specified by a single function, the *value function* $v(k)$, which is reminiscent of- although different from - the value function in optimal control. We will show that the value function satisfies two equivalent equations, the integrated Eq. IE and the differentiated Eq. DE, the latter one resembling the classical Hamilton–Jacobi–Bellman (HJB) equation of optimal control. This similarity is reassuring since it shows how standard methods from control theory can be adapted to analyze the impact of time inconsistency. Unfortunately, the similarity is superficial only, since (DE) is a non-local equation (and not a partial differential equation like (HJB)) and we will demonstrate that its solutions exhibit different qualitative behavior. In the knife-edge case where the discount rate is constant, the non local term in (DE) collapses, and (DE) becomes identical to (HJB). Consequently, when the discount rate is constant, the equilibrium strategies are also optimal from the perspective of all successive decision makers.

Given a Markov strategy $\sigma(k)$, continuously differentiable and convergent, we shall be dealing with the Cauchy problem (8), (9). The value $k_0(t)$ depends on current time t, initial data k, and the strategy σ. To stress this dependence, it is convenient to write $k_0(t) = \mathcal{K}(\sigma; t, k)$ where $\mathcal{K}$ is the *flow* associated with the differential Eq. 8. It is defined by

$$\frac{\partial \mathcal{K}(\sigma; t, k)}{\partial t} = f(\mathcal{K}(\sigma; t, k)) - \sigma(\mathcal{K}(\sigma; t, k)) \tag{20}$$

$$\mathcal{K}(\sigma; 0, k) = k. \tag{21}$$

The following theorem characterizes the equilibrium strategies and its proof is given in Appendix A. There are two parts in this characterization: a functional equation on the value function $v(k)$ and an instantaneous optimality condition determining current consumption. In the following, i is the inverse of marginal utility u':

$$u'(c) = y \Longleftrightarrow c = i\,(y)$$

Theorem 5 *Let (I, σ) be a convergent strategy. If it is an equilibrium strategy, then the value function*

$$v(k) := \int_0^\infty R(t) u\,(\sigma\,(\mathcal{K}\,(\sigma; t, k)))\,dt \tag{22}$$

satisfies, for all $k \in I$, the functional equation

$$v\,(k) = \int_0^\infty R(t) u\left(i \circ v'\left(\mathcal{K}\left(i \circ v'; t, k\right)\right)\right) dt \tag{IE}$$

and the instantaneous optimality condition

$$u'\,(\sigma\,(k)) = v'(k) \tag{23}$$

Conversely, if $I \subset [0, \infty)$ is an interval and a function $v : I \to R$ is twice continuously differentiable, satisfies (IE), and $\sigma = i \circ v'$ is non-negative and converges to $\bar{k} > 0$, then (I, σ) is an equilibrium strategy.

The instantaneous relation (23) expresses the usual tradeoff between the utility derived from current consumption and the utility value of saving. This is a standard condition in a world where there is one commodity that can be used for investment or consumption. It can be rewritten as follows:

$$\sigma\,(k) = i\left(v'(k)\right) \tag{24}$$

Equation IE is an equation for the unknown function $v(k)$. Let us spell out what it means. Given a candidate function v, we must first solve the Cauchy problem (20), (21) with $\sigma = i \circ v'$. Second, we calculate the right-hand side of Eq. IE, which is an integral along the trajectory of capital stock. The final result should be equal to $v(k)$. Equation IE is therefore a fundamental characterization of the equilibrium strategies and it takes the form of a functional equation on v.

In order to contrast the equilibrium dynamics with the dynamics resulting from using the optimal control approach, the following proposition gives an alternative characterization, the differentiated Eq. DE, which resembles the usual Euler equation from optimal control and its proof is given in Appendix B. As we mentioned in the introduction, it is related to (but different from) Eq. 5 of [9], and to dynamic programming relations in the discrete time framework. It seems to us, however, that there is no equivalent of the integrated Eq. IE in the previous literature.

Theorem 6 *Let $I \subset [0, \infty)$ be an interval and $v : I \to R$ be a C^2 function such that the strategy $\sigma = i \circ v'$ is non-negative and converges to $\bar{k} > 0$. Then v satisfies the integrated Eq. IE if and only if it satisfies the following functional equation*

$$-\int_0^\infty R'(t) u \circ i\left(v'\left(\mathcal{K}\left(i \circ v'; t, k\right)\right) dt = \sup_c \left[u(c) + v'(k)(f(k) - c)\right]\right. \tag{DE}$$

together with the boundary condition:

$$v\left(\bar{k}\right) = u\left(f\left(\bar{k}\right)\right) \int_0^\infty R(t) dt \tag{BC}$$

We may rewrite the differentiated Eq. DE in a more familiar way:

$$\rho(k)v(k) = \sup_c \left[u(c) + v'(k)(f(k) - c)\right] \tag{25}$$

where:

$$\rho(k) = -\frac{\int_0^\infty R'(t)u\left(\sigma\left(\mathcal{K}\left(\sigma; s, k\right)\right)\right) dt}{\int_0^\infty R(s)u\left(\sigma\left(\mathcal{K}\left(\sigma; s, k\right)\right)\right) ds}$$

is interpreted as an effective discount rate. Equation 25 then tells us that, along an equilibrium path, the relative changes in value to the consumer must be equal to the effective discount rate. The effective discount rate is here endogenous to the model and its presence reflects the strategic behavior of the current decision maker resulting from internalizing the behavior of future decision makers.

With exponential discounting, $R(t) = e^{-\delta_0 t}$, the effective discount rate is constant, $\rho(k) = \delta_0$ for all k, and (DE) is the ordinary (HJB) equation

$$\delta_0 v(k) = \sup_c \left[u(c) + v'(k)\left(f(k) - c\right)\right]. \tag{26}$$

Next, consider piecewise exponential discounting, $R(t) = e^{-\delta_0 t}$ for $t \leq \tau$ and $R(t) = e^{-\delta_1 t}$ for $t > \tau$, with $\tau > 0$. Equation DE is now:

$$\delta_0 v(k) = \sup_c \left[u(c) + \left(\delta_0 - \delta_1\right) g(k) + v'(k)\left(f(k) - c\right)\right] \tag{27}$$

where

$$g(k) = \int_\tau^\infty e^{-\delta_1 t} u \circ i\left(v'\left(\mathcal{K}\left(i \circ v'; t, k\right)\right)\right) dt.$$

This integral is clearly positive, so the extra term $(\delta_0 - \delta_1)g(k)$ has the sign of $(\delta_0 - \delta_1)$. If $\delta_0 > \delta_1$, for instance, it yields additional incentives to save, relative to the exponential case.

Neither Eq. IE nor Eq. DE are of a classical mathematical type. If it were not for the integral term, Eq. DE would be a first-order partial differential equation of known type (Hamilton-Jacobi), but this additional term, which is non-local (an integral along the trajectory of the flow (20) associated with the solution v), creates a loss of regularity in the functional equation that generates mathematical complications. As a result, existence and uniqueness problems arise as they typically do in dynamic games.

5 Existence and multiplicity of equilibrium strategies

5.1 Reduction to a system of ODEs

From now on, we shall use the following specifications:

$$R(t) = \lambda \exp\left(-\delta t\right) + (1 - \lambda) \exp\left(-\rho t\right) \tag{28}$$

$$u(c) = \ln c \tag{29}$$

Using a logarithmic utility simplifies the computations, but the results extend to general power utilities $u(c) = c^{1-\theta} / (1 - \theta)$, with $\theta > 0$, and presumably to more general utilities as well. The use of quasi-exponential discount factors, on the other hand, is quite essential, because it will reduce Eq. IE to a system of two differential equations, as was already observed by Karp [9, Eqs. 7, 8].

Without loss of generality, we shall assume that:

$$\rho > \delta \tag{30}$$

Obviously $R(0) = 1$. If $\lambda > 0$, then $R(t)$ does not change sign: $R(t) > 0$ for every $t > 0$. The rate of time preference is:

$$r(t) = \frac{\delta\lambda \exp(-\delta t) + \rho(1-\lambda)\exp(-\rho t)}{\lambda \exp(-\delta t) + (1-\lambda)\exp(-\rho t)} \tag{31}$$

and we have:

$$r(0) = \lambda\delta + (1-\lambda)\,\rho \ \text{(spot rate)}$$
$$r(\infty) = \delta \ \text{(long term rate)}$$

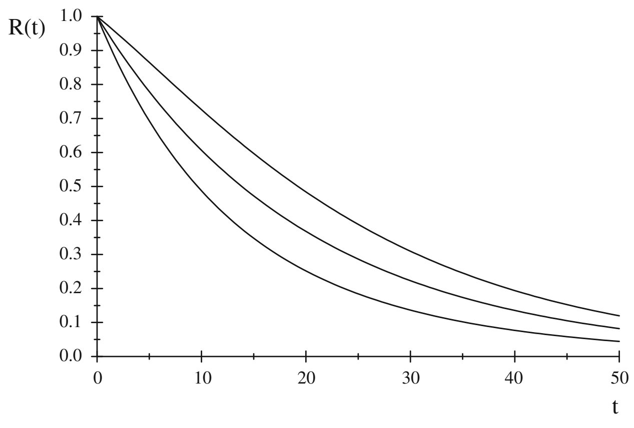
There are four distinct cases:

Case 1 $0 < \lambda < 1$. Then $r(0) > r(\infty)$, and the discount rate decreases with maturity. This is the *hyperbolic* case, well documented in the psychological literature (see [6]), which reflects preference for immediate gratification. We have $r(t) > 0$ for all $t > 0$.
Case 2 $1 < \lambda < \frac{\rho}{\rho-\delta}$. Then $0 < r(0) < r(\infty)$, and the discount rate increases with maturity. In view of the remark by Phelps and Pollack which we quoted in the beginning, this can be seen as *extreme altruism*. We have $r(t) > 0$ for all $t > 0$.
Case 3 $\lambda > \frac{\rho}{\rho-\delta}$. Then $r(0) < 0 < r(\infty)$. The discount rate increases from negative to positive values: we have $r(t) < 0$ for $t < \frac{1}{\rho-\delta}\ln\frac{\rho}{\delta}\frac{\lambda-1}{\lambda}$.
Case 4 $\lambda < 0$. Then $R(t)$ becomes negative when $t \longrightarrow \infty$

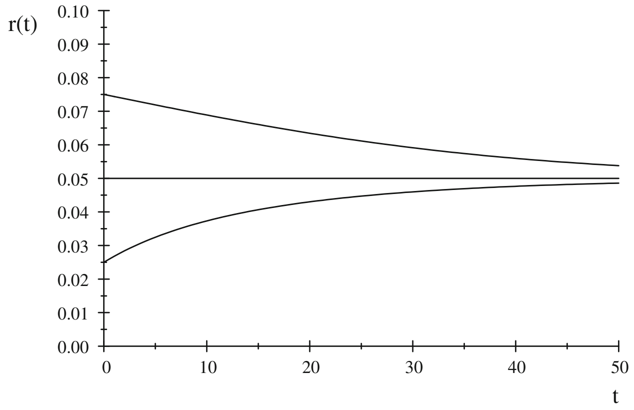
The two last case have no economic interpretation that we can see, so in the sequel we will be assume:

$$0 < \lambda < \frac{\rho}{\rho-\delta}$$

We plot the discount factor $R(t)$ for $\rho = 0.1$, $\delta = 0.05$, $\lambda = 0.5$ (case 1), $\lambda = 1$ (exponential case, constant discount) and $\lambda = 1.5$ (case 2). The highest curve corresponds to the hyperbolic case (case 1), the lowest one to extreme altruism (case 2) , and the exponential sits in the middle.



We draw the same picture for discount rates:



To simplify notations, we set:

$$a = (\delta + \rho)/2 \quad \text{and} \quad b = (\delta - \rho)/2,$$

Proposition 7 *Let $I \subset [0, \infty)$ be an interval and $v : I \to R$ be a C^2 function such that the strategy $\sigma(k) = 1/v'(k)$ converges to $k_\infty \in I$. Then v satisfies (DE) and (BC) if and only if there exists a C^1 function $w : I \to R$, such that (v, w) satisfies the system:*

$$\left(f - \frac{1}{v'}\right) v' - \ln v' = av + bw, \tag{32}$$

$$\left(f - \frac{1}{v'}\right) w' - (2\lambda - 1) \ln v' = bv + aw \tag{33}$$

with the boundary conditions:

$$v(k_\infty) = \left(\frac{\lambda}{\delta} + \frac{1-\lambda}{\rho}\right) \ln f(k_\infty) = \left(\frac{\lambda}{a+b} + \frac{1-\lambda}{a-b}\right) \ln f(k_\infty) \tag{34}$$

$$w(k_\infty) = \left(\frac{\lambda}{\delta} - \frac{1-\lambda}{\rho}\right) \ln f(k_\infty) = \left(\frac{\lambda}{a+b} - \frac{1-\lambda}{a-b}\right) \ln f(k_\infty) \tag{35}$$

Note that, by Theorems 5 and 6, $\sigma = 1/v'$ then is an equilibrium strategy that converges to k_∞. The proof is given below. Note that it gives explicit formulas for v and w, namely

$$v(k) = \int_0^\infty \left(\lambda e^{-\delta t} + (1-\lambda) e^{-\rho t}\right) \ln (\sigma (\mathcal{K}(\sigma; t, k)))\, dt, \tag{36}$$

$$w(k) = \int_0^\infty \left(\lambda e^{-\delta t} - (1-\lambda) e^{-\rho t}\right) \ln (\sigma (\mathcal{K}(\sigma; t, k)))\, dt \tag{37}$$

5.2 Multiplicity of equilibrium strategies

For the next result, we define $\bar{k}$ as follows:

$$f'\left(\bar{k}\right) = \frac{1}{\frac{\lambda}{\delta} + \frac{1-\lambda}{\rho}}$$

Note that, since $\lambda > 0$, we have $\frac{\lambda}{\delta} + \frac{1-\lambda}{\rho} > 0$, so this defines $\bar{k}$ uniquely. If $\lambda\delta + (1-\lambda)\rho > 0$, we also define $\underline{k}$ as follows:

$$f'\left(\underline{k}\right) = \lambda\delta + (1-\lambda)\rho$$

Theorem 8 *Assume f is C^3 for $k > 0$. Then there is a non-empty open interval A such that:*

(a) *if (I, σ) is an equilibrium strategy converging to $k_\infty \in I$, then $k_\infty \in \bar{A}$*
(b) *for any $k_\infty \in A$, there is an open interval I containing k_∞ and an equilibrium strategy (I, σ)converging to k_∞*

The interval A is given by:

Case 1 $0 < \lambda < 1$*: then* $A = \left]\underline{k},\ \ \bar{k}\right[$
Case 2 $1 < \lambda < \frac{\rho}{\rho-\delta}$*: then* $A = \left]\bar{k},\ \ \underline{k}\right[$

Note that $\lambda\delta + (1-\lambda) = r(0)$, the spot rate of time preference.

In the exponential case, when $\delta = \rho$, or $\lambda = 1$, we find the classical relation $f'(k_\infty) = \rho$. In the general case, we find a continuum of possible equilibrium strategies, and corresponding asymptotic growth rates, and their range is fully characterized (except for the endpoints). So the proof should be in two parts: first showing that every possible k_∞ is in that range, and then showing that every point in that range is a possible k_∞. Later on, we will show that the equilibrium strategy converging to $\bar{k}$ has further properties that makes it the rational choice among the many possible equilibrium strategies.

The proof of the Theorem is given in the Appendix C. Note for future reference that it gives the additional information:

$$v'(k_\infty) = \frac{1}{f(k_\infty)} \tag{38}$$

$$w'(k_\infty) = \frac{f'(k_\infty) - a}{bf(k_\infty)} \tag{39}$$

We do not know if there are equilibrium strategies converging to the endpoints $\underline{k}$ and $\bar{k}$. The proof in Appendix D.2 carries through without changes, but the linear stability condition in Appendix D.1 does not. In other words, we can find functions (v, w) satisfying the conditions of Proposition 7, but we do not know if the strategy $\sigma = i\left(v'\right)$ converges to $\underline{k}$ or $\bar{k}$.

5.3 There are no subgame perfect strategies

Consider an equilibrium strategy σ converging to k^*. By definition, if the decision-maker at time t applies a new equilibrium strategy σ' and if subsequent decision-makers stick to σ, he/she will be punished. The question is whether this threat is credible, that is, whether subsequent decision-makers will earn more by switching to σ' or sticking to σ.

Definition 9 Let (I, σ) and $\left(I, \sigma'\right)$ be two equilibrium strategies converging to k_∞ and k'_∞. We shall say that σ is *eventually dominated by* σ' if, for any starting point $k \in I$, there is some $t > 0$ such that:

- if strategy σ has been applied up to time $s > t$, then the decision-maker at time s prefers σ' to σ
- if strategy σ' has been applied on the interval $\left[s,\ s'\right]$, with $t < s < s'$, then the decision-maker at time s' prefers σ' to σ

The following result shows that all strategies in the interior of A are eventually dominated (except perhaps for those converging to the endpoints).

Proposition 10 *If* $0 < \lambda < 1$*, and* $k^* < k_\infty < \bar{k}$*, then any equilibrium strategy converging to* k^* *is eventually dominated by an equilibrium strategy converging to* k_∞*. If* $\lambda > 1$ *and* $\underline{k} < k_\infty < k^*$*, then any equilibrium strategy converging to* k^* *is eventually dominated by an equilibrium strategy converging to* k_∞*.*

In the case $0 < \lambda < 1$, this result was first observed in Karp [9] Proposition 3: more conservative rules (in the sense that they increase the steady state levels of capital and consumption) eventually dominate less conservative ones. In other words, current generations will be willing to reduce their level of consumption in order to increase the welfare of future ones, provided the effort is not too substantial. In the case, $\lambda > 1$, the opposite holds: less conservative rules dominate more conservatives ones. The only non-dominated strategies, $k^* = \bar{k}$ in the first case and $k^* = \underline{k}$ in the second, may not be implementable because they do not satisfy the stability condition.

To prove Proposition 10, we will consider the equilibrium value as a function both of the initial point k_0, as before, and of the terminal point k_∞. Write:

$$V(k_0, k_\infty) = \int_0^\infty R(t) u\left(\bar{\sigma}\left(\mathcal{K}(\bar{\sigma}; t, k_0)\right)\right) dt$$

where $\bar{\sigma}$ is an equilibrium strategy converging to k_∞. We will need the following obvious lemma:

Lemma 11 *Let* $f(x, y)$ *be a* C^1 *function of two variables such that:*

$$f(x, x) = \varphi(x)$$
$$\frac{\partial f}{\partial x}(x, x) = \psi(x)$$

Then:

$$\frac{\partial f}{\partial y}(x, x) = \varphi'(x) - \psi(x)$$

Apply this to V, at a point $k_0 = k^* = k_\infty$. We have, by formula (38):

$$V\left(k^*, k^*\right) = \int_0^\infty R(t) \ln f\left(k^*\right) dt = \left(\frac{\lambda}{\delta} + \frac{1-\lambda}{\rho}\right) \ln f\left(k^*\right) = \frac{1}{f'\left(\bar{k}\right)} \ln f\left(k^*\right)$$
$$\frac{\partial V}{\partial k_0}\left(k^*, k^*\right) = \frac{1}{f\left(k^*\right)}$$

and hence:

$$\frac{\partial V}{\partial k_\infty}\left(k^*, k^*\right) = \frac{1}{f'\left(\bar{k}\right)} \frac{f'\left(k^*\right)}{f\left(k^*\right)} - \frac{1}{f\left(k^*\right)}$$
$$= \frac{1}{f\left(k^*\right)} \left(\frac{f'\left(k^*\right)}{f'\left(\bar{k}\right)} - 1\right)$$

There are now two cases to consider:

- $0 < \lambda < 1$. Then $f'(k^*) > f'(\bar{k})$ on the whole existence interval, and $\frac{\partial V}{\partial k_\infty}(k^*, k^*) > 0$. It follows that there is some $\varepsilon > 0$ and some $k_\infty > k^*$ such that, for all k_0 with $|k_0 - \bar{k}| < \varepsilon$, we have:

$$V(k_0, k_\infty) > V(k_0, k^*)$$

- $\lambda > 1$. Then $f'(k^*) < f'(\bar{k})$ on the whole existence interval, and $\frac{\partial V}{\partial k_\infty}(k^*, k^*) < 0$. From the above, we see that there is some $\varepsilon > 0$ and some $k_\infty < k^*$ such that, for all k_0 with $|k_0 - \bar{k}| < \varepsilon$, we have:

$$V(k_0, k_\infty) > V(k_0, k^*)$$

This concludes the proof.

6 Intergenerational equity

In the classical Ramsey model, the same rate of time preference is applied to the short to middle term (where the present generation is alive) and in the middle to long term (when we are all dead, and costs/benefits accrue to people yet unborn). This means that this parameter is put to two different uses:

- for weighing consequences to me of my own actions
- for weighing consequences to others of my own actions

In a seminal paper, Sumaila and Walters [16] separate the (psychological) impatience of the present generation from its (ethical) concern for future generations. We will give a continuous-time version of their model. Consider a population with a birth rate $\alpha > 0$ and a death rate $\omega > 0$, so that the growth rate is $\gamma = \alpha - \omega$. Denote by N the population at time $t = 0$, so that the population at time t is $N \exp(\gamma t)$. All individuals, born and unborn, have the same utility function, and the same individual rate of time preference r. The social planner, operating at time 0, evaluates the expected lifetime utility of individuals at their birth, and aggregates the utility of all individuals, born and unborn, using a Pareto weight $\exp(-\pi s)$ for those born at the time s.

Assume now the planner at time 0 is considering an investment which will result in a public good being available at time t, bringing utility u to all individuals alive at that time. There are $N \exp(\gamma t)$ of them, $Ne^{-\omega t}$ of whom were alive at time 0, and $N\alpha e^{\gamma s} e^{-\omega(t-s)} ds = N\alpha e^{\alpha s} e^{-\omega t} ds$ of whom were born between s and $s + ds$. The social welfare function, evaluated at time 0 is:

$$\left(Ne^{-\omega t} e^{-rt} + \int_0^t e^{-\pi s} N\alpha e^{-\omega t} e^{\alpha s} e^{-r(t-s)} ds \right) u = NR(t)u$$

The planner's discount rate is therefore:

$$R(t) = \frac{\pi - r}{\pi - \alpha - r} e^{-(\omega + r)t} - \frac{\alpha}{\pi - \alpha - r} e^{-(\pi - \gamma)t}, \quad \text{with } \gamma = \alpha - \omega \tag{40}$$

We shall assume that $\pi > \gamma$ (the discount rate applied to future generations' utilities is greater than the growth rate of the population) so that both discount rates on the right-hand side of (40) are positive. We then distinguish three cases:

6.1 Case 1: $\gamma < \pi < \alpha + r$

This is the altruistic case, when we treat future generations not very differently from ourselves. Then $\pi - \gamma < r + \omega$, and we are in the quasi-exponential case (28), with

$$\delta = \pi - \gamma, \quad \rho = r + \omega, \quad \lambda = \frac{\alpha}{\alpha + r - \pi}$$

Clearly $0 < \lambda < 1$, so we are in the hyperbolic case (discount rate decreases with maturity). With these specifications, we find, as above, that the existence interval is $A = \left]\underline{k}, \bar{k}\right[$, with:

$$f'\left(\underline{k}\right) = \lambda\delta + (1 - \lambda)\,\rho = r - \gamma$$
$$f'\left(\bar{k}\right) = \frac{1}{\frac{\lambda}{\delta} + \frac{(1-\lambda)}{\rho}} = \frac{r + \omega}{\pi + \omega}\,(\pi - \gamma)$$

6.2 Case 2: $\pi > r + \alpha$ and $\gamma < r$

This is the case when we discount the utilities of future generations at a much higher rate than our own: the same event, happening at time t and bringing utility u, will be worth $e^{-rt}u$ for me today if the beneficiary is myself, but only $e^{-\pi t}$ if the beneficiary is someone else. Since $\pi > r + \alpha$, we have $r + \omega < \pi - \gamma$, and we are in the quasi-exponential case (28). with

$$\delta = r + \omega, \quad \rho = \pi - \gamma, \quad \lambda = \frac{\pi - r}{\pi - \alpha - r}$$
$$\frac{\rho}{\rho - \delta} = \frac{\pi - \gamma}{\pi - \alpha - r}$$

Clearly $\lambda > 1$, so we are in the non-hyperbolic case (discount rate increases with maturity). If $\gamma < r$, we have $\lambda < \frac{\rho}{\rho - \delta}$, and the existence interval is $A = \left]\underline{k}, \bar{k}\right[$, with:

$$f'\left(\underline{k}\right) = \lambda\delta + (1 - \lambda)\,\rho = r - \gamma$$
$$f'\left(\bar{k}\right) = \frac{1}{\frac{\lambda}{\delta} + \frac{(1-\lambda)}{\rho}} = \frac{r + \omega}{\pi + \omega}\,(\pi - \gamma)$$

6.3 Case 3: $\pi > r + \alpha$ and $\gamma > r$

The inequality $\gamma > r$ means that the growth rate of the population is higher than the individual rate of time preference. This leads to the pathological case when $\lambda > \frac{\rho}{\rho - \delta}$. The existence interval is $A = \left]\underline{k}, \infty\right[$, with:

$$f'\left(\bar{k}\right) = \frac{r + \omega}{\pi + \omega}\,(\pi - \gamma)$$

6.4 Summing up

We have shown that any capital level k_∞ between $\underline{k}$ and $\bar{k}$ (Case 1), between $\bar{k}$ and $\underline{k}$ (Case 2 with $\gamma < r$), or higher than $\bar{k}$ (Case 2 with $\gamma > r$) is the steady state of some equilibrium strategy. We do not know, however, how far away from k_∞ that strategy is defined. In particular, for the planner at time $t = 0$, the range of strategies available in equilibrium will depend on her initial level of capital, so that she may not be able to reach all the $k_\infty \in A$ as steady states.

In addition, when the economy moves closer to k_∞, the initial strategy will be dominated by another one with a lower (in Case 1) or higher (in Case 2) steady state. It is to be expected that, from that time on, future generations will defect from the strategy chosen at time $t = 0$, and that the economy will then move slowly towards $\underline{k}$ (in Case 1), $\bar{k}$ (in Case 2 with $\gamma < r$) or towards infinity (in Case 2 with $\gamma > r$). In other words, the equilibrium strategies we have found are not subgame perfect.

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Appendix A: Proof of Theorem 5

A.1 Preliminaries

Before proceeding with the proof of the theorem, let us mention some facts about the flow $\mathcal{K}$ defined by (20), (21). To make notations simpler, we will use $\mathcal{K}(t,k)$ instead of $\mathcal{K}(\sigma; t, k)$ when the omission of the dependency on σ causes no ambiguity.

Note first that, since Eq. 20 is autonomous, i.e. the right-hand side does not depend explicitly on time, the solution which takes the value k at time 0 coincides with the solution which takes the value $\mathcal{K}(t,k)$ at time $t \geq 0$. This is the well-known semi-group property, namely:

$$\mathcal{K}(s, \mathcal{K}(t,k)) = \mathcal{K}(s+t, k) \tag{41}$$

Next, consider the linearized equation around a prescribed solution $t \to \mathcal{K}(t,k)$ of the nonlinear system (20), namely:

$$\frac{dk_1}{ds} = \left(f'(\mathcal{K}(s,k)) - \sigma'(\mathcal{K}(s,k))\right) k_1(s) \tag{42}$$

This is a linear equation, so the flow is linear. The value at time t of the solution which takes the value k at time 0 is $\mathcal{R}(t)k$, where the function $\mathcal{R} : R \to (0,\infty)$ satisfies:

$$\frac{d\mathcal{R}}{dt} = \left(f'(\mathcal{K}(s,k)) - \sigma'(\mathcal{K}(s,k))\right)\mathcal{R}(t) \tag{43}$$

$$\mathcal{R}(0) = 1. \tag{44}$$

From standard theory, it is well known that, if f and σ are C^k, then $\mathcal{K}$ is C^{k-1}, and:

$$\frac{\partial \mathcal{K}(t,k)}{\partial k} = \mathcal{R}(t)$$

$$\frac{\partial \mathcal{K}(t,k)}{\partial t} = -\mathcal{R}(t)\left(f(k) - \sigma(k)\right).$$

where $\mathcal{R}(t)$ is computed by setting $k_0(t) = \mathcal{K}(t,k)$ in formulas (43) and (44)

Let us now turn to the actual proof of Theorem 5.

A.2 Necessary condition

Given a Markov equilibrium strategy σ, we define the associated value function $v(k)$ as in formula (22)

$$v(k) := \int_0^\infty R(t)u\left(\sigma\left(\mathcal{K}(t,k)\right)\right)dt \tag{45}$$

Differentiating with respect to k, we find that:

$$\begin{aligned} v'(k) &= \int_0^\infty R(t)u'\left(\sigma\left(\mathcal{K}(t,k)\right)\right)\sigma'\left(\mathcal{K}(t,k)\right)\frac{\partial\mathcal{K}}{\partial k}(t,k)dt \\ &= \int_0^\infty R(t)u'\left(\sigma\left(\mathcal{K}\left(t,k\right)\right)\right)\sigma'\left(\mathcal{K}(t,k)\right)\mathcal{R}(t)dt \end{aligned}$$

Since σ is an equilibrium strategy, the maximum of $P_1\left(k,\sigma,c\right)$ with respect to c must be attained at $\sigma(k)$. The function P_1 itself is given by formula (18), where $k_0(t) = \mathcal{K}(t,k)$ and where k_1 is defined by (15) and (16), so that $k_1(t) = \mathcal{R}(t)\left(\sigma(k) - c\right)$. Substituting into (18), we get:

$$\begin{aligned} P_1\left(k,\sigma,c\right) &= u(c) - u\left(\sigma(k)\right) \\ &\quad + \int_0^\infty R(t)u'\left(\sigma\left(\mathcal{K}(t,k)\right)\right)\sigma'\left(\mathcal{K}(t,k)\right)\mathcal{R}(t)\left(\sigma(k)-c\right)dt \end{aligned}$$

Since u is concave and differentiable, the necessary and sufficient condition to maximize $P_1(k,\sigma,c)$ with respect to c is

$$u'(c) = \int_0^\infty R(t)u'\left(\sigma\left(\mathcal{K}\left(t,k\right)\right)\right)\sigma'\left(\mathcal{K}(t,k)\right)\mathcal{R}(t)dt$$

which is precisely $v'(t,k)$, as we just saw. Therefore, the equilibrium strategy must satisfy

$$u'\left(\sigma(k)\right) = v'(k)$$

and, substituting back into Eq. 45, we get Eq. IE.

A.3 Sufficient condition

Assume now that there exists a function v satisfying (IE), and consider the strategy $\sigma = i \circ v'$. Given any consumption choice $c \in R$, the payoff to the decision-maker at time 0 is

$$\begin{aligned} P_1\left(k,\sigma,c\right) &= u(c) - u\left(\sigma(k)\right) \\ &\quad + \int_0^\infty R(t)u'\left(\sigma\left(\mathcal{K}(t,k)\right)\right)\sigma'\left(\mathcal{K}(t,k)\right)\mathcal{R}(t)\left(\sigma(k)-c\right)dt \\ &= u(c) - u\left(\sigma(k)\right) + v'(k)\left(\sigma(k)-c\right) \\ &= u(c) - u\left(\sigma(k)\right) - u'\left(\sigma(k)\right)\left(c-\sigma(k)\right) \\ &\leq 0, \end{aligned}$$

where the first equality follows from the definition of $\mathcal{R}$, the second equality is obtained by differentiating v with respect to k, the third equality follows from the definition of σ, and the last inequality is due to the concavity of u. Observing that $P_1(k,\sigma,\sigma(k)) = 0$, we see that the inequality $P_1(k,\sigma,c) \leq 0$ implies that $c = \sigma(k)$ achieves the maximum so that σ is an equilibrium strategy.

Appendix B: Proof of Theorem 6

Let a function $v : R \to R$ be given. Consider the function $\varphi : R \to R$ defined by

$$\varphi(k) = v(k) - \int_0^\infty R(t)u\left(\sigma\left(\mathcal{K}\left(\sigma; t, k\right)\right)\right) dt \tag{46}$$

where $\sigma(k) = i\left(v'(k)\right)$. Consider $\psi(t, k)$, the value of φ along the trajectory $t \longrightarrow \mathcal{K}\left(\sigma; t, k\right)$ originating from k at time 0, that is

$$\begin{aligned}\psi(t, k) &= \varphi\left(\mathcal{K}\left(t, k\right)\right)\\ &= v\left(\mathcal{K}(t, k)\right) - \int_0^\infty R(s)u\left(\sigma\left(\mathcal{K}\left(s, \mathcal{K}\left(t, k\right)\right)\right)\right) ds\\ &= v\left(\mathcal{K}(t, k)\right) - \int_0^\infty R(s)u\left(\sigma\left(\mathcal{K}\left(s+t, k\right)\right)\right) ds\\ &= v\left(\mathcal{K}(t, k)\right) - \int_t^\infty R(s-t)u\left(\sigma\left(\mathcal{K}\left(s, k\right)\right)\right) ds\end{aligned}$$

where we have used formula (41).

We compute the derivative of this function with respect to t:

$$\begin{aligned}\frac{\partial\psi}{\partial t}(k, t) &= v'\left(\mathcal{K}(t, k)\right)\left[f\left(\mathcal{K}\left(t, k\right)\right) - i\left(\sigma\left(\mathcal{K}(t, k)\right)\right)\right]\\ &\quad+u\left(\sigma\left(\mathcal{K}(t, k)\right)\right) + \int_t^\infty R'(s-t)u\left(\sigma\left(\mathcal{K}\left(s, k\right)\right)\right) ds\end{aligned}$$

From the definition of i, we have

$$u\left(i\left(v'\left(\mathcal{K}(t, k)\right)\right)\right) - v'\left(\mathcal{K}\left(t, k\right)\right) i\left(v'\left(\mathcal{K}\left(t, k\right)\right)\right) = \sup_c\left\{u(c) - v'\left(\mathcal{K}\left(t, k\right)\right) c\right\}$$

Substituting in the preceding equation, and recalling that $\sigma = i \circ v'$ gives

$$\begin{aligned}\frac{\partial\psi}{\partial t}(k, t) &= \sup_c\left\{u(c)+v'\left(\mathcal{K}\left(t, k\right)\left(f\left(\mathcal{K}(t, k)\right) - c\right)\right)\right\} + \int_t^\infty R'(s-t)u\left(\sigma\left(\mathcal{K}\left(s, k\right)\right)\right) ds\\ &= \sup_c\left\{u(c)+v'\left(\mathcal{K}\left(t, k\right)\left(f\left(\mathcal{K}(t, k)\right) - c\right)\right)\right\} + \int_0^\infty R'(s)u\left(\sigma\left(\mathcal{K}\left(s, \mathcal{K}(t, k)\right)\right)\right) ds\end{aligned}$$

where the second equality is obtained by using a change of variable and formula (41). If (DE) holds, then the right-hand side of the last equation is identically zero along the trajectory, so that $\psi(k, t) = \psi(k)$ does not depend on t. Letting $t \longrightarrow \infty$ in the definition of ψ, we get:

$$\begin{aligned}\psi(k) &= \lim_{t\longrightarrow\infty}\left\{v\left(\mathcal{K}(t, k)\right) - \int_0^\infty R(s)u\left(\sigma\left(\mathcal{K}\left(s+t, k\right)\right)\right) ds\right\}\\ &= v\left(\bar{k}\right) - \int_0^\infty R(s)u\left(\sigma\left(\bar{k}\right)\right) ds = v\left(\bar{k}\right) - u\left(f\left(\bar{k}\right)\right)\int_0^\infty R(s)ds\end{aligned}$$

and hence, if (BC) holds then, $\psi = \varphi = 0$ and Eq. IE holds.

Conversely, if $v(k)$ satisfies Eq. IE, then the same lines of reasoning shows that Eq. DE and the boundary condition are satisfied.

Appendix C: Proof of Proposition 7

We begin with a Lemma:

Lemma 12 *Let $\sigma(k)$ be any convergent Markov strategy. Denote its steady state by k_∞. Let $h : [0, \infty] \longrightarrow R$ be any C^1 function with exponential decay at infinity. The following are equivalent:*

$$I(k) = \int_0^\infty h(t)u\left(\sigma\left(\mathcal{K}\left(\sigma; t, k\right)\right)\right) dt \tag{47}$$

$$I'(k)\left(f(k) - \sigma(k)\right) + \int_0^\infty h'(t)u\left(\sigma\left(\mathcal{K}\left(\sigma; t, k\right)\right)\right) dt + h(0)u\left(\sigma(k)\right) = 0$$

$$I\left(k_\infty\right) = \int_0^\infty h(t)\ln f\left(k_\infty\right) \tag{48}$$

Proof Consider the function $\psi(k, t)$ defined by:

$$\psi(k, t) = I\left(\mathcal{K}(t, k)\right) - \int_t^\infty h(s-t)u\left(\sigma\left(\mathcal{K}\left(\sigma; s, k\right)\right)\right) ds$$

Differentiating with respect to t we have:

$$\frac{\partial \psi}{\partial t} = I'\left(\mathcal{K}\left(\sigma; t, k\right)\right)\left(f\left(\mathcal{K}\left(\sigma; t, k\right) - \sigma\left(\mathcal{K}\left(\sigma; t, k\right)\right)\right)\right) + \int_t^\infty h'(s-t)u\left(\sigma\left(\mathcal{K}\left(\sigma; s, k\right)\right)\right) ds + h(0)u\left(\sigma(k)\right)$$

If (47) holds, then ψ is identically zero, and so is its derivative $\frac{\partial \psi}{\partial t}$, so (48) holds. Conversely, if (48) holds, then $\frac{\partial \psi}{\partial t}$ vanishes, and $\psi(k, t) = \psi(k)$ does not depend on t, so that:

$$\psi(k) = \lim_{t \longrightarrow \infty}\left\{I\left(\mathcal{K}(t, k)\right) - \int_t^\infty h(s-t)u\left(\sigma\left(\mathcal{K}\left(\sigma; t, k\right)\right)\right) ds\right\} = I\left(k_\infty\right) - \int_0^\infty h(s)u\left(k_\infty\right) dt$$

as desired. □

Now for the proof of Proposition 7.

Proof Suppose that $v(k)$satisfies (DE), (BC) and $\sigma(k) = 1/v'(k)$ converges to some k_∞. Substituting the expression of $R(t)$ into (DE), we get:

$$\sup_c\left[u(c)+v'(k)\left(f(k)-c\right)\right] = -\int_0^\infty\left[\left(\lambda e^{-\delta t} + (1-\lambda)\, e^{-\rho t}\right)\right]\ln\left(\sigma\left(\mathcal{K}\left(\sigma; t, k\right)\right)\right) dt \tag{49}$$

Setting $u(c) = \ln c$ in the left-hand side of (49) leads to:

$$\sup_c\left[\ln c + v'(k)\left(f(k) - c\right)\right] = \left(f(k) - \frac{1}{v'(k)}\right) v'(k) - \ln v'(k)$$

Define $v(k)$ and $w(k)$ as in formulas (36) and (37). Equation 49 becomes:

$$\left(f(k) - \frac{1}{v'}\right) v' - \ln v' = av(k) + bw(k) \tag{50}$$

Writing this into Eq. 50 gives (32). To get (33), we use Lemma 12 with

$$\sigma(k) = i\left(v'(k)\right) = 1/v'(k)$$
$$h(t) = \lambda e^{-\delta t} - (1-\lambda)\, e^{-\rho t}$$

so that $h(0) = (2\lambda - 1)$. We get an equation for $w(k)$:

$$w'(k)\left(f(k) - \frac{1}{v'(k)}\right) - \int_0^\infty \left(\lambda\delta e^{-\delta t} - (1-\lambda)\,\rho e^{-\rho t}\right) u\left(\sigma\left(\mathcal{K}(\sigma; t, k)\right)\right) dt - (2\lambda - 1)\ln v' = 0 \tag{51}$$

which is easily seen to coincide with (33), and a boundary condition

$$w\left(k_\infty\right) = \left(\frac{\lambda}{\delta} - \frac{1-\lambda}{\rho}\right)\ln f\left(k_\infty\right)$$

So v and w satisfy the Eqs. 32 and 33, together with the boundary conditions (34) and (35).

Conversely, suppose v_1 and w_1 satisfy the equations and the boundary conditions, with the strategy $\sigma_1 = 1/v_1'$ converging to k_∞, so that:

$$\begin{aligned}\left(f - \frac{1}{v_1'}\right) v_1' - \ln\left(v_1'\right) &= av_1 + bw_1\\ v_1\left(k_\infty\right) &= \left(\frac{\lambda}{\delta} + \frac{1-\lambda}{\rho}\right)\ln f\left(k_\infty\right)\\ \left(f - \frac{1}{v_1'}\right) w_1' &= bv_1 + a_1 w_1\\ w_1\left(k_\infty\right) &= \left(\frac{\lambda}{\delta} - \frac{1-\lambda}{\rho}\right)\ln f\left(k_\infty\right)\end{aligned} \tag{52}$$

Consider the functions:

$$v_2(k) = \int_0^\infty \left(\lambda e^{-\delta t} + (1-\lambda)\, e^{-\rho t}\right)\ln\left(\sigma_1\left(\mathcal{K}\left(\sigma_1; t, k\right)\right)\right) dt \tag{53}$$

$$w_2(k) = \int_0^\infty \left(\lambda e^{-\delta t} - (1-\lambda)\, e^{-\rho t}\right)\ln\left(\sigma_1\left(\mathcal{K}\left(\sigma_1; t, k\right)\right)\right) dt \tag{54}$$

Applying Lemma 12 with $I = v_2$ and $I = w_2$ successively, we have:

$$v_2'(k)\left(f - \sigma_1\right) + \int_0^\infty \left(\lambda e^{-\delta t} + (1-\lambda)\, e^{-\rho t}\right)\ln\left(\sigma_1\left(\mathcal{K}\left(\sigma_1; t, k\right)\right)\right) dt + \ln\left(\sigma_1(k)\right) = 0$$
$$v_2\left(k_\infty\right) = \left(\frac{\lambda}{\delta} + \frac{1-\lambda}{\rho}\right)\ln f\left(k_\infty\right)$$
$$w_2'(k)\left(f - \sigma_1\right) + \int_0^\infty \left(\lambda e^{-\delta t} - (1-\lambda)\, e^{-\rho t}\right)\ln\left(\sigma_1\left(\mathcal{K}\left(\sigma_1; t, k\right)\right)\right) dt = 0$$
$$w_2\left(k_\infty\right) = \left(\frac{\lambda}{\delta} - \frac{1-\lambda}{\rho}\right)\ln f\left(k_\infty\right)$$

and hence:

$$\begin{aligned}v_2'(k)\left(f - \sigma_1\right) + \ln\left(\sigma_1(k)\right) &= a_1 v_2 + b_1 w_2\\ w_2'\left(f - \sigma_1\right) &= a_2 v_2 + b_2 w_2\end{aligned} \tag{55}$$

Substracting (55) from (52), and setting $v = v_1 - v_2,\ w = w_1 - w_2$, we get:

$$\begin{aligned}\left(f - \frac{1}{v_1'}\right) v' &= av + bw \\ v\left(k_\infty\right) &= 0 \\ \left(f - \frac{1}{v_1'}\right) w' &= bv + aw \\ w\left(k_\infty\right) &= 0\end{aligned} \tag{56}$$

Obviously $v = w = 0$ is a solution. In Lemma 13 (see below), we show that it is the only one, so $v_1 = v_2$ and $w_1 = w_2$. Equation 53 then becomes:

$$v_1(k) = \int_0^\infty \left(\lambda e^{-\delta t} + (1-\lambda)\, e^{-\rho t}\right) \ln\left(\sigma_1\left(\mathcal{K}\left(\sigma_1; t, k\right)\right)\right) dt$$

which is precisely Eq. IE. Since v_1 satisfies (IE), it satisfies (DE) and (BC). □

Lemma 13 *If* (v, w) *is a pair of continuous functions, which are continuous on a neighbourhood* Ω *of* k_∞*, continously differentiable for* $k \neq k_\infty$*, and which solve* (56) *for* $k \neq k_\infty$*, then* $v = 0$ *and* $w = 0$*.*

Proof Set $f(k) - 1/v_1'(k) = \varphi(k)$, and note that $\varphi(k) \to \infty$ when $k \to k_\infty$. The system can be rewritten as:

$$\begin{pmatrix}\varphi v' \\ \varphi w'\end{pmatrix} = \begin{matrix} a & b \\ b & a\end{matrix} \begin{pmatrix} v \\ w\end{pmatrix}$$

The characteristic equation of the matrix on the right-hand side is:

$$\lambda^2 - 2a\lambda + a^2 - b^2 = 0$$

and the roots are $\lambda = \frac{a+b}{2} = \delta$ and $\lambda = \frac{a-b}{2} = \rho$ Changing basis, we can rewrite the system as:

$$\begin{pmatrix}\varphi V' \\ \varphi W'\end{pmatrix} = \begin{pmatrix} \rho & 0 \\ 0 & \delta\end{pmatrix} \begin{pmatrix} V \\ W\end{pmatrix}$$

where V and W are suitable linear combinations of v and w. The solutions are:

$$V(k) = C_1 \exp\left(\frac{\rho}{\varphi(k)}\right), \quad W(k) = C_2 \exp\left(\frac{\delta}{\varphi(k)}\right)$$

where C_1 and C_2 are constants. Since $1/\varphi(k) \to \pm\infty$ when $k \to k_\infty$, and both signs occur, one for $k < k_\infty$ and the other for $k > k_\infty$, the only way we can get $V\left(k_\infty\right) = W\left(k_\infty\right) = 0$ is by setting $C_1 = C_2 = 0$. □

Appendix D: Proof of Theorem 8

D.1 Part (a)

Assume that there exists a local equilibrium strategy $\sigma(k) = 1/v'(k)$ converging to some k_∞, with v and w of class C^2.

The evolution of the capital stock is given b:

$$\frac{dk}{dt} = f(k) - \frac{1}{v'(k)}$$

and since v is C^2, we may linearize it around k_∞ to get:

$$\frac{dx}{dt} = \left(f'(k_\infty) + \frac{v''(k_\infty)}{v'(k_\infty)^2} \right) x$$

Convergence to k_∞ requires that

$$f'(k_\infty) + \frac{v''(k_\infty)}{v'(k_\infty)^2} \leq 0 \tag{57}$$

We have to compute $v'(k_\infty)$ and $v''(k_\infty)$. The boundary conditions (34) and (35) give:

$$av(k_\infty) + bw(_\infty) = \ln f(k_\infty)$$

Equation 32 then gives $f(k_\infty) v'(k_\infty) - 1 - \ln v'(k_\infty) = \ln f(k_\infty)$, the only solution of which is

$$v'(k_\infty) = \frac{1}{f(k_\infty)}$$

Differentiating (32), and evaluating it at $k = k_\infty$, we get:

$$w'(k_\infty) = \frac{\left(f'(k_\infty) - a\right)}{bf(k_\infty)}$$

To compute the second derivative of v at k_∞, we differentiate (33), using the fact that v and w are C^2, and we evaluate it at $k = k_\infty$. We get:

$$\frac{v''(k_\infty)}{v'(k_\infty)^2} \left[f'(k_\infty) - a - (2\lambda - 1) b \right] = b^2 - \left(a - f'(k_\infty)\right)^2$$

and hence:

$$f'(k_\infty) + \frac{v''(k_\infty)}{v'(k_\infty)^2} = \frac{(a - (2\lambda - 1) b) f'(k_\infty) + b^2 - a^2}{f'(k_\infty) - a - (2\lambda - 1) b}$$

Substituting:

$$\frac{a^2 - b^2}{(a - (2\lambda - 1) b)} = \frac{1}{\frac{\lambda}{\delta} + \frac{1-\lambda}{\rho}}$$
$$a + (2\lambda - 1) b = \lambda\delta + (1 - \lambda) \rho$$

we find that inequality (57) can be rewritten as follows:

$$\frac{f'(k_\infty) - \frac{1}{\frac{\lambda}{\delta} + \frac{1-\lambda}{\rho}}}{f'(k_\infty) - \lambda\delta + (1 - \lambda) \rho} \leq 0$$

which will be satisfied provided $f'(k_\infty)$ lies between the roots of the numerator and denominator. In other words, we must have $k_\infty \in A$. Note (38) and (39) for future reference.

D.2 Part (b)

We will prove that the system (32), (33) has a C^2 solution (v, w) satisfying (34), (35) and defined in some neighbourhood Ω of k_∞. By Proposition 7, this will prove that v satisfies (DE) and (BC) on Ω. If in addition $\underline{k} < k_\infty < \bar{k}$, the equilibrium strategy $\sigma(k) = 1/v'(k)$ converges to k_∞, as we observed in the preceding subsection, so that it is a local equilibrium strategy converging to k_∞, as announced.

Introduce a new function $\mu(k)$ defined by:

$$\mu(k) := av(k) + bw(k) - \ln f(k)$$

Note that, according to (34) and (35):

$$\mu(k_\infty) = 0$$

Take $(v(k), \mu(k))$ instead of $(v(k), w(k))$ as unknowns. The first equation becomes:

$$fv' - 1 - \ln fv' = \mu$$

The equation $x - \ln(1 + x) = \mu$ has two solutions, $x_-(\mu) < 0 < x_+(\mu)$ for $\mu > 0$, none for $\mu < 0$, and the single solution $x = 0$ for $\mu = 0$. In other words, there are two smooth branches coming out of 0 (note that $x_-(\mu)$ behaves like $-\sqrt{\mu}$ and $x_+(\mu)$ like $\sqrt{\mu}$ for μ close to 0). Dropping the subscripts $+$ and $-$, we find that the system (32), (33) describes in fact two dynamical systems in the region $\mu > 0$, that is:

$$av(k) + bw(k) - \ln f(k) > 0$$

while the region $\mu < 0$ is forbidden. The two dynamical systems are described by the equations:

$$\frac{dv}{dk} = \frac{1 + x(\mu)}{f(k)} \quad \mu(k) \geq 0 \tag{58}$$

$$\frac{d\mu}{dk} = \frac{1}{f(k)} \frac{1 + x(\mu)}{x(\mu)} D_0(\mu, v, k) + a\frac{1 + x(\mu)}{f(k)} - \frac{f'(k)}{f(k)} \tag{59}$$

$$D_0(\mu, v, k) = \left(b^2 - a^2\right) v + a\mu + [a - (2\lambda - 1) b] \ln f(k) + (2\lambda - 1) b \ln(1 + x) \tag{60}$$

where x is either x_+ or x_- according to which determination is chosen. We are looking for a pair $(v(k), \mu(k))$ of solutions of (58), (59), defined and C^2 in a neighbourhood of k_∞, with $\mu(k) \geq 0$ in that neighbourhood and $\mu(k_\infty) = 0$.

We will get rid of the indeterminacy between x_+ or x_- by taking x as the independent variable. Then

$$\mu = x - \ln(1 + x)$$

$$d\mu = \left(1 - \frac{1}{1 + x}\right) dx = \frac{x}{1 + x} dx$$

becomes a function of x, and the Eqs. 58, 59 and 60 then become:

$$\frac{dk}{dx} = f(k) \frac{x^2}{1 + x} \frac{1}{D(x, v, k)}, \tag{61}$$

$$\frac{dv}{dx} = x^2 \frac{1}{D(x, v, k)} \tag{62}$$

with:

$$D(x,v,k) = (1+x) D_0(\mu,v,k) + ax(1+x) - xf'(k)$$
$$= (1+x)\left[2ax + [(2\lambda-1)b - a]\ln\frac{(1+x)}{f(k)} + \left(b^2-a^2\right)v\right] - xf'(k)$$

Note that, by the boundary condition (34):

$$D(0,v_\infty,k_\infty) = (a-(2\lambda-1)b)\ln f(k_\infty) + \left(b^2-a^2\right)v_\infty$$
$$= \left[a-(2\lambda-1)b + \left(b^2-a^2\right)\left(\frac{\lambda}{a+b}+\frac{1-\lambda}{a-b}\right)\right]\ln f(k_\infty) = 0$$

We are looking for a solution of that system such that $k(0) = k_\infty$. But for $x = 0$ and $k = k_\infty$, both the numerator and denominator on the right-hand sides of (61) and (62) vanish. In other words, we are dealing with a singularity. To blow up the singularity, we introduce a new variable s, and consider the new system (which is now autonomous and three-dimensional):

$$\frac{dx}{ds} = D(x,k,v), \quad x(0) = 0 \tag{63}$$
$$\frac{dk}{ds} = f(k)\frac{x^2}{1+x}, \quad k(0) = k_\infty \tag{64}$$
$$\frac{dv}{ds} = x^2, \quad v(0) = v_\infty \tag{65}$$

where there are now three unknown functions $(x(s), k(s), v(s))$, defined near $s = 0$. We have $D(0, v_\infty, k_\infty) = 0$.

The linearized system near $(0, k_\infty, v_\infty)$ is:

$$\frac{d}{ds}\begin{pmatrix} x \\ v \\ k \end{pmatrix} = A\begin{pmatrix} x \\ k \\ v \end{pmatrix} \tag{66}$$

with (all derivatives to be computed at $(0, k_\infty, v_\infty)$):

$$A := \begin{pmatrix} \frac{\partial D}{\partial x} & \frac{\partial D}{\partial v} & \frac{\partial D}{\partial k} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\frac{\partial D}{\partial x}(0,k_\infty,v_\infty) = a + (2\lambda-1)b - f'(k_\infty)$$
$$\frac{\partial D}{\partial v}(0,k_\infty,v_\infty) = b^2 - a^2$$
$$\frac{\partial D}{\partial k}(0,k_\infty,v_\infty) = -[(2\lambda-1)b - a]f'(k_\infty)$$

The matrix D has at most one non-zero eigenvalue.

D.2.1 Case 1: $a + (2\lambda-1)b \neq f'(k_\infty)$

In that case, the eigenspace associated with the zero eigenvalues is the plane E_0 defined by the equation:

$$\left(a + (2\lambda-1)b - f'(k_\infty)\right)x + \left(b^2-a^2\right)v - ((2\lambda-1)b - a)f'(k_\infty)k = 0$$

The matrix can obviously be put in diagonal form. By the central manifold theorem (see for instance [1], Theorem 1),[2] there exists a map $h(v, k)$, defined in a neighborhood $\mathcal{O}$ of (v_∞, k_∞) such that $h(v_\infty, k_\infty) = 0$ and the manifold $\mathcal{M}$ defined by the equation $x = h(v, k)$ is invariant by the flow. In other words, the function $h(v, k)$ satisfies the functional identity:

$$h^2 \frac{\partial h}{\partial v} + f(k) \frac{h^2}{1+h} \frac{\partial h}{\partial k} = D(h, v, k)$$

The map h and the central manifold $\mathcal{M}$ are as smooth as f': they are C^2, for instance, if f is C^3. Note that $\mathcal{M}$ is two-dimensional and tangent to E_0 at $(0, v_\infty, k_\infty)$:

$$\frac{\partial h}{\partial v}(v_\infty, k_\infty) = -\frac{b^2 - a^2}{a + (2\lambda - 1)b - f'(k_\infty)}$$
$$\frac{\partial h}{\partial k}(v_\infty, k_\infty) = \frac{(2\lambda - 1)b - a}{a + (2\lambda - 1)b - f'(k_\infty)} f'(k_\infty).$$

Note also that there is another invariant manifold N, which is tangent to the eigenvector associated with the non-zero eigenvalue. Each of these invariant manifolds gives a different type of solution to the differential system.

We are interested in the solutions which lie on the central manifold $\mathcal{M}$. They can be found by substituting $x = \alpha k + \beta v + h(k, v)$ in the equations, yielding

$$\frac{dk}{ds} = f(k) \frac{h(k, v)^2}{1 + h(k, v)}, \quad k(0) = k_\infty \tag{67}$$
$$\frac{dv}{ds} = h(k, v)^2, \quad v(0) = v_\infty \tag{68}$$

while x is found by substituting $x(s) = h(k(s), v(s))$.

Eliminating the variable s from (67) and (68), we get

$$\frac{dv}{dk} = \frac{f(k)}{1 + \alpha k + \beta v + h(k, v)}, \quad v(k_\infty) = v_\infty$$

The solution of this initial-value problem is $v(k) = \psi(k)$, where $\psi(k_\infty) = v_\infty$ and ψ is C^2 if h is C^2, that is, if f is C^3. Substituting in $x = h(k, v)$, we get $x(k) = \alpha k + \beta v + h(k, \psi(k))$. Finally, $\mu(k) = x(k) - \ln(1 + x(k))$ is also C^2, and so is $bw(k) = \mu(k) - av(k) + \ln f(k)$.

D.2.2 Case 2: $a + (2\lambda - 1)b = f'(k_\infty)$

In that case, the computations change somewhat. Substituting in the equations, we get:

$$D(x, v, k) = (1 + x)\left[2ax + \left(f'(k_\infty) - 2a\right)\ln\frac{(1+x)}{f(k)} + \left(b^2 - a^2\right)v\right] - xf'(k)$$
$$= \alpha(x, v.k)\,x^2 + \beta(x, v.k)(k - k_\infty) + \gamma(x, v.k)(v - v_\infty)$$

[2] For the sake of commodity, Carr requires that the linearized matrix at the fixed point be block diagonal.

where α, β, γ are smooth functions which are non-zero.at $(0, v_\infty, k_\infty)$. Proceeding as above, we introduce a new variable s, and we replace the system (63), (64) (65) with the following:

$$\frac{dx}{ds} = \alpha(x, v.k)\, x^2 + \beta(x, v.k)(k - k_\infty) + \gamma(x, v.k)(v - v_\infty), \quad x(0) = 0 \tag{69}$$

$$\frac{dk}{ds} = f(k)\frac{x^2}{1+x}, \quad k(0) = k_\infty \tag{70}$$

$$\frac{dv}{ds} = x^2, \quad v(0) = v_\infty \tag{71}$$

where there are now three unknown functions $(x(s), k(s), v(s))$, defined near $s = 0$. We can no longer apply the central manifold theorem, because all the eigenvalues of the linearized system vanish. So we introduce a new change of variables, namely:

$$K(s) := \frac{k(s) - k_\infty}{x^2(s)}, \quad V(s) := \frac{v(s) - v_\infty}{x^2(s)}$$

which yields:

$$\frac{dK}{ds} = \frac{d}{ds}\frac{k - k_\infty}{x^2} = \frac{f(k)}{1+x} - 2\alpha x K - 2\beta x K^2 - 2\gamma(1+x)xKV$$

and:

$$\frac{dV}{ds} = \frac{d}{ds}\frac{v - v_\infty}{x^2} = 1 - 2\alpha x V - 2\beta x K V - 2\gamma(1+x)xV^2$$

so that the system (69), (70), (71) becomes:

$$\frac{dx}{ds} = \alpha x^2 + \beta x^2 K + (1+x)\gamma x^2 V, \quad x(0) = 0 \tag{72}$$

$$\frac{dK}{ds} = \frac{f}{1+x} - 2\alpha x K - 2\beta x K^2 - 2\gamma(1+x)xKV, \quad K(0) = K_\infty \tag{73}$$

$$\frac{dV}{ds} = 1 - 2\alpha x V - 2\beta x K V - 2\gamma(1+x)xV^2, \quad V(0) = V_\infty \tag{74}$$

with $f = f\left(k_\infty + x^2 K\right), \alpha = \alpha\left(x, v_\infty + x^2 V, k_\infty + x^2 K\right)$, and so forth. The system (72), (73), (74) clearly has no zeroes for $x = 0$, so that, by the standard Cauchy-Lipschitz theorem for solving ordinary differential equations, it has a unique (local) solution for any values $K_\infty > 0$, V_∞, and the solution depends smoothly on the initial values. Any such solution $(x(s), K(s), V(s))$ will yield a solution $(x(s), k(s), v(s))$ of the original system (69), (70), (71) by setting:

$$k(s) = K_\infty + x(s)^2 K(s), \quad v(s) = V_\infty + x(s)^2 V(s) \tag{75}$$

Since $\frac{dK}{ds}(0) = f(k_\infty) \neq 0$, one can take K instead of s as the independent variable near $s = 0$ (corresponding to $K = K_\infty$), thereby getting new functions $\tilde{x}(K)$ and $\tilde{V}(K)$ instead of $x(s)$ and $V(s)$. The equations become:

$$\frac{d\tilde{x}}{dK} = \frac{\alpha\tilde{x}^2 + \beta\tilde{x}^2 K + \gamma(1+\tilde{x})\tilde{x}^2\tilde{V}}{\frac{f}{1+\tilde{x}} - 2\alpha\tilde{x}K - 2\beta\tilde{x}K^2 - 2\gamma(1+\tilde{x})\tilde{x}K\tilde{V}}, \quad \tilde{x}(K_\infty) = 0 \tag{76}$$

$$\frac{d\tilde{V}}{dK} = \frac{1 - 2\alpha\tilde{x}\tilde{V} - 2\beta\tilde{x}K\tilde{V} - 2\gamma(1+\tilde{x})\tilde{x}\tilde{V}^2}{\frac{f}{1+\tilde{x}} - 2\alpha\tilde{x}K - 2\beta\tilde{x}K^2 - 2\gamma(1+\tilde{x})\tilde{x}K\tilde{V}}, \quad \tilde{V}(K_\infty) = V_\infty \tag{77}$$

The right-hand side is smooth at $(0, K_\infty, V_\infty)$ and $\frac{d\tilde{V}}{dK} = 1$ at this point. So problem (76), (77) has a unique (local) solution, and it is as smooth as the right-hand side, that is to say, as smooth as f'' (because of the function $\delta(k)$), so we need $f(k)$ to be C^2. If $\left(\tilde{x}(K), \tilde{V}(K)\right)$ is such a solution, we write it back into the formulas (75):

$$v(K) = v_\infty + \tilde{x}(K)^2\tilde{V}(K)$$
$$k(K) = k_\infty + \tilde{x}(K)^2 K$$

so that:

$$\frac{v(K) - v_\infty}{k(K) - k_\infty} = \frac{\tilde{V}(K)}{K} \longrightarrow \frac{1}{f(k_\infty)} \quad \text{when } K \longrightarrow 0$$

It follows from the inverse function theorem that v is a smooth function of k, with:

$$v(k_\infty) = v_\infty, \quad \frac{dv}{dk}(k_\infty) = \frac{1}{f(k_\infty)}$$

To sum up, we have shown that in the case where $\lambda = 1/2$, there is a continuum of smooth (at least C^2) $(k(x), v(x))$ solutions of the system (61), (62), each of whom leads to a smooth pair of function $(v(k), w(k))$ satisfying (32), (33), with $\sigma(k_\infty) = 0$ and $v(k_\infty) = v_\infty$.

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