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The geometry of global production and factor price equalisation

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A R T I C L E I N F O

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A B S T R A C T

We consider a production economy where commodities are partitioned into K irreproducible factors and L reproducible goods, and the production technologies have constant returns to scale. We examine the geometry of the global production set in the space of commodities, and we derive theorems of non-substitution type. We define the "factors values" of the different goods, we use them to characterize the efficient production plans, and we investigate in detail the relations between the prices of goods and the prices of factors. We show that the prices of factors uniquely determine the prices of goods, and that, generically, equalising the prices of $2K$ goods equalises the prices of factors.

1. Introduction

We consider in this paper, an *abstract* economy in which commodities can be partitioned into L produced *goods* and K non-reproducible *factors* of production, henceforth referred to simply as 'goods' and 'factors'. Goods are produced from goods and factors. The production sector, on which we focus attention, consists of a collection of technologies with *constant returns to scale*, and *without joint production*.

Our investigation aims at improving upon the present understanding of the geometry of the global production set in the commodity space R^{K+L}. Particular attention will be paid to the subset of *non-specialized* and *efficient* production plans. The analysis is conducted both in the *primal* setting, the space of commodities, where we look at the global production set Y and its frontier, and in the *dual* setting, the space of prices, where we investigate the set of price systems which support efficient production. Combining the primal and dual viewpoints enables us to provide both a geometric and analytical characterization of non-specialized and efficient production plans. We define the *factors* values of each good in order to interpret our findings. We then investigate the connection between prices of goods and the prices of factors. We show that the prices of factors *globally* and *uniquely* determine the prices of goods, and we then revisit in our setting the problem known in the literature as *factor price equalisation*.

This paper undoubtedly belongs to the field of production theory. It can however be related with a larger set of preoccupations which have surfaced in the economic literature over a long period. Indeed, special cases of the abstract model we are considering appear in different lines of research, some of which, associated for example with the names of Sraffa, Leontieff, and Heckscher–Ohlin, are relatively old. Like Leontieff and Sraffa (1960), we focus attention on the 'production of commodities by means of commodities' with constant returns to scale. Like Heckscher and Ohlin, and the standard models

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of international trade based on their work (see for example Dixit and Norman, 1980 and the references therein), we emphasize the dichotomy between factors and goods. Indeed, our model can be viewed as the production sector of a generalized Heckscher–Ohlin economy, in which the number of goods or factors is not restricted to two and in which production requires, not only factors, but intermediary goods as well.

Although these lines of research have developed outside the Walrasian tradition (and sometimes against it), the concerns of the paper can also be related with the present state of general equilibrium theory. On the consumption side, since the discussion on the Sonnenschein–Mantel–Debreu Theorem (see Shafer and Sonnenschein, 1982 and the references therein) in the 1980s, it has been recognized that the abstract theory used in general equilibrium is, somehow, too general; effort has then been directed towards developing more focused settings which may be empirically more relevant, for instance in the direction indicated by Hildenbrand (1994). In a similar way, the analysis presented in this paper can be viewed as part of a research program aimed at constructing a *theory of production of intermediate scope*, lying somewhere between the most general Arrow–Debreu model and the simplified and sometimes simplistic modelling of production adopted in fields of application.

Fields of applications that might potentially benefit from a theory of production of intermediate generality include macroeconomics and growth theory. More obvious examples of possible fruitful connections are for example, computable general equilibrium models and trade theory. *Computable general equilibrium models*, which are widely used in applied research, generally have constant returns to scale production sectors, often fitting, as special cases, the framework of our model. Also, the present development of international trade has revived the interest for the phenomenon of factor price equalisation.[1] The present production model provides *a core building block of elementary trade theory*, that would take into account what Ethier (1984) called "higher dimensional issues".[2]

The results of the present paper are as follows:

- Section 2 presents the model and the objects under scrutiny.
- Section 3.1 takes what we call the primal viewpoint. It provides a geometrical description of the set of efficient and non-specialized production plans. We view Theorem 3 as a general non-substitution theorem. In the case there is only one factor of production, and this factor is available in a given quantity v, the classical Arrow–Samuelson Theorem states that the set $Y_{NSE}(.,v)$ of goods bundles which can be produced efficiently, in a non-specialized way, is a portion of a hyperplane in the space of L goods. With K factors, $K < L$, Theorem 3 generalizes this result: it states that the set $Y_{NSE}(.,v)$ then consists of a union of $(L-K)$-dimensional "facets". If $K = 1$, we get the Arrow–Samuelson Theorem back. If $1 < K < L$, we get constraints on the geometry of $Y_{NSE}(.,v)$.
- Section 3.2 switches from the primal viewpoint of Section 3.1 to the dual viewpoint. It investigates the set of remuneration plans which support efficient and non-specialized production plans. In our context, the analysis rests upon the key concept of the "factor values" matrix: the factor k value of good ℓ is the total amount of factor k directly or indirectly used in the production of good ℓ. The price of good ℓ is the sum over factors of the factor price multiplied by the factor value. Preliminary connections between the algebraic dual viewpoint and the geometric primal viewpoints are exhibited.
- Section 4 makes a detailed analysis of the connections between the primal and the dual viewpoints introduced in Section 3.
- Section 4.1 is centered on Theorem 13 which states that the (relative) prices of factors determine the (relative) prices of goods globally and uniquely through a mapping that we denote by φ. The local property according to which, in a constant returns framework, the prices of goods can be locally written as a function of the prices of factors, which is classical and follows from a routine application of the implicit function theorem, is then shown to be global. This statement provides a key technical tool for the remaining of the analysis, and this is why we call it a Preparation Theorem.
- This section also computes the Jacobian matrix of the mapping φ exhibited in the Preparation Theorem: it shows that it is equal to the transpose of the factor value matrix. The corresponding result is straightforward when goods are produced only from factors and in this case may be related with the elementary findings of Stolper–Samuelson (see for example Dixit and Norman, 1980). Our result echoes in a more general multi-dimensional context the Stolper–Samuelson Theorem.
- In Section 4.2, we provide an algebraic counterpart to the geometric description of non-specialized efficient production plans we gave in Section 3.1: in a sense, we present the equations of the facets exhibited in Theorem 3. Indeed, Theorem 17 provides a comprehensive description of the non-specialized efficient frontier, from the joint knowledge of the factor values matrix defined in Section 3.2, and of the mapping φ defined in Section 4.1. Proposition 21 is a key tool for comparative static analysis.

[1] Economic historians have reassessed the empirical importance of the phenomenon of factor price equalisation (O'Rourke et al., 1996; O'Rourke and Williamson, 2000). And one of the hottest empirical debate in the last 20 years has borne on the extent to which factor price equalisation explains the increase of wages differentials in the US: see for example Freeman (1995), for a lively account of empirical aspects of the debate and of its theoretical background. Although the paper does not claim immediate relevance to such debates, it provides some hopefully fresh view on one of its basic perspectives.

[2] This is obviously not a new subject: see Samuelson (1953), Mac Kenzie (1955), Jones and Scheinkman (1977), Neary (1985), for a somewhat limited sample of the most significant earlier contributions. Note too, that although the "new" theory of international trade stresses new reasons for gains to trade, in addition to the indirect exchange of factors, it does not dismiss the effect of trade on factor prices. On the whole, the interest in the effect which trading goods may have on factor prices seems to have been rekindled rather than weakened by the contemporary development of international trade.

- Section 4 focuses on the factor price equalisation problem. The question is: can two remunerations plans, associated with two different efficient plans but with the same goods prices, be associated with different factor prices?
- Corollary 7 re-appraises an early result of Mac Kenzie (1955), originally stated in the case when only finitely many production activities were allowed. We then join the line of investigation started by Gale and Nikaido (1965), which assumes that $K = L$, so that there are as many factors as goods. In that case, one naturally looks for a bijection between the set of factor prices in R_+^K and the set of goods prices in R_+^L; in other words, one asks whether the map φ defined above is invertible. It is well known that this is not true in general: it requires an additional condition, known in the Heckscher–Ohlin case ($K = L = 2$) as *no factor-intensity reversal*. We adapt to our framework the most powerful (i.e. with the weakest generalized non-intensity reversal) result of that tradition, with $K = L \geq 2$, which is due to Mas Colell (1979b): this is Proposition 23.
- In a more general context where $L > K$, Theorem 24 asserts that, in general, equalising the price of $2K$ goods will be enough to equalise the price of factors; in particular, if $L \geq 2K$, equalising the prices of all goods will equalise the prices of factors (Corollary 25). This result holds *without any further assumption*, and notably without any non-intensity reversal, as in the Hecksher–Ohlin or Nikaido traditions. On the other hand, it is only true in general (*generically*, to use the technical term; its precise meaning will be provided later on in the paper).

Note that the questions answered in Proposition 23 (as well as in all papers of the Gale–Nikaido tradition concerned with generalized non-intensity reversal) are questions in the theory of production and not directly trade theory questions: they concern what happens in the production sector of a given economy. For example, absence of intensity reversal in production does not guarantee factor price equalisation in a trade context, because of the possibility of specialization. The same remark applies to our results in Theorems 24 and 26: they are inputs and not outputs of trade theory. We however argue later that they are closer to direct applications to trade theory than standard earlier results.

2. Model and preliminary analysis

2.1. Setting and assumptions

Consider a static production economy with constant returns to scale, in which one distinguishes between produced goods and non-reproducible factors of production (henceforth referred to simply as goods and factors). There will be L goods and K factors; we denote a bundle of the former by $x = (x^1, \ldots, x^L)$, and of the latter by $v = (v^1, \ldots, v^K)$. In this paper, we focus on the case where $L \geq K$. As usual, we shall denote by R_+^N the set of points with non-negative coordinates, $x^n \geq 0$, and by R_{++}^N the set of points with positive coordinates, $x_n > 0$. The transpose of a matrix M will be denoted by ${}^tM$.

The dual variables will be denoted by $(p_1, \ldots, p_L)$ (goods prices) and $(w_1, \ldots, w_K)$ (factors remunerations), respectively. The bundle $(x, v) \in R^{L+K}$ will be denoted by y, and the system $(p, w) \in R^{L+K}$ by q. We shall refer to y as a *production plan* and to q as a *remuneration plan*.

Factors cannot be produced but are used to produce goods. Goods must be produced, and are also used to produce other goods. Throughout this paper, we will assume that all technologies available for producing goods exhibit constant returns to scale; such technologies typically use as inputs some factors and some other goods. In this paper, we will assume that:

- each technology has a unique output: there is no joint production,
- all factors and all goods are used in production.

In this setting, the production technology of good ℓ will be associated with a *production function* f_ℓ: $R_+^{L-1} \times R_+^K \to R_+$, so that $x^\ell = f_\ell(-x', -v)$ is the quantity of good ℓ that will be produced by using efficiently the available technology and the input vector (x', v), which has negative components (note that there is no ℓ-component in x')

Given $x \in R^L$, we shall denote by $x^{-\ell} \in R^{L-1}$ the goods bundle obtained from x by deleting the ℓ th component. Assuming free disposal, the production set Y_ℓ for good ℓ is then given by:

$$Y_\ell = \{(x, v) \in R^{K+L} | v \in -R_+^K, x^{-\ell} \in -R_+^{L-1}, x^\ell \leq f_\ell(-x^{-\ell}, -v)\}$$

Our first condition subsumes standard assumptions concerning production functions, concavity, and constant returns to scale.

<u>Condition 1</u>. For every ℓ,the function f_ℓ is concave and positively homogenous of degree one with respect to all variables.

The next condition means that all goods other than ℓ and all factors are required in producing ℓ:

<u>Condition 2</u>. For every ℓ, the function f_ℓ is positive on $R_{++}^{L-1} \times R_{++}^K$ and vanishes on the boundary.

We shall also need some smoothness and non-degeneracy conditions on the production functions.

<u>Condition 3</u>. For every ℓ, the function f_ℓ is twice continuously differentiable on $R_{++}^{L-1} \times R_{++}^K$, its first derivatives are all strictly positive:

$$Df_\ell : R_{++}^{L-1} \times R_{++}^K \to R_{++}^{L-1} \times R_{++}^K$$

and the matrix of second derivatives, $D^2 f_\ell$, has corank 1 at every point.

The condition on the gradient Df_ℓ means that the production functions are strictly increasing with respect to all inputs. Since f_ℓ is positively homogeneous of degree one, Df_ℓ is positively homogeneous of degree zero, so that, by the Euler identity, we have $D^2 f_\ell(z)z = 0$ for all $z \in R^{L-1}_{++} \times R^K_{++}$. This means that z is in the kernel of $D^2 f_\ell(z)$, which therefore must be at least one-dimensional. Our assumption means that there is nothing else in the kernel, which is then exactly one-dimensional; it is a standard addition to the smoothness assumption of Condition 1.

The global production set in the economy is the convex cone in R^{K+L}:

$$Y = \sum_\ell Y_\ell$$

We now introduce the last condition:
Condition 4: Y is closed, and $(R^L_{++} \times R^K) \cap Y \neq \varnothing$.

This assumption means that, given sufficient amounts of factors, there is some production plan such that every good is produced in positive *net* quantity. In other words, the economy is not unproductive to the point where, even with large amounts of factors, it would be impossible to have a positive net production of all goods together. Indeed, in the case of a Leontieff economy, where there is a single factor and for each production sector a single technology, this is the standard condition ensuring that the input–output matrix is "productive".

Note that Condition 4, together with free disposal and constant returns to scale, implies that, given sufficient amounts of factors, *any* goods bundle can be produced, namely:

$$(R^L_{++} \times R^K) \cap Y = R^L_{++} \tag{1}$$

Strictly speaking, our assumptions rule out from the analysis two of the more popular models of production economies: the Heckscher–Ohlin model and the Leontieff model. In the Heckscher–Ohlin model, there are two goods and two factors, but no intermediate goods are used in production: goods are produced with factors only. In the Leontieff model, there is a single factor, and for each production sector a single technology: these technologies display strict complementarities, which are incompatible with Condition 3. However, our model covers cases as close as desired to these two special models. Simple intuition then rightly suggests that most of the results we shall derive here will also apply to them. In the Conclusion, we shall discuss more in detail the potential generality as well as the limitations of our argument.

2.2. Non-specialized and efficient production plans

2.2.1. Definitions

Any production plan $y = (x, v) \in Y$, the global production set, can be written (possibly in many different ways) as:

$$y = \sum y_\ell$$

with $y_\ell = (x_\ell, v_\ell) \in Y_\ell$.

This paper focuses on production plans where all ℓ technologies are active. Formally, these are the y such that there is some decomposition with $x^\ell_\ell > 0$ for all ℓ in the above formula. The set of such production plans is a subset of Y, denoted by Y_{NS}, and called the *non-specialized* [3] global production set; the $y \in Y_{NS}$ will be referred to as NS-production plans. In other words, $Y_{NS} \subset Y$ is the set of all production plans resulting in positive *gross* production of every good. We shall denote by Y_{NSE} its *efficient frontier*. Of course, some of these goods may be used in production of other goods, so that the net production may end up being negative. We shall also denote by Y^+_{NSE} the subset of efficient production plans resulting in positive *net* production of every good:

$$Y_{NS} = \left\{ \sum_{\ell=1}^{L} y_\ell | y_\ell = (x_\ell, v_\ell) \in Y_\ell, x^\ell_\ell > 0 \quad \forall \ell \right\}$$

$$Y_{NSE} = \{y \in Y_{NS} | Y \cap \{y + R^{L+K}_+\} = y\}$$

$$Y^+_{NSE} = \{y = (x, v) \in Y_{NSE} | x^l > 0 \quad \forall \ell\}$$

Condition 4 implies that Y^+_{NSE} is not empty.

There are three possible approaches to the study of production:

[3] This should not be confused with the non-specialization of countries in international trade. Non-specialization of world production covers cases when countries specialize as well as cases when they do not.

- The first one relies on examining the various ways to produce one unit of good ℓ. Geometrically, this amounts to studying the section of Y_ℓ with the hyperplane $x^\ell = 1$, namely:

$$Y_\ell^1 = \{(x, v) \in R^{K+L} | v \in -R_+^K, x^{-\ell} \in -R_+^{L-1}, 1 \le f_\ell(-x^{-\ell}, -v)\}$$

- The second one focuses on investigating the production possibilities of the economy when the factors endowments are given. Geometrically, this amounts to studying the section of the global production set with the subspace corresponding to the total endowment $v \in -R_+^K$, namely:

$$Y(.,v) = \{x | (x, v) \in Y\}$$

  and, with similar notation, the sections $Y_{NS}(.,v)$, $Y_{NSE}(.,v)$ and $Y_{NSE}^+(.,v)$.
- The third one consists in fixing the net quantities x of goods produced, rather than the total amount v of factors used. The section:

$$Y(x,.) = \{v' | (x', v') \in Y, x' \ge x\}$$

  then represents the set of factor endowments that would allow to produce more than the vector x of goods. They are obviously convex, and they are also non-empty because of Condition 4.

2.2.2. Efficiency

The standard decentralisation theorem asserts that an efficient global production plan is the sum of profit-maximizing production plans, profit being measured with an appropriate supporting price vector.

Let us then take a point $y = \sum_\ell y_\ell \neq 0$ on the efficient frontier of Y (without assuming yet that it is non-specialized), so that $y \in Y_{NSE}$. There is some non-trivial price vector $q = (p, w)$, which we call a *remuneration plan supporting* y, such that:

$$qy = \text{Max}\{qz | z \in Y\} \tag{2}$$

$$qy_\ell = \text{Max}\{qz_\ell | z_\ell \in Y_\ell\} = 0, \text{ for every } \ell. \tag{3}$$

Conditions 1 and 2 then imply, in a straightforward way:

$$q \in R_+^{L+K} \text{ and } qy = 0 \tag{4}$$

and since we have $y_\ell \neq 0$ for some ℓ, Condition 3 implies that q is *unique* (up to a positive, multiplicative constant).

Remuneration plans will often be scaled by choosing the first good as numeraire, so that $p_1 = 1$.

$$E_1 = \{q = (p, w) \in R_{++}^{L+K} | p_1 = 1\} \tag{5}$$

Our first insights are summarized as follows:

Proposition 1. *Let $y \in Y_{NSE}$ be an efficient NS-production plan and denote by $\pi(y) \in E_1$ the unique normalized remuneration plan supporting it. Then:*

1. $\pi(y)y = 0$
2. y can be uniquely written as $y = \sum_\ell \lambda_\ell(y)\gamma_\ell(y)$, with $\gamma_\ell(y) \in Y_\ell^1$; the $\lambda_\ell(y) > 0$ are the activity levels at which technology ℓ has to operate.
3. $\pi(y)\gamma_\ell(y) = 0$ for all ℓ.

Proposition 1 is derived in Appendix A, where the mappings $\lambda_\ell, \gamma_\ell$ are formally defined. The geometrical interpretation is as follows. Consider:

$$T(y) = \{z | \pi(y)z = 0\}$$

$T(y)$ is the tangent hyperplane to the convex cone Y at y. It goes through 0, the vertex of the cone, through y, the contact point, and contains the $\gamma_\ell(y)$.

We now proceed to illustrate the above concepts as well as our first insights in very simple cases. The first case has only one factor, (call it labour) $K = 1$, and two goods, $L = 1, 2$, the production of which has constant returns to scale. The case is both well known and easy to visualize, as in Fig. 1.

The quadrant underneath (resp. to the left of) the positive quadrant depicts the transformation possibilities of sector 1 (resp. sector 2) when it can use one unit of the scarce factor. When one unit of scarce factor is shared in proportions $t, 1 - t$,

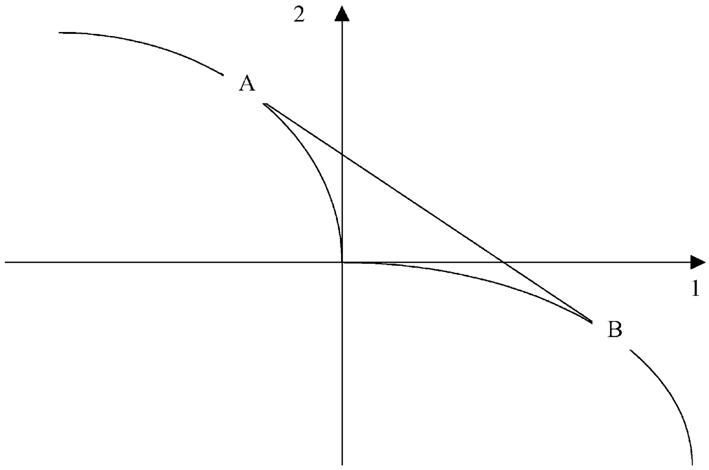

Fig. 1. The set of global production plans when there are two goods and one factor, and the total availability of the latter is prescribed.

the corresponding production plan obtains as the convex combination of two production plans in the two quadrants. The line segment AB visualizes the upper limit of such attainable production plans.

In our framework, Fig. 1 represents $Y(.,1)$, the subset of global production plans when the available quantity of factor is 1: this is the two-dimensional section of the global production set $Y \subset R^{L+K} = R^3$ with the plane $v = 1$. The production set Y itself is the cone generated from this section in R^3. Its efficient frontier is the cone generated by the upper boundary. The set of efficient NS-production plans, Y_{NSE}, is the cone in R^3 generated from the line segment AB: it is contained in a plane. This is a remarkable and well-known property, and we will come back to it later.

Note that this primal view of the efficient set has a dual counterpart: there is a single supporting plane $T(y)$ to Y_{NSE}, and the associated production prices, given by $\pi(y)$, are known to be proportional to the "labour values" of the goods (see also next sections).

Such a geometry is not general. Take a two goods and two factors model of Heckscher–Ohlin type. The global production set is obviously still a cone. Its section by the plane $(v_1 = 1, v_2 = 1)$ will again be convex, but strictly so. The boundary of its section in the positive quadrant will not be a straight line, as in Fig. 1, but, as is well known in the standard Heckscher–Ohlin model, a strictly convex set.

3. Global economic efficiency

3.1. The primal viewpoint and the geometry of production

3.1.1. The geometry of Y_{NSE}

Let us now focus attention on the NS-efficient frontier Y_{NSE}, bearing in mind the preceding examples.

We already know, from Proposition 1, that Y_{NSE} is smooth, in the sense that at every point y of Y_{NSE} there is a unique normalized supporting remuneration plan $\pi(y)$, and hence a well-defined and unique tangent hyperplane $T(y)$. The next result tells us that Y_{NSE} is made up of L-dimensional cones, smoothly glued together.

Take an efficient NS-production plan y, and consider the (unique) normalized y-allocation $(\gamma_\ell(y), \lambda_\ell(y))$, $1 \le \ell \le L$. Set:

$$C(y) = \left\{ \sum_{\ell} \mu_\ell \gamma_\ell(y) | \mu_\ell > 0, 1 \le \ell \le L \right\} \tag{6}$$

This is an L-dimensional polyhedral cone in R^{K+L}, with vertex at 0, and edges carried by the $\gamma_\ell(y)$. In the case $K = 1, L = 2$ we investigated before, $C(y)$ is the two-dimensional cone in (x_1, x_2, v)-space, and its section by $v = 1$ is the line segment AB we depicted in Fig. 1.

We know that $y \in C(y)$ (just take $\mu_\ell = \lambda_\ell(y)$). It follows from the previous section that $C(y)$ in contained in $T(y) = \{z | \pi(y)z = 0\}$, the tangent hyperplane to Y at y. The next proposition provides our first "primal" insight into the geometry of the set Y_{NSE}. The proof is deferred to Appendix B.

Proposition 2. $C(y) = T(y) \cap Y_{NSE}$.

Clearly $Y_{NSE} = \cup_{y \in Y_{NSE}} C(y)$. In other words, the set Y_{NSE} of NS-efficient production plans *is a disjoint* (*possibly infinite*) *union of L-dimensional polyhedral cones*, each one being the intersection of Y_{NSE} with some tangent hyperplane. We will not be comment further at his stage. Note only that both in the case illustrated in Fig. 1 and in the standard 2×2 Heckscher–Ohlin model, Y_{NSE} consists of a single two-dimensional polyhedral cone.

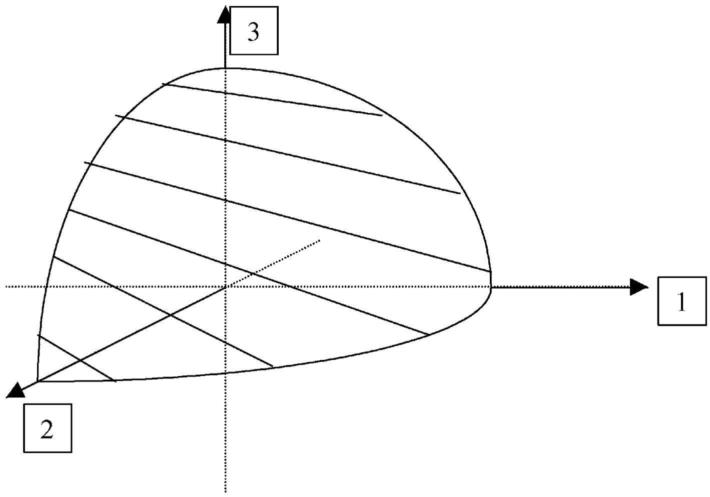

Fig. 2. The set of all goods which can be produced in an economy where there are 3 goods and 2 factors, the latter being available in prescribed quantities. We get a convex set, which is bounded by a developable surface.

3.1.2. Non-substitution theorems

For $v \in -R^K_{++}$, we have defined earlier the section $Y(.,v) = \{x|(x, v) \in Y\}$. With obvious notations, we will consider the associated subsets $Y_{NS}(.,v)$, $Y_{NSE}(.,v)$ and $Y^+_{NSE}(.,v)$.

A *facet* of $Y_{NSE}(.,v)$ is the intersection of $Y_{NSE}(.,v)$ with a tangent hyperplane; it is necessarily a closed convex set containing the points of contact. We define its *dimension* to be the dimension of its affine span (the smallest affine subspace which contains it), and we recall that its *relative interior* is its interior relative to its affine span.

Our main result in this subsection, which generalizes the classical non-substitution theorems, is the following. The proof is given in Appendix C.

Theorem 3. *Every $x \in Y_{NSE}(.,v)$ belongs to a single facet. This facet has dimension $(L - K)$, and x belongs to its relative interior.*

There are two limiting situations, $K = 1$ (one factor) and $K = L - 1$ (many factors). In the first case, we get the classical non-substitution theorem of Arrow and Samuelson:

Corollary 4. *When there is only one factor of production, available in total quantity $-v \in R^K_{++}$, the set $Y(.,v) \cap R^L_+$ of all goods bundles which can be produced is bounded by an affine hyperplane.*

Proof. The set $Y_{NS}(.,v) \cap R^L_+$ is bounded by $Y_{NSE}(.,v) \cap R^L_+$. By Theorem 3, every $y \in Y_{NSE}(.,v)$ belongs to a single $(L - 1)$-dimensional facet, and is in its relative interior. If there were two distinct facets, they would both have to be $(L - 1)$-dimensional, and they would meet along an $(L - 2)$-dimensional facet. But there are no such facets, so the only possibility is that there is single $(L - 1)$-dimensional facet. The result follows. □

To say what happens when $K = L - 1$, we must recall that a hypersurface in R^L is *ruled* if it is a union of straight lines, and that it is *developable* if it is ruled and, along each of these straight lines, all points have the same tangent hyperplane.

Corollary 5. *If there are $L - 1$ factors of production, available in total quantity $-v \in R^K_{++}$, the set $Y(.,v) \cap R^L_+$ of all goods bundles which can be produced is bounded by a developable hypersurface.*

Corollary 5 does not seem to have appeared in previous literature, although it would (probably) follow from Travis (1972) in the case $L = 3$, $K = 2$, and from Young (2006) in the case $L = 4$, $K = 3$. It is visualized on Fig. 2, for $L = 3$, $K = 2$. The figure provides a support for intuition in the general situation. It is suggestive in showing why the dimension of the facets, here one, is compatible with the fact that supporting prices vary smoothly on the efficiency surface. This is a key point in the general understanding of the problem.

Another interesting case is $L = 2K$: there are twice as many goods as factors. Corollary 6 provides another point of view on the general situation, in what will appear later on as a borderline case.

Corollary 6. *If $K = L/2$, every $x \in Y_{NSE}(.,v) \cap R^L_+$ belongs to the relative interior of some K-dimensional facet of $Y(v) \cap R^L_+$.*

Corollary 7, which originates with Mac Kenzie (1955), is another consequence of Theorem 3. It is the first occurrence in this paper of a factor price equalisation result: *two different facets in $Y_{NSE}(.,v)$ have different normal price vectors p.*

Corollary 7. *If (p, w_1), (p, w_2) are two remuneration plans in $Y_{NSE}(.,v)$ associated with x, x', then $w_1 = w_2$.*

Proof. Mac Kenzie's proof, although given in a different framework (he allows only a finite number of activities) would still work here. We see it as a simple consequence of Theorem 3. Indeed, suppose that there exist such x, x', in two different facets. The statement implies necessarily: $px + w_1 v > px' + w_1 v$, so that $px > px'$. But symmetrically, the reverse first inequality also holds with w_2 instead w_1, so that one would have $px' > px$, a contradiction. □

3.1.3. The geometry of $Y(x,.)$

We consider $Y(x,.) \subset R^K$, the set of factor endowments that allow to produce x:

$$Y(x,.) = \{v' | (x', v') \in Y, x' \geq x\}$$

By Condition 4, it is non-empty. The following result will be useful in the sequel; we leave the proof to the reader.

Lemma 8. *The set $Y(x,.)$ is strictly convex.*

3.2. The dual viewpoint and production prices

The dual viewpoint on efficiency consists of describing the efficiency frontier indirectly, through the set of supporting prices and cost minimization. Mathematically speaking, we describe the polar cone of the production set rather than the production cone itself.

3.2.1. Cost minimization and factor values

Let a remuneration plan $q = (p, w) \in R^{L+K}_{++}$ be given. Say we wish to produce one unit of good ℓ. To do it at minimal cost requires solving the optimization problem:

$$\text{Min}\{px^{-\ell} + wv | x^{-\ell} \in R^{L-1}_{+}, \quad v \in R^K_+, \quad f_\ell(x^{-\ell}, v) \geq 1\}. \tag{7}$$

It follows from our assumptions, notably the strict concavity of f_ℓ, that this problem has a unique solution for every q. In other words, there is *a unique cost-minimizing bundle of goods and factors for the production of one unit of good ℓ*. We denote this solution by:

$$(a_\ell(p, w), b_\ell(p, w)) \in R^L_{++} \times R^K_{++} \tag{8}$$

with $a^\ell_\ell = 0$. We shall refer to the (a^j_ℓ, b^k_ℓ) as the *Leontieff coefficients* associated with the remuneration plan $q = (p, w)$. The unit production cost of good ℓ is then

$$c_\ell(p, w) = pa_\ell(p, w) + wb_\ell(p, w) = \sum_{j \neq \ell} p_j a^j_\ell(p, w) + \sum_k w_k b^k_\ell(p, w) \tag{9}$$

which is the optimal value achieved in problem (7). It follows from Conditions 1 and 2 that the function c_ℓ is positively homogeneous of degree one and concave. Using the non-degeneracy in Condition 3 and the inverse function theorem, we see that it is C^1 as well.

In a remuneration plan, production prices must be equal to costs (average and marginal costs are equal in our constant returns to scale setting) so that the following relation holds:

$$p_\ell = c_\ell(p, w) = pa_\ell(p, w) + wb_\ell(p, w), \quad \forall \ell \tag{10}$$

In order to describe this system of equations in a more compact way, let us denote by $A(p, w)$ the $L \times L$ matrix with $a_\ell(p, w)$ as ℓ th column (so that all diagonal coefficients are zero) and by $B(p, w)$ the $L \times K$ matrix with $b_\ell(p, w)$ as ℓ th column. They are respectively $L \times L$ and $K \times L$ matrices, the ℓ th column of which consists of the input vector, respectively in goods and factors, used for the production of one unit of good ℓ in the chosen allocation. In a Leontieff economy with strict complementarities, A and B are given a priori, while here they arise endogenously, but not necessarily uniquely.

Eq. (10) then become:

$$p(I - A(p, w)) = wB(p, w) \tag{11}$$

If the above formula holds true for some $p \in R^L_{++}$ and $w \in R^K_{++}$, then the transpose $A^*(p, w)$ is necessarily productive,[4] and the same is true of $A(p, w)$ itself, so that the matrix $I - A(p, w)$ is invertible with positive entries. Hence, one will write (11) as:

$$p = wB(p, w)(I - A(p, w))^{-1} \tag{12}$$

$$= w\left[B(p, w) + B(p, w)\sum_{n \geq 1} A^n(p, w)\right] \tag{13}$$

Note then, that if the bundle x has to be produced with the techniques B, A, the vectors of factors directly required for production is Bx, and the vectors of goods directly required is Ax; the vector of factors indirectly required for producing these goods is BAx, and the quantity of goods indirectly required is A^2x, the production of which also requires BA^2x of factors, and so forth. So the right-hand side is the price of all factors directly or indirectly used in production.

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[4] See for example Horn and Johnson (1990), or Mas Colell et al. (1995).

In a one-dimensional setting, $K = 1$, where the only factor is called labour, this formula expresses the standard finding that the labour value of a commodity is the sum of direct labour plus (total) labour indirectly incorporated in the production of the commodity.

In the present setting, where there are many factors, let us introduce the $K \times L$ matrix:

$$F(p, w) = B(p, w)(I - A(p, w))^{-1} \tag{14}$$

so that the above formula becomes $p = wF(p, w)$. In view of formula (13) and its interpretation, one may understand the term F_ℓ^k at the k th line and ℓ th column as the total quantity of factor k used *directly or indirectly* in the production of good ℓ, when the cost-minimizing technique is used and when the inputs prices are (p, w). The k th line gives the quantity of factor k used by the L goods: when the factor is labour, this is called the "labour values" vector. Similarly, the ℓ th column gives the quantities of factors directly or indirectly required for the production of one unit of good ℓ.

We propose to call F the *"factor values" matrix*. The relation $p = wF(p, w)$ then can be stated as follows: the price of any good is equal to the sum over all factors of the factor price multiplied by the factor value of the good.

3.2.2. Relating the primal and the dual viewpoint

The proof of the next result requires a careful comparison of the dual and primal notation which is made in Appendix D. If $y \in Y_{NSE}$ is an efficient NS-production plan, recall that $\pi(y) \in E_1$ the unique normalized remuneration plan supporting it.

Proposition 9. *If $y = (x, v) \in Y_{NSE}$, supported by a remuneration plan $q = (p, w) = \alpha\pi(y)$, for some $\alpha > 0$, then*

$$p = wF(p, w) \tag{15}$$

and there exists some $\lambda \geq 0$ such that:

$$x = (I - A(p, w))\lambda, \tag{16}$$

$$v = -B(p, w)\lambda \tag{17}$$

Conversely, if $q = (p, w) \in R_{++}^{L+K}$ and $p = wF(p, w)$, then any $y = (x, v)$ satisfying (16) and (17)for some $\lambda \geq 0$ belongs to Y_{NSE}, and there exists $\alpha > 0$ such that:

$$(p, w) = \alpha\pi(y) \tag{18}$$

Corollary 10. *In both cases, one has:*

$$F(p, w)x = -v \tag{19}$$

Formula (19) gives another interpretation of the factor values matrix $F(p, w)$: *for any efficient production plan, the quantity of factors used in the production of a goods bundle equals the sum over the goods of the factors values of any good multiplied by the quantity of this good in the bundle*. This statement is clearly illustrated by Fig. 2. Each of the lines that generate $Y_{NSE}^{+}(., v)$ in the figure is the intersection of two hyperplanes, the first one expressing that, say, the total "labor" values of the three goods equals available labour, the second one expressing, say, that the "land value" of the three goods bundle equals the available land.

Formula (19) also has an interesting mathematical interpretation, as the equation for the facet of x in $Y_{NSE}(., v)$ Recall from Proposition 2 that every $y = (x, v)$ in Y_{NSE} belongs to the L-dimensional cone $C(y)$ defined by formula (6). Recall also from Theorem 3 the definition of a facet of $Y_{NSE}(., v)$. The facet containing x will be denoted by $\text{Facet}_v(x)$

Corollary 11. *Take $\bar{y} = (\bar{x}, \bar{v})$ in Y_{NSE} and let $\bar{q} = (\bar{p}, \bar{w})$ be an associated remuneration plan. Then:*

$$C(\bar{y}) \subset \{(x, v) \in Y_{NSE} | F(\bar{p}, \bar{w})x + v = 0\} \tag{20}$$

$$C(\bar{y}) \supset \{(x, v) | x \in R_{++}^{L}, F(\bar{p}, \bar{w})x + v = 0\} \tag{21}$$

$$\text{Facet}_{\bar{v}}(\bar{x}) \cap R_{++}^{L} = \{x \in R_{++}^{L} | F(\bar{p}, \bar{w})x + \bar{v} = 0\} \tag{22}$$

Proof. Take a point $y = (x, v)$ such that $y \in C(\bar{y}) = T(\bar{y}) \cap Y_{NSE}$. Then the normal to Y_{NSE} at y and $\bar{y}$ must be the same, so $\bar{q}$ must also be a remuneration plan for y. It follows from Proposition 9 that $F(\bar{p}, \bar{w})x = -v$. This proves (20).

Conversely, take a point $y = (x, v)$ with $x \in R_{++}^{L}$ such that $F(\bar{p}, \bar{w})x = -v$. Setting $\lambda = (I - A(p, w))^{-1}x \in R_{++}^{L}$, we apply Proposition 9 so that $y \in Y_{NSE}$. Since $\bar{q}$ is a remuneration plan for y, it must be normal to Y_{NSE} at y, so $y \in C(\bar{y})$. This proves formula (21). Formula (22) then follows from Condition (1) and constant returns to scale. □

4. The role of factors prices

In this section, we will go into the structure of the set of all possible remuneration plans $q = (p, w)$. We will first show that the factors prices determine the goods prices, and then we will show that the factors prices determine the whole geometry of production.

This section, and the following one, will rely on the Leontieff matrices $A(p, w)$ and $B(p, w)$, and the factors values matrix $F = B(I - A)^{-1}$, as defined in the preceding section. In fact, all subsequent results shall hold, *provided only that the matrix*

$A(p, w)$ is productive, so that $I - A(p, w)$ is invertible with positive entries. This covers both the case developed in this paper, where all goods are used in production, so that all non-diagonal entries of A are positive, and the Hecksher–Ohlin case, when no goods are used in production, so that $A = 0$.

4.1. The Preparation Theorem

Let us come back to Eq. (15), which characterize the couples (p, w) which are remuneration plans:

$$p = wB(p, w)(I - A(p, w))^{-1} = wF(p, w)$$

A standard counting argument, number of equations versus number of unknowns, may suggest a one-to-one correspondence between the factor prices and the good prices when $K = L$, or between the factor prices and a subset of the good prices when $K < L$. Note that even if such an intuition were right locally, it would not follow that it is right globally; furthermore, it could also be wrong locally, when the appropriate Jacobian matrix fails to be invertible. Both possibilities are illustrated by the examples we provide in Section 4. There is, however, some truth in the counting argument, and it is captured by the implicit function theorem, along the lines which we develop now.

We will show that if $(\bar{p}, \bar{w})$ is a solution of the above equations, and if we change $\bar{w}$ to $w = \bar{w} + dw$, then, there exists a unique $p = \bar{p} + dp$ that satisfies Eq. (16). By the implicit function theorem, the set of (p, w) that satisfies the equation then is the graph of some map $p = \varphi(w)$, at least in some neighbourhood of $(\bar{p}, \bar{w})$. We will show that this function is well-defined, and can be extended to a the whole of R^K_{++}.

Let us proceed. Recall that $\pi(y)$ is normalized by setting its first component equal to 1, so that $\pi(y) \in E_1$. The set Σ of all possible remuneration plans can be described in two ways:

$$q = (p, w) \in \Sigma \Leftrightarrow \exists \mu > 0, \exists y \in Y_{NSE} : q = \mu\pi(y) \Leftrightarrow p = wF(p, w) \tag{23}$$

We begin, as announced, by showing that Σ is locally the graph of a map φ. This local property has been known in the literature of the 1970s and 1980s although rarely identified as such.[5]

Lemma 12. *Let $q = (p, w) \in \Sigma$. There exists a neighbourhood $\mathcal{N}$ of w in R^K_{++}, and a smooth map φ: $\mathcal{N} \to R^L$ such that $\varphi(w) = p$ and:*

$$\{(\varphi(t), t) | t \in \mathcal{N}\} \subset \Sigma$$

Proof. Let $q = (p, w) \in \Sigma$. So Eq. (23) hold true in a neighborhood of Σ. By the envelope theorem, applied to the optimisation problem (7), this last equation can be differentiated as:

$$\mathrm{d}p = \mathrm{d}wB(q)(I - A(q))^{-1}.$$

The conclusion follows from the implicit function theorem applied at q. □

This defines φ locally only. In Appendix E, we show that it can be defined globally. More precisely, we show that the projection map from Σ to R^K_{++} is globally one-to-one. We show first that above each w there can be at most one p (Lemma 28). This is reminiscent of a standard argument in the analysis of the Malinvaud–Taylor algorithm.[6] We then show that the local extension can be made global (Lemmas 29 and 30).

The end result will be an essential tool for the sequel.

Theorem 13 (Preparation Theorem). *Σ is the graph of a map $\varphi : R^K_{++} \to R^L_{++}$, which is smooth and homogeneous of degree one.*

In economic terms, the theorem asserts that *relative prices of factors globally determine the relative prices of goods*. We believe it is novel in the present context, where both goods and factors are used in production.

Corollary 14. *For every ℓ, the function φ_ℓ is concave.*

Proof. $\varphi_\ell(w) = [wF(\varphi(w), w)]_\ell$ is the total cost of producing one unit of good ℓ when the remuneration scheme is $(\varphi(w), w)$. Denoting by e_ℓ the bundle consisting of one unit of good ℓ, we have:

$$\varphi_\ell(w) = \min_v \{-wv | v \in Y(., e_\ell)\}$$

and the right-hand side is a concave function of w. □

Also, by the envelope theorem, we have:

[5] See for instance Woodland (1983), Chapter 5 and specially pp. 115, 116. Naturally, in the case where goods are produced only from factors (as is the case in the Gale–Nikaido tradition mentioned below) both the local and the global property have been known for long: φ consists of the collection of standard cost functions (costs depending only on w) and from standard production theory.

[6] See Malinvaud (1967). We owe the idea as well as relevant references to Michael Jerison.

Theorem 15. *The Jacobian matrix of $\varphi : R^K_{++} \to R^L_{++}$ at w is given by:*

$$\partial\varphi(w) = F^*(\varphi(w), w)$$

where the right-hand side is the transpose of the matrix of factor values evaluated at $(\varphi(w), w)$

This property has a very intuitive interpretation: *the marginal change in the price of good ℓ* induced by a marginal change in the price of factor k is equal to the factor k value of good ℓ. In the case where goods are produced only from factors, it is clear that $\varphi'(\bar{w})$ obtains from the set of cost-minimizing input bundles: the factor k value of good ℓ then is the quantity of factor k directly used for the production of one unit of good ℓ. If we particularize this statement to a 2×2 world, the previous statement leads to the Stopler–Samuelson theorem. One may therefore view Proposition 15 as *a generalized Stopler–Samuelson theorem* (although in the standard 2×2 statement, the direction product prices towards factor prices is emphasized) Note that the analysis of "friends" and "enemies" along the lines of Jones and Scheinkman (1977) carries on here, with the more complex interpretation that the present setting calls for.

4.2. *From the geometry to the algebra of production*

Combining Theorem 9 with the Preparation Theorem, we see that the equation $F(\varphi(w), w)x = -v$ relates points (x, v) on the NS-efficient frontier with supporting remuneration plans $(\varphi(w), w)$, which themselves are fully determined by the factor prices w. In this subsection, we formalize this intuition and derive some consequences.

4.2.1. *Determining the sections $Y(.,v)$*

It is natural to ask what are the factors endowments v which are compatible with certain factor prices w. Here, as earlier, $F(p, w)$ is the factors values matrix, and $F_\ell(p, w)$ is its ℓ th column.

Proposition 16. *Fix $w \in R^K_{++}$. The set of factor endowments $-v \in R^K_{++}$, such that there is some $x \in R^L_{++}$ for which $(x, v) \in Y_{NSE}$ is supported by the remuneration plan $(\varphi(w), w)$, is the open convex cone $K(w) \subset R^K_{++}$ spanned by the vectors $F_\ell(\varphi(w), w)$, $1 \le \ell \le L$. The multivalued mapping $w \to K(w)$ is continuous.*

Knowing $K(w)$ and $\varphi(w)$ leads us to a simple description of the NS efficiency frontier Y_{NSE}, and more particularly of Y^+_{NSE}. Let us show first how they determine the sections $Y^+_{NSE}(.,v)$:

Theorem 17. *We have:*

$$Y^+_{NSE}(.,v) = \cup_{w \in K^{-1}(v)}\{x \in R^L_{++} | F(\varphi(w), w)x + v = 0\}$$

Theorem 17 provides us with the equations of $Y^+_{NSE}(.,v)$. For each $w \in K^{-1}(v)$, the equation $F(\varphi(w), w)x = -v$ defines an affine subspace in R^L with dimension $L - K$. Its intersection with R^L_{++} is a facet of $Y^+_{NSE}(.,v)$, and as w varies, the facet moves, thereby spanning all of $Y^+_{NSE}(.,v)$.

In words, these equations state that *the factors values of any production plan is equal to the quantity of factors available*. This statement is clearly illustrated in the previous Fig. 2. Each of the lines that generate $Y^+_{NSE}(.,v)$ in this figure is the intersection of two planes, the first one expressing that the total "labour" value of the three goods equals available labour, and the second one expressing that, say, the total "land" value of the three goods equals available land.

Corollary 18. *We have:*

$$\begin{aligned} Y_{NSE} \quad &\subset \quad \cap_{w \in K^{-1}(v)}\{(x, v) \,\big|\, F(\varphi(w), w)x + v = 0\} \\ \emptyset \quad &\neq \quad \cap_{w \in K^{-1}(v)}\{(x, v) \,\big|\, x \in R^L_{++}, F(\varphi(w), w)x + v = 0\} \subset Y_{NSE} \end{aligned}$$

For the proof of these results, we shall need the following lemma:

Lemma 19. *Given (p, w), define $K(p, w)$ to be the set of factor endowments $-v \in R^K_{++}$ for which there is an $x \in R^L_{++}$ such that (p, w) is a remuneration plan supporting (x, v). Then $K(p, w)$ is a convex cone.*

Proof. Take two points $-v_1$ and $-v_2$ in $K(p, w)$, so that $y_1 = (x_1, v_1)$ and $y_2 = (x_2, v_2)$ satisfy $(p, w) = \alpha\pi(y_1) = \alpha\pi(y_2)$ with $x_i \in R^L_{++}$. Then

$$F(p, w)x_1 + v_1 = F(p, w)x_2 + v_2 = 0$$

and $F(p, w)(tx_1 + sx_2) + tv_1 + sv_2 = 0$ for any $t, s \ge 0$. So $(\varphi(w), w)$ supports $(tx_1 + sx_2, tv_1 + sv_2)$, and since $tx_1 + sx_2 \in R^L_{++}$, we have $ts_1 + sv_2 \in K(p, w)$, and the result follows. □

We then proceed as follows. Suppose $(\varphi(w), w)$ is a remuneration plan supporting some (x, v) in Y_{NS}, with $x \in R^L_{++}$. Then $(x, v) \in Y_{NSE}$, and by Proposition 9, we have $F(\varphi(w), w)x + v = 0$. This implies that $-v$ belongs to the convex cone generated by the $F_\ell(\varphi(w), w)$, $1 \le \ell \le L$, in R^K_{++}. By the definition of F, for each ℓ, the production plan $(e_\ell, F_\ell(\varphi(w), w))$[7] is supported by $(\varphi(w), w)$. Conversely, if v belongs to the cone generated by the $F_\ell(\varphi(w), w)$, then, by the preceding Lemma, there is some $x \in R^L_{++}$ such that $(\bar{p}, \bar{w})$ is a remuneration plan supporting (x, v). Proposition 16 follows; the continuity of $K(v)$ is obvious. Theorem 17 then follows by Corollary 11.

4.2.2. Determining the sections $Y(x,.)$

For the sake of completeness, we state the results that follow from the preceding analysis concerning the section $Y(x,.)$, that is, the set of all factors bundles which can be used to produce x.

Theorem 20. *We have:*

$$Y(x,.) = \{F(\varphi(w), w)x | w \in R^K_{++}\} + R^K_{++}$$

and $F(\varphi(w), w)x$ is the unique solution of the problem:

$$\min\{-wv' | v' \in Y(x,.)\}$$

We skip the proof, just noting that the key ingredient is the fact that the cost-minimizing bundle v for producing x when factors prices are w and goods prices are p is $-F(p, w)x$.

4.2.3. Comparative statics

Finally, we give some static comparative results, by examining the local co-movements of a NS-production plan and the supporting remuneration plan. The following proposition summarizes the result:

Proposition 21. *Let $y = (x, v)$ be a NS-efficient production plan, and (p, w) some supporting remuneration plan, so that $p = \varphi(w)$ Then $(x + dx, v + dv)$ still belongs to Y_{NSE}, and $(p + dp, w + dw)$ still is an associated remuneration plan if and only if:*

1 $pdx + wdv = 0$,
2 $\mathrm{d}p = \mathrm{d}wF(\varphi(w), w)$,
3 $\mathrm{d}wM(w, x) = dv - [F(\varphi(w), w)]\,\mathrm{d}x$, where

$$M(w, x) := \frac{\partial}{\partial w}[F(\varphi(w), w)x] \tag{24}$$

is a symmetric negative semi-definite $K \times K$ matrix of rank $K - 1$.

Proof. The first statements easily follow from our previous analysis. From the above result, M is the second derivative of the function $w \to wF(\varphi(w), w)x = \varphi(w)x$, which is concave and positively homogeneous of degree one. The last statement then follows directly from Theorem 15. □

Corollary 22. *If $dw = 0$, that is, if the prices of factors do not change, we get $dv = F(\varphi(w), w)\,dx$.*

What is called Ribzcinsky's theorem (where the reverse direction from endowments towards outputs is classically emphasized) may be viewed as a particular case of the above statement in a 2×2 world with no factor-intensity reversal and where goods are produced from factors only.

5. Factor price equalisation

The question of factor price equalisation can be raised independently from its possible impact on international trade: in a given production economy, is it the case that both (p, w_1) and (p, w_2) can be remuneration plans, so that the same price for goods is compatible with different prices of factors? In other words, could there exist two distinct (here unconnected) economies with the same available technologies, two distinct total vector of factor endowments, the same prices for goods, but different prices for factors?

5.1. Classical findings: number of goods equal number of factors

The literature has investigated that question in many different ways. For example Mac Kenzie asks a more demanding property: can it be the case that (p, w_1) and (p, w_2) are distinct remuneration plans when the total endowment of factors is the same? As we have seen, he answered in the negative, a result that is more easily recovered from the present analysis (Corollary 7). In other words, one cannot find two similar economies, both in terms of technologies and initial endowments, which have the same goods prices and different factor prices.

[7] Here, again, we denote by e_l the bundle consisting of one unit of good l.

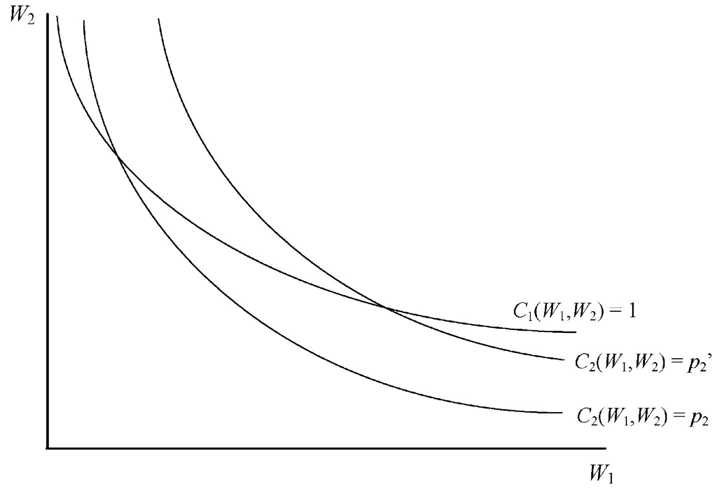

Fig. 3. The Heckscher–Ohlin case: two goods, one of which is the numéraire, and two factors, the prices of which are given on the axes. We have given the isocost curves, ie the pairs of factor prices which will allow to produce one unit of good at presrcribed cost (1 in the case of good 1). We have drawn two isocost curves for good 2, and because of non-intensity reversal, each of them intersects the isocost curve for good 1 at one point only.

Let us come back to the initial and more difficult question: can (p, w_1) and (p, w_2) be different remuneration plans corresponding to different factor endowments v? It has been known for some time that the answer to the above question is no in the 2×2 Hecksher–Ohlin model, whenever the so-called *factor intensity of goods* is not reversed in the production process. Under the no factor-intensity reversal condition, the prices of goods uniquely determine the price of factors: this is the factor price equalisation property. We have illustrated it in Fig. 3, which is drawn under the assumption that there is no intensity reversal and that good 1 is more intensive in factor 2. We have drawn the curves $c_1(w_1, w_2) = 1$ and $c_2(w_1, w_2) = p_2$ for two values of p_2.

Such a univalence theorem has been extended to the case where *the number of factors equals the number of goods*, and goods are produced only from factors, in a literature starting from Gale and Nikaido (1965) and which was active in the seventies (see for instance Nikaido, 1972). The literature has emphasized a *generalized* no factor-intensity reversal condition, a weaker form of which appears in Mas Colell (1979b). We will now show that Mas Colell's findings hold in the extended setting we consider here.

Assume that $K = L$ and consider the function $\varphi: R^K_{++} \to R^L_{++}$, whose existence has been shown in the Preparation Theorem. As in Mas Colell (1979b), let us consider the share matrix $S(\varphi)$ with entries:

$$s_{\ell k} = \frac{w_k}{\varphi_\ell} \frac{\partial \varphi_\ell}{\partial w_k}$$

As we have seen above, $S(\varphi)$ can be computed explicitly in terms of A and B. The extended non-intensity reversal assumption that we consider asserts that the determinant $(\det S)$ *never vanishes*: the reader will note that in the one-dimensional case this is equivalent to no factor-intensity reversal as discussed above.

Proposition 23. *Assume that $K = L$ and that* $\det S(\varphi(w))$*is b* ounded away from zero on R^K_{++}. *Then $\varphi: R^K_{++} \to R^L_{++}$ is onto and one-to-one.*

Proof. The assumption means that there is some $\varepsilon > 0$ such that $|\det S(\varphi(w))| \geq \varepsilon$ for all $w \in R^K_{++}$. Once φ is shown to exist, the proof, and indeed the statement of Mas Colell (1979b) applies, and the result follows. For the reader's convenience, let us mention that the proof consists in considering the map $\psi: R^K \to R^L$ defined by $\psi(u) = \log \varphi(\exp u_1, \ldots, \exp u_2)$. By assumption, the determinant of the Jacobian matrix $D\psi$ is bounded away from zero. By Hadamard's global version of the implicit function theorem, ψ is onto and one-to-one (see also Mas Colell, 1979a). □

5.2. Factor price equalisation in general

However, the univalence property does not hold true when the no factor-intensity reversal condition does not hold, even if $K = L$. Fig. 4, which exhibits iso-cost curves in the space of the factor prices, when $K = L = 2$, shows an example of intensity reversal, which is in no way pathological. Note that here, as well as in the previous picture, the analysis exploits the fact – the generality of which is established by Theorem 3 – that the set of normalized (the normalization obtains by putting the price of good one equal to one) remuneration plans (p, w) is a one-dimensional object (because there are only two factors) that may be indexed by the ratio of factor prices.

Here, the prices of goods do not any longer uniquely determine the prices of factors: every p is associated with two different w. Therefore, plotting relative prices of goods in terms of relative prices of factors, we get a local maximum (or minimum), as in Fig. 5(a).

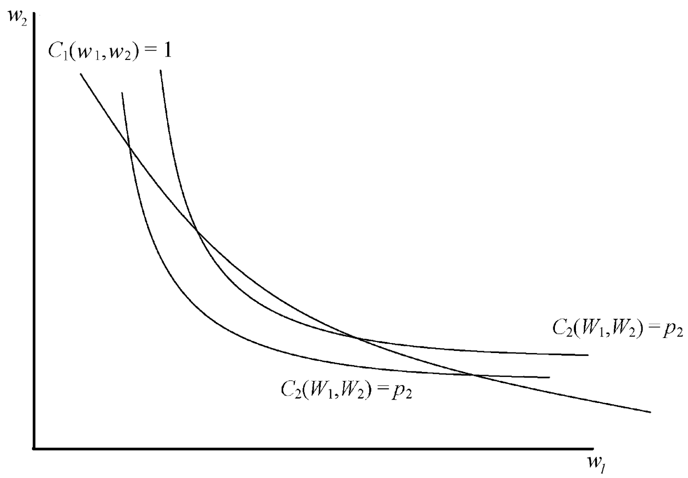

Fig. 4. The Heckscher–Ohlin situation (two goods and two factors) again, but this time with intensity reversal: the isocost curves for goods 1 and 2 intersect at several (two) points.

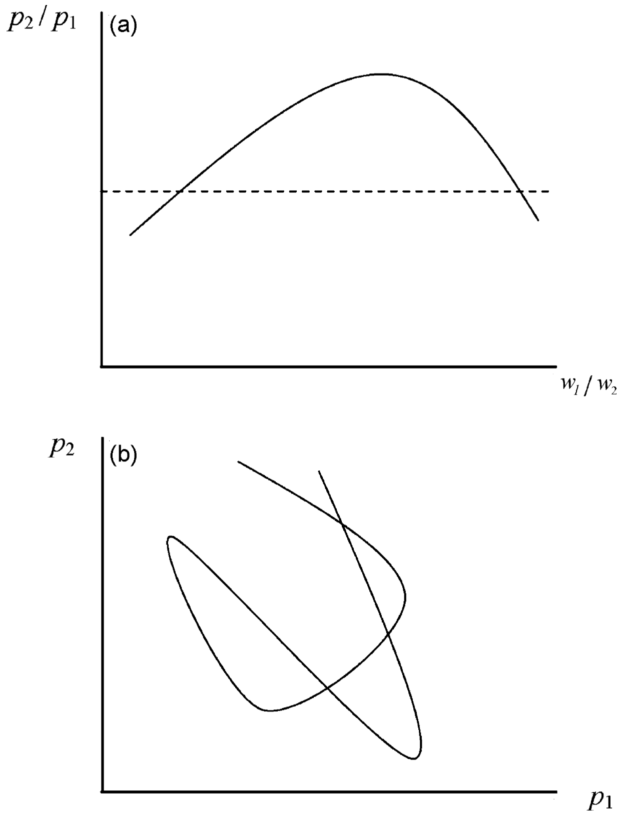

Fig. 5. Mapping relative prices of factors into relative prices of goods. Figure (a) is the Heckscher–Ohlin case with intensity reversal. Figure (b) is the case when there are two factors and three goods, one of them (good 3) being taken as numéraire. In both these situations, relative prices of goods do not determine the relative prices of factors

Similarly, in the case when there are three goods, so that $K = 2$, $L = 3$, mapping relative prices of goods in terms of relative prices of factors gives a map from R into R^2 and Fig. 5(b) shows a picture with self-intersections. Each self-intersection corresponds to a point where the map is not one-to-one: there are some p associated with two different w, that is, the prices of goods in such a case do not determine the prices of factors. Although the picture rightly suggests that the non-univalence situation is less frequent than in the previous example, it is non-pathological (only strong assumptions, such as no factor-intensity reversal, can rule it out) and robust (if it holds for some production economy, it holds for neighbouring ones). Note that the picture is not derived from a fully explicit example, and claims only to be suggestive, but, it is easy, for instance, in a three goods economy where intensity reversal occurs for every pair of goods, to visualize (in the two-dimensional space of factor prices where iso-costs functions are drawn) a situation where three iso-cost curves intersect in two different points and which is not destroyed by a small change of the underlying production sector. Note that a self-intersection of the above

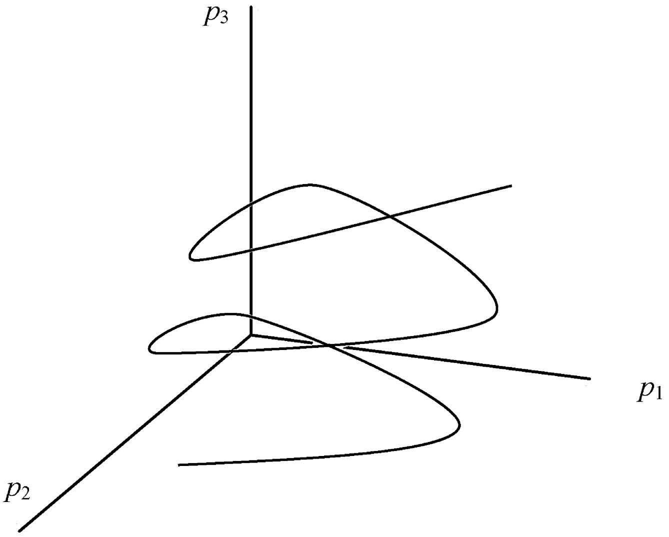

Fig. 6. This is the case when there are two factors, but four goods or more, one of which is taken as numéraire. The curve is the set of all possible prices of the remaining goods when the prices of factors vary, and it does not have self-intersections, so that the relative prices of goods determine the relative prices of factors.

curve in w_1, w_2 satisfies $\varphi(w_1) = \varphi(w_2)$, $w_1 \neq w_2$: we have two equations and two unknowns, and the fact suggests that this system of two equations has a solution should not be expected to be pathological.

Coming back to the $K = 2$ example, and remembering the fact that the set of remuneration plans is one-dimensional, we note that the likelihood of one p being associated with two w becomes weaker when $L = 4$. With four goods, or more, the set under scrutiny would be a parametric curve in R^3, or in a higher dimensional space, and it is extremely unlikely that such a curve would display self-intersections (see Fig. 6).

We can also appeal to the above equations determining a self-intersection. The system $\varphi(w_1) = \varphi(w_2)$, $w_1 \neq w_2$, has three equations and two unknowns. Hence, the prices of goods should be expected to determine the prices of factors whenever $K \geq 4 = 2L$. Indeed, we will show that this is the case.

The idea of the proof can be related to this counting of equations and unknowns. In the general case, there are $(L-1)$ equations, one for each (normalized) good price. On the other hand, there are $(K-1)$ unknowns for w_1 and $(K-1)$ for w_2, in total $2(K-1)$ unknowns. If there are more equations than unknowns, that is, if $L-1 > 2(K-1)$, or $L \geq 2K$, then a solution is unlikely.

5.3. *A generic theorem*

Using Thom's Transversality Theorem (see Abraham and Robbin, 1967 or Aubin and Ekeland, 1984), it is possible to frame the above discussion in a rigourous mathematical statement. The production economy is entirely characterized by the family $f = (f_1, \ldots, f_L)$ of production functions. Pick some $r \geq 2$, and let us denote by $\mathcal{F}^r$ the set of all $f = (f_1, \ldots, f_L)$ which satisfies conditions 1 to 4, each f_ℓ being r times continuously differentiable. In Appendix F we prove the following:

Theorem 24. *Generically in $\mathcal{F}^r$, equalising the prices of $2K$ goods equalises the prices of factors:*

$$[\varphi_\ell(w^1) = \varphi_\ell(w^2), \quad 1 \leq \ell \leq 2K] \Rightarrow w^1 = w^2. \tag{25}$$

Corollary 25. *If $L \geq 2K$, generically in $\mathcal{F}^r$, equalising the prices of all goods equalises the prices of factors:*

$$\varphi(w_1) = \varphi(w_2) \Rightarrow w^1 = w^2$$

In other words, if $L \geq 2K$, for "almost all" $f \in \mathcal{F}^r$, factor prices equalisation holds in the economy where $f = (f_1, \ldots, f_L)$ are the production functions, *whatever the total factor endowments*. Since there is no equivalent of Lebesgue measure in infinite-dimensional spaces, such as $\mathcal{F}^r$ the expression "almost all" is meaningless and has to be replaced by a similar but different notion: this is the idea of "genericity" which is explained in Appendix F. Note, however, that if we took a parametric approach, and chose some specification of the production functions, depending on finitely many parameters $\theta_1, \ldots, \theta_N$, then we would be back to the usual interpretation of "almost all", so that property (25) would hold for all values of $\theta_1, \ldots, \theta_N$, except for a set of measure zero in R^N.

The proof of Theorem 24 is given in Appendix F. It builds on the above intuition, that is, if there are more equations than unknowns, then system (25) should have no solution, and if it happens to have one, it should be a pathological situation which could be corrected by an arbitrarily small perturbation of the function φ. But one can perturb φ only through perturbing f, and the main difficulty in the proof is that the dependence of φ on the $(f_1, \ldots, f_L)$ is not explicit. One then has to go through

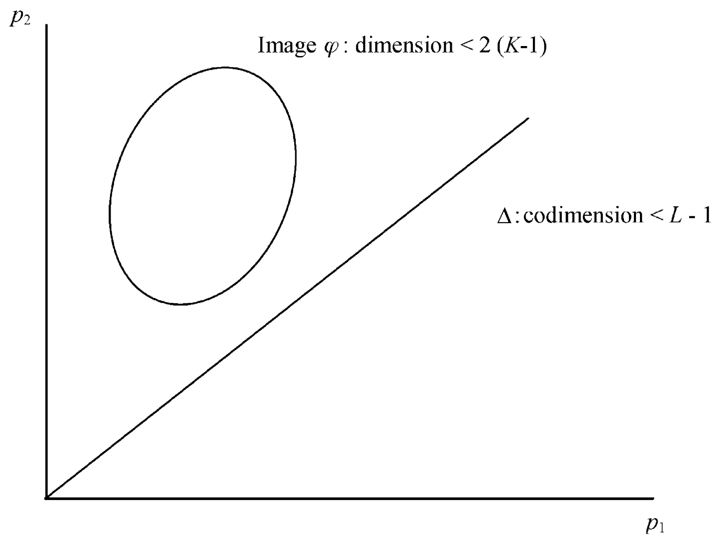

Fig. 7. Illustrating the proof of Theorem 24.

the implicit function theorem, and to prove that a certain number of determinants do not vanish, which gives the proof a rather technical character.

Let us now give a direct, and informal, description of the proof. Normalize goods prices by restricting them to the subspace E_1 defined by (5), and normalize factors prices by restricting them to the subspace:

$$W_1 = \{w \in R^K_{++} | w_1 = 1\}$$

Now consider the $(2K-1)$ dimensional space:

$$\Delta = \{(w^1, w^2) | w^1 \neq w^2\} \subset W_1 \times W_1$$

and define a map $\Phi : \Delta \to E_1 \times E_1$ by $\Phi(w^1, w^2) = (\varphi(w^1), \varphi(w^2))$.

In $E_1 \times E_1$, we consider the diagonal:

$$D = \{(p_1, p_2) \in E_1 \times E_1 | p_1 = p_2\}$$

As illustrated in Fig. 7, the codimension of D is $L-1$ (because E_1 has dimension $L-1$), and the dimension of the image of Φ is at most $2(K-1)$ (because W_1 has dimension $K-1$). If $2(K-1) < L-1$, that is, if $2K \leq L$, the image of Φ will not intersect Δ in general: this is a non-linear version of the geometrical fact that a randomly chosen linear subspace of dimension N will not intersect another randomly chosen linear subspace of codimension $> N$. It follows that, when $2K \leq L$, the equality $\varphi(w^1) = \varphi(w^2)$ cannot occur when $w^1 \neq w^2$.

If $L < 2K$, there are two possible cases: $K < L < 2K$ and $L \leq K$. In the first one, we still have a factor price equalisation theorem, but a weaker one. It is contained in the following geometrical result, which is also proved in Appendix F.

Theorem 26. *If $K < L < 2K$, generically in $\mathcal{F}^r$, the cone*

$$\mathcal{C} = \{(w^1, w^2) \in R^{2K}_{++} | w^1 \neq w^2, \varphi(w^1) = \varphi(w^2)\}$$

is either empty, or a submanifold of dimension $(2K - L)$

6. Possible applications

In the course of this paper, we have already provided some direct applications of our results. There are many more potential applications to be developed, and we will indicate here some of them.

6.1. The properties of Walrasian equilibria

Consider a set of economies, along which characteristics such as endowments, preferences, and production functions, may vary. The *e* quilibrium correspondence associates with any such economy the set of its equilibria. This correspondence carries a lot of qualitative information about equilibria: how many are there, in which cases is there only one, how does their number change across economies. In general settings, such as the one adopted by Kehoe (1982, 1983), we get broad answers by using powerful techniques from topology: typically, one would find that the number of equilibrium is generically odd, so that equilibria appear or disappear in pairs. In more focused settings, such as the one we described in this paper, we can be much more precise, and do some comparative statics.

Assume there are H consumers, with private endowments v_h inelastically supplied, $1 \le h \le H$. Suppose individual preferences remain fixed but v_h is changed to $v_h + \mathrm{d}v_h$. The factors prices then change from w to $w + \mathrm{d}w$. Using previous results and in particular Proposition 21, together with elementary findings of demand theory, we can write the following formula, which relates dw and dv:

$$\mathrm{d}wM\left(w, \sum_h x_h\right) = \mathrm{d}v - F(\varphi(w), w)[(\partial_p D^c)\,\mathrm{d}wF(\varphi(w), w) - \sum_{k=1}^{K}\left(\sum_{h=1}^{H}(\partial_R D_h)[(v_h^k - F_k(\varphi(w), w)x_h)]\,\mathrm{d}w_k\right) + wdv_h(\partial_R D_h)].$$

Here $x_h = D_h(\varphi(w), wv_h)$ is household's h consumption, $M(w,.)$ is defined in formula (24), F is the factor values matrix, F_k its k th line, v_h is the individual endowment of agent h and x_h, $(\partial_p D^c)$ is the matrix of compensated aggregate demand and $(\partial_R D_h)$ is the vector of income effects for agent h. Note that, when there is only one consumer, the matrix operating on dw is:

$$M(w,.) + F(\varphi(w), w)(\partial_p D^c)^t F(\varphi(w), w) \tag{26}$$

from which one can extract a $(K-1)\times(K-1)$ negative definite and invertible matrix. In the case of several consumers, the additional term:

$$F(\varphi(w), w)\sum_{k=1}^{K}\left(\sum_{h=1}^{H}[\partial_R D_h](v_h^k - F_k(\varphi(w), w)x_h)\,\mathrm{d}w_k\right) \tag{27}$$

links the quantity of factor k held by consumer h with the quantity of factor k that he/she indirectly consumes: heterogeneity then matters a lot, as in an exchange economy linked to the individual income effects. Understanding this term will be crucial to study the bifurcations of the Walras correspondence. For instance, it would help to identify sets of economies with no bifurcations, thereby throwing light on the question of uniqueness, and help to appraise the effect of heterogeneity on the incomes effects on the number of equilibria.

Another avenue of research concerns the Walras tâtonnement. It seems that the original idea of Walras himself was to consider a process whereby tâtonnement bear only on the prices of factors, while the prices of goods are immediately adjusted to the prices implied by the new factors prices (see Reyberol, 1999). Formally, this is equivalent to a tâtonnement process on the excess demand function

$$Z(w) = F(\varphi(w), w)\sum_h D_h(\varphi(w), w)v_h - v$$

for fixed v. First investigations show that again the income effects associated with the additional term (27) play a crucial role in the analysis; we find conditions for the existence of a Keynes effect, that is, situations where a decrease in wages triggers an increase in unemployment.

6.2. The standard theory of trade: comparative statics

Let us sketch some comparative statics analysis of a simple trade model. Consider two countries that have the same production sector, hence the same $\varphi(w)$, but different vector endowments v^a and v^b. Suppose that there are representative consumers a and b, which could be assumed to have the same homothetic preferences if need be, so that there is a representative consumer for the world. The interesting questions then are similar to the ones we raised in the preceding section, and again, the answers are contained in the *trade equilibrium correspondence*. Here is a sample of such questions:

Q1—for a given set of economies, does there exists a non-specialized trade equilibrium, where all goods are produced at the same price in every country (which would then coincide with the world equilibrium with factor mobility), does there exist several and how many? how do they vary with the data and how do we switch to specialization?
Q2—In a non-specialized trade equilibrium, are the price of factors equalised across countries? When is it the case in a specialized trade equilibrium?
Q3—Can specialized equilibria and non-specialized trade equilibria coexist for given data?

We claim that the above analysis, or its suggested extensions, provides appropriate tools for a systematic investigation of the above questions. For instance, Theorem 24 sheds light on the question of factor price equalisation: when enough goods are traded (more than $2K$), then factor price equalisation obtains, even in countries specialize in other goods. When there are fewer traded goods, $K < L < 2K$, the study of the critical economies where factor price equalisation does not hold provides interesting information on the graph of the trade correspondence and on the possible coexistence of specialized and non-specialized equilibria.

6.3. Other extensions and applications

The approach of Bidard (1990) provides a natural entry to the case of joint production, where we hope our results can be extended. Another potential extensions is to the non-standard trade theories: when prices are not competitive, but oligopolistic, prices are still related to marginal costs, and the relations between prices of goods and prices of factor, although different from the ones we have described here, are not dissimilar. Finally, let us mention the theory of taxation, which has been often developed (either with a view to reform or to describe the equilibrium manifold) under the assumption of decreasing returns, when constant returns to scale is a better assumption.

We also believe that our results provide starting points for an improved investigation of several other subjects, such as (a) the analysis of observable consequences of general equilibrium theory, in the line of previous attempts, limited to exchange economies, as in Brown and Matzkin (1996) or Chiappori et al. (2000), and (b) the intertemporal theory of production, rather than the atemporal one we consider here. In an intertemporal context where capital depreciates within the period, goods at period t are produced from goods produced at period $t-1$ and from factors available at period t. Most of our results are expected to apply in this setting, and allow us to derive dynamic non-substitution theorems as well as to exhibit the intertemporal prices of goods as a function of the (time-varying) relative prices of factors and of inter-period interest rates. For instance, the deformation of prices due to technical progress can be assessed. When capital does not depreciate immediately, some additional work is required to adapt the present results.

6.4. Conclusion

We first provide a final assessment of our results, in order to make clearer their position in the complex literature to which they add.

How restrictive is the assumption that all goods are used in production? Let us first note both the assumption that all goods are used in production and the assumption that no intermediate good is used in production are polar cases, and the fact that the second one has been more popular in previous literature than the first does not make it more realistic. The best approximation to reality lies somewhere in between, where each production technology requires specific intermediate goods. The basic requirement for most our conclusions to hold, is that the $L \times L$ matrix $I - A(p, w)$ be invertible, where $A = (a_\ell^k)$ and $a_\ell^k(p, w)$ is the quantity of good k used in the production of one unit of good ℓ, when p is the price of goods, w the price of factors, and the cost-minimizing technology is used (with the convention that $a_\ell^\ell = 0$). If this is the case, then the Preparation Theorem holds, by the arguments in Appendix E, and our results would follow as well.

In other words, the most general setting for our results would have been the assumption that the matrix $I - A(p, w)$ is invertible for every (p, w), which would have covered both polar cases and most intermediate cases simultaneously. We chose not to do so because we did not consider $A(p, w)$ as primary economic data, and we chose to describe the model in terms of production functions and sets. Introducing varying possibilities as to which intermediate goods where used to produce which consumption goods would have greatly complicated the model, and perhaps obscured the fundamental message.

We now summarize. The results of Section 3.1 proved in our polar case hold in the other polar case (where again they do not seem to have been known) as well as the intermediate cases. The results of Sections 3.2 and 4.1 have the same status, although this time they are straightforward and well known in the limit case where goods are produced only from factors. We claim our description of the geometry of the global production set in Section 2 to be original, although some of its aspects have been understood and presented in special contexts. The analytical perspectives of section 3 provides a generalisation of results known in the case when goods are produced from factors to the case when intermediate goods are needed. To the best of our knowledge, the results in section 4, but Proposition 23, are novel. The genericity argument has been presented in the polar case when all other goods are needed for producing any good, but it would extend to all intermediate cases, provided only that the matrix $I - A(p, w)$ be invertible. Genericity should then be understood within the family of admissible economies, that is, within the family production functions which satisfy the given structural assumptions.

In conclusion, although "general equilibrium", as a subject aiming at maximal generality in the understanding of the systemic aspects of economic interactions, is no longer very active, its intellectual apparatus is extremely alive in many specific fields where it has been either partly adapted (standard or new trade theories) and/or simplified to the extreme (with a representative consumer). Also a lot of influential work on policy analysis rests on the use of computable general equilibrium models that adopt specific modelling options that both allow to simplify the general analysis and make it empirically plausible. In all cases, it may be argued that an equilibrium theory of intermediate generality is needed, not mainly to provide specific applications of the intellectual products of the most abstract theory, but essentially and crucially to improve upon the present specific models and increase their policy relevance. The present paper is an attempt of going further into a theory of production of intermediate generality, aiming at the objectives we just described. As we argued above, the most systematic attempts in this direction have been made in international trade theory until the eighties; however, the results presented here are neither uniquely motivated by trade, nor primarily applicable to this field. They concern a broader research program, and aim at providing some useful harbour to additional investigation into this program.

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Appendix A. A detailed primal view of the problem

Take a NS-production plan $y = \sum y_\ell$, with $y_\ell = (x_\ell, v_\ell) \in Y_\ell$. We have:

$$v^k = \sum_\ell v_\ell^k, \quad v_\ell^k < 0 \quad \forall \ell \tag{28}$$

$$x^i = \sum_l x_\ell^i, \quad x_\ell^\ell > 0 \quad \forall \ell \tag{29}$$

$$x_\ell^\ell \leq f_\ell(-x_\ell^{-\ell}, -v_\ell), \quad \forall \ell \tag{30}$$

Since we have constant returns to scale, the set Y_{NS} is a convex cone. Normalize the production plans in each sector by considering the inputs required to produce one unit of good. Set:

$$Y_\ell^1 = \{y = (x, v) \in Y_\ell | x_\ell^\ell = 1\}$$

so that any vector $y \in Y_\ell^1$ describes one of the many ways to produce one unit of good ℓ.

Definition 27. Let $y \in Y_{NSE}$. A family $(\hat{y}_\ell, \lambda_\ell)$, $1 \leq \ell \leq L$ will be called a *normalized y-allocation* if:

$$\hat{y}_\ell = (\hat{x}_\ell, \hat{v}_\ell) \in Y_\ell^1 \tag{31}$$

$$\lambda_\ell > 0 \quad \forall \ell \tag{32}$$

$$y = \sum_\ell \lambda_\ell \hat{y}_\ell \tag{33}$$

Eq. (31) is the normalization condition. It is clear from the definitions that for any NS-production plan, as defined by Eqs. (28)–(30), there is a corresponding normalized allocation. The economic interpretation is straightforward. The $(-\hat{x}_\ell^{-\ell}, -\hat{v}_\ell)$ are the inputs required to produce one unit of good ℓ, with the chosen technology, and the numbers λ_ℓ then denote the level at which this technology has to be set in order to produce (jointly with the others) the bundle y.

If a NS-production plan y is efficient, we have seen in Section 2.2.2 that there exists a unique remuneration plan, $q \in E_1$ such that:

$$qy_\ell = \text{Max}\{qy | y \in Y_\ell\} = 0 \quad \forall \ell \tag{34}$$

and a unique normalized y-allocation $(\hat{y}_\ell, \lambda_\ell)$. Since only technology ℓ can produce good ℓ, the $\hat{y}_\ell$, $1 \leq \ell \leq L$, are linearly independent, and $q\hat{y}_\ell = 0$ for all ℓ. This enables us to define maps γ_ℓ, λ_ℓ and π from Y_{NSE} to Y_ℓ^1, R_{++} and E_1 by:

$$\gamma_\ell(y) = \hat{y}_\ell = (\hat{x}_\ell, \hat{v}_\ell) \quad \text{with}\, \hat{x}_\ell^\ell = 1 \tag{35}$$

$$\lambda_\ell(y) = \lambda_\ell > 0 \tag{36}$$

$$\pi(y) = q = (p, w) \quad \text{with}\, p_1 = 1 \tag{37}$$

It follows from the definitions that:

$$y = \sum_\ell \lambda_\ell(y)\gamma_\ell(y), \quad \gamma_\ell^\ell(y) = 1 \tag{38}$$

$$\pi(y)\gamma_\ell(y) = \text{Max}\{\pi(y)z_\ell | z_\ell \in Y_\ell\} = 0 \quad \forall \ell \tag{39}$$

In other words, $\gamma_\ell(y)$ is the unique (and efficient) way of producing one unit of good ℓ when the global production plan $y = (x, v)$ is aimed for. The activity level at which technology ℓ has to operate, using this production plan, is specified by $\lambda_\ell(y) > 0$, so that $y = \sum_\ell \lambda_\ell(y)\gamma_\ell(y)$. The normalized remuneration plan supporting this production plan is $\pi(y)$, with $\pi(y)y = 0$ and $\pi_1(y) = 1$.

Later on, we shall also consider the map:

$$\gamma : Y_{NSE} \to Y_1^1 \times \cdots \times Y_L^1$$

defined by $\gamma = (\gamma_1, \ldots, \gamma_L)$.

Appendix B. Proof of Proposition 2

Consider the closure of $C(y)$:

$$\bar{C}(y) = \left\{ \sum_\ell \mu_\ell \gamma_\ell(y) | \mu_\ell \geq 0, \ 1 \leq \ell \leq L, \right\}$$

It is a closed convex cone, containing each of the $\gamma_\ell(y)$. Because of Eq. (39), each of the $\gamma_\ell(y)$ also belongs to the hyperplane $T(y)$. It follows immediately that $\bar{C}(y) \subset T(y)$. As a consequence of our constant returns to scale assumption, all of $\bar{C}(y)$ is contained in the global production set Y. Hence $\bar{C}(y) \subset T(y) \cap Y$, and $C(y) \subset T(y) \cap Y_{NSE}$.

Conversely, if $z \in T(y) \cap Y_{NSE}$, then there is a normalized z-allocation $(\gamma_\ell(z), \lambda_\ell(z))$, $1 \le \ell \le L$. By Eq. (39), each $\gamma_\ell(z)$ maximizes $\pi(y)'z_\ell$ subject to $z_\ell \in Y_\ell$, and it follows from uniqueness that $\gamma_\ell(z) = \gamma_\ell(y)$ and $z \in C(y)$. Hence $T(y) \cap Y_{NSE} \subset C(y)$.

Appendix C. Proof of Theorem 3

If $y = (x, v) \in Y_{NSE}$, denote as usual by $T(y)$ the tangent hyperplane in R^{K+L} to Y_{NSE} at y, and by $T(x)$ the tangent hyperplane in R^L to $Y_{NS}(.,v)$ at x. Since $Y_{NSE}(.,v)$ is the intersection of Y_{NSE} with $R^L \times \{v\}$, we must have:

$$T(x) = T(y) \cap (R^L \times \{v\}).$$

Denoting $\text{Facet}_v(x)$ the facet containing x, we have:

$$\text{Facet}_v(x) = Y_{NSE}(.,v) \cap T(x) = Y_{NSE}(.,v) \cap T(y) \cap (R^L \times \{v\}) \tag{40}$$

and using Proposition 2 we find that

$$\text{Facet}_v(x) = C(y) \cap (R^L \times \{v\}) \tag{41}$$

$C(y)$ is an L-dimensional cone in R^{L+K}, and $R^L \times \{v\}$ an L-dimensional affine subspace transversal to the cone. Their intersection is an $(L-K)$-dimensional cone, as announced.

Appendix D. Proof of Theorem 9

Apply Proposition 1 to $y = (x, v)$. Property 2 states that we have the decomposition $y = \sum_\ell \lambda_\ell(y)\gamma_\ell(y)$, where $\gamma_\ell = (\hat{x}_\ell, \hat{v}_\ell)$ and $\hat{x}_\ell^\ell = 1$. Property 3 states each of the $\gamma_\ell = (\hat{x}_\ell, \hat{v}_\ell)$ is supported by q. In other words,

$$q\gamma_\ell = \text{Max}\{px_\ell + wv_\ell | 1 \le f_\ell(-x_\ell^{-\ell}, -v_\ell)\} = 0$$

with $x_\ell^{-\ell} = (x_\ell^1, \ldots, x_\ell^{\ell-1}, x_\ell^{\ell+1}, \ldots x_\ell^L)$ as usual. We rewrite the objective function as follows:

$$px_\ell + wv_\ell = p_\ell x_\ell^\ell - \left(\sum_{i \ne \ell} p_i(-x_\ell^i) + \sum_k w_k(-v^k)\right)$$

The first term on the right-hand side is just p_ℓ, a constant, so that in fact $-\hat{x}_\ell^{-\ell}$ is the set of inputs which minimizes the second term, which is the cost of producing one unit of good ℓ. In other words, $(-\hat{x}_\ell^{-\ell}, -\hat{v}_\ell)$ solves problem (7), and using the notations (8), we have:

$$\gamma_\ell(y) = (-a_\ell^1(q), \ldots, -a_\ell^{\ell-1}(q) 1 - a_\ell^{\ell+1}(q), \ldots, -a_\ell^l(q), -b_\ell^1(q), \ldots, -b_\ell^K(q)) \tag{42}$$

or, equivalently:

$$a_\ell^i(p, w) = -\hat{x}_\ell^i > 0 \quad \forall i \ne \ell \tag{43}$$

$$a_\ell^\ell(p, w) = 0 \tag{44}$$

$$b_\ell^k(p, w) = -\hat{v}_\ell^k > 0 \quad \forall k \tag{45}$$

Eq. (33) can then be written:

$$x^\ell = \lambda_\ell - \sum_{i \ne \ell} a_\ell^i(p, w)\lambda_\ell, \quad v^k = -\sum_\ell b_\ell^k(p, w)\lambda_\ell$$

which gives Eqs. (16) and (17). Note that (16) can be inverted, so that $\lambda(y) = (I - A(q))^{-1}x$. Substituting into (17), we get formula (19):

$$-v = B(p, w)\lambda = B(p, w)(I - A(p, w))^{-1}x = F(p, w)x$$

To prove the second part of the proposition, suppose $p = wB(p, w)(I - A(p, w))^{-1}$ and (x, v) satisfies (16) and (17) for some $\lambda \ge 0$. The ℓ th columns of the matrices $A(q)$ and $B(q)$ define a cost-minimizing bundle for the production of one unit of good ℓ, and then, for each ℓ, a profit maximising $y_\ell \in Y_\ell$ (with zero profit). This implies that for any $\lambda \in R_{++}^L$, the production plan y defined by (16) and (17) is the sum of profit maximising bundles; therefore it is efficient and belongs to Y_{NSE}. The uniqueness of (the direction of) q follows from the same argument as above.

Appendix E. Proof of the Preparation Theorem

We begin by proving that the projection map ψ from Σ to R^K_{++} is one-to-one.

Lemma 28. *If* (p^1, w) *and* $(p^2, w) \in \Sigma$, *then* $p_1 = p_2$

Proof. Suppose otherwise, so that $p_1 \neq p_2$. Introducing the cost functions, as in Eq. (9) we have $p_1 = c(p_1, w)$ and $p_2 = c(p_2, w)$. Consider

$$t = \text{Max}_\ell(p^1_\ell / p^2_\ell)$$

and assume without loss of generality that $t > 1$. Set $p^3 = t\,p^2$, so that $p^3_\ell \geq p^1_\ell\ \forall \ell$, and $p^3_k = p^1_k$ for some k. But $tp^2 = c(tp^2, tw)$ by homogeneity, and hence, using the fact that the cost is a decreasing function of prices:

$$p^3_\ell = c_\ell(p^3, tw) > c_\ell(p^1, w) = p^1_\ell\ \forall \ell$$

Hence $p^3_k > p^1_k$, a contradiction. □

We then prove that the projection map ψ from Σ to R^K_{++} covers all of R^K_{++}. We begin with a technical result. By Condition 4, there is some $y \in Y_{NSE} \cap (R^L_{++} \times R^K) \subset Y^{++}_{NSE}$. Take a supporting price vector, for instance $q = \pi(y)$, and set $A = A(q)$ and $B = B(q)$

Lemma 29. *Take* $q^0 = (p^0, w^0) \in \Sigma$. *Then,* $p^0 \leq w^0 B(I - A)^{-1}$

Proof. Consider the goods price vector p^1 defined by $p^1 = w^0 B(I - A)^{-1}$. We have $p^1 = w^0 B + p^1 A$, meaning that the vector p^1 gives the unitary costs of production if the input prices are (p^1, w^0) and the plan y is used. Set $p^2 = w^0 B(p^1, w^0) + p^1 A(p^1, w^0)$. The vector p^2 gives the unitary costs of production if the prices are (p^1, w^0) and a cost-minimizing plan is used; these costs must be less than the costs of production using any other plan, so that $p^2 \leq p^1$.

The algorithm then proceeds: set $p^3 = w^0 B(p^2, w^0) + p^2 A(p^2, w^0)$; then $p^3 \leq p^2$, and so on. The sequence p^n is a decreasing sequence of positive vectors, so that it must converge to some p satisfying $p = (w^0)B(p, w_0) + pA(p, w^0)$, and by Lemma 27 we must have $p = p^0$. We have proved that $p^1 \geq p^2 \geq \cdots \geq p = p^0$. □

Lemma 30. *The projection map* ψ *from* Σ *to* R^K_{++} *covers all of* R^K_{++}

Proof. We shall prove that the image $\psi(\Sigma)$ is open and closed in R^K_{++}. Openness follows from Lemma 12. To show closedness, consider a sequence w^n in $\psi(\Sigma) \cap R^K_{++}$, converging to some $w \in R^K_{++}$. Take the corresponding sequence p^n such that $(p^n, w^n) \in \Sigma$, so that $w^n = \psi(p^n, w^n)$. By the preceding lemma the p^n are uniformly bounded, so there is a convergent subsequence to some p, and $p' = w' B(p, w)(I - A(p, w))^{-1}$ by continuity. Since $w \in R^K_{++}$ and the matrices B and $(I - A)^{-1}$ have positive coefficients, we must have $p \in R^L_{++}$. So the image $\psi(\Sigma)$ is closed in R^K_{++}. Since R^K_{++} is connected, the only subsets which are both open and closed are the empty set $\varnothing$ and R^K_{++} itself. The result follows. □

The map ψ can be inverted on R^K_{++}, yielding the Preparation Theorem.

Appendix F. Proof of Theorems 24 and 26

F.1. Generic properties

Consider a property $P(\theta)$, depending on $\theta \in \Theta$. In our context, θ indexes parameters that generate different specifications of a model. We would like a suitable mathematical notion to translate the idea that $P(\theta)$ holds true "in general", that is, for "most" specifications of the model. In the case when the parameter space Θ is imbedded in R^N the standard way to formalize this idea is to say that $P(\theta)$ is true almost everywhere, that is, except on a subset of Lebesgue measure zero. In the case when Θ is infinite-dimensional, there is no analogue to the Lebesgue measure, and another formalization must be sought: it is the notion of genericity, which is due to Rene Thom; see Abraham and Robbin (1967) and Aubin and Ekeland (1984) for a full exposition. Θ is assumed to be a complete metric space. A property $P(\theta)$ is *generic* if the set of points θ where it does *not* hold is contained in a countable union of closed subsets with empty interiors. Equivalently, a property $P(\theta)$ is g enericif we have:

$$\Omega = \{\theta | P(\theta) \text{ is true}\} \supset \cap_{n=1}^{\infty} U_n$$

where each $U_n \subset \Theta$ is open and dense in Ω.

If two properties P_1 and P_2 are generic, so are $P_1 \wedge P_2$ (P_1 and P_2) and $P_1 \wedge P_2$ (P_1 or P_2). As a consequence, if P is generic, then its negation (non $-\ P$) cannot be generic. More generally, if a sequence of properties P_n are all generic, then so is $\wedge_n P_n$. In other words, generic properties behave in the same way as properties that are true almost surely, although there is no underlying measure to support them.

F.2. Defining appropriate sets and maps

Let us denote by $\mathcal{F}^r$ the set of all $f = (f_1, \dots, f_L)$ which satisfies Conditions 1–4, each f_ℓ being r times continuously differentiable. Note that there is no natural norm on this space (because our assumptions allow f_ℓ or its derivatives to become unbounded when its arguments go to zero or infinity), but nevertheless there is a natural metric, the precise description of which is given in what follows. In the statements of Theorems 24, and 26, genericity is understood with respect to $\mathcal{F}^r$, for some $r \geq 2$.

Assume $L > K$. Set $J = L + 1 - K$ and denote by $\mathcal{C}^r$ the space of all real-valued functions f on R^J_{++} which are positively homogeneous of degree one, r times continuously differentiable on R^J_{++}, and vanish on the boundary. Let $S = \left\{ z \in R^J_{++} | \sum z_j < 1 \right\}$ be the standard simplex in R^J and S^0 its relative interior. Choose a family Ω_n of open subsets of S^0 such that $\overline{\Omega_n} \subset \Omega_{n+1}$ and $S = \cup \Omega_n$. Endow $\mathcal{C}^r$ with the following family of semi-norms:

$$q^0(f) = \max\{|f(z)|| z \in S\}$$
$$q^n_{i_1 \dots i_K}(f) = \max \left\{ \left| \frac{\partial^m f}{\partial^{i_1} z_1 \dots \partial^{i_J} z_J}(w) \right| | w \in \Omega_n \right\}$$

where $i_1 + \dots + i_J = m \leq r$. This is a countable family of semi-norms, which we relabel simply as $\{q_k\}_{k \geq 1}$. It is well known that the distance:

$$d(f, g) = \sum_k \frac{1}{k^2} \max\{q_k(f - g), 1\}$$

turns $\mathcal{C}^r$ into a separable and complete metric space. To say that $f_n \to f$ in $\mathcal{C}^r$ means that f_n converges to f uniformly on S (and hence uniformly on compact subsets of R^J_+, including the boundary) and that all partial derivatives of f_n converge to the corresponding derivatives of f, uniformly on compact subsets of S (and hence uniformly on compact subsets of R^J_{++}, excluding the boundary). Every neighbourhood of f in $\mathcal{C}^r$ contains a neighbourhood of the form $\{g | q_k(f - g) \leq \varepsilon\}$ for some k and ε.

Let us denote by $\mathcal{F}$ the set of all $f = (f_1, \dots, f_L)$ which satisfies Conditions 1 to 4. Set $\mathcal{F}^r = \mathcal{F} \cap \mathcal{C}^r$, and endow $\mathcal{F}^r$ with the induced topology. Since $r \geq 2$, $\mathcal{F}^r$ is an open subset of $\mathcal{C}^r$, so that Baire's category Theorem holds on $\mathcal{F}^r$.

In the above, we have defined maps $A : R^{K+L}_{++} \to \mathcal{L}(R^L, R^L)$, $B : R^{K+L}_{++} \to \mathcal{L}(R^K, R^L)$ (see section 3.2) and $\varphi : R^K_{++} \to R^L_{++}$ (see Theorem 13). We will also need the truncated map $\bar{\varphi} : R^K \to R^{2K}$ defined by:

$$\bar{\varphi}(w) = (\varphi_1(w), \dots, \varphi_{2K}(w))$$

These maps depend on $f \in \mathcal{F}$, and we shall henceforth write them A_f, B_f and φ_f, $\bar{\varphi}_f$ to stress this dependence. Note that they are all positively homogeneous, A and B of degree 0, and $\bar{\varphi}_f$ of degree one.

For the purposes of the proof, it will be convenient to normalize prices by setting $w_1 = 1$. We set:

$$W_1 = \{w \in R^K_{++} | w_1 = 1\}$$

Consider the $(2K - 1)$-dimensional space:

$$\Delta = \{(w^1, w^2, \lambda) \in W_1 \times W_1 \times R_{++} | w^1 \neq w^2\}$$

and the map Ψ: $\mathcal{F}^r \times \Delta \to R^{2K}$ defined by

$$\Psi(f, w^1, w^2, \lambda) = \lambda \bar{\varphi}_f(w^1) - \bar{\varphi}_f(w^2) \quad (46)$$

F.3. Transversality

Proposition 31. *Generically in $\mathcal{F}^r$, the partial map $\Psi_f : \Delta \to R^{2K}$ defined by $\Psi_f(w^1, w^2, \lambda) = \Psi(f, w^1, w^2, \lambda)$ is transversal to the origin in R^L.*

Again, we refer to Abraham and Robbin (1967) and to Aubin and Ekeland (1984) for a definition of transversality and a statement of the Thom transversality theorem. Saying that Ψ_f is transversal to the origin means that either $\Psi_f^{-1}(0)$ is empty, or that at every $(w^1, w^2, \lambda) \in \Psi_f^{-1}(0)$, the tangent map $D\Psi_f$ is onto. Theorems 24 and 26 both will follow from this proposition.

We shall first prove a weaker result. Fix a compact subset $C \subset W_1$, set:

$$\Delta^C = \{(w^1, w^2, \lambda) \in C \times C \times R_{++} | w^1 \neq w^2\}$$

and denote by Ψ^C_f the restriction of Ψ_f to Δ^C.

Lemma 32. *For every compact subset $C \subset W_1$, every $f \in \mathcal{F}^r$ has an open neighbourhood $\mathcal{N}(f)$ such that, generically with respect to $g \in \mathcal{N}(f)$, the partial map $\Psi^C_g : \Delta^C \to R^L$ is transversal to the origin in R^L.*

Proof. Let C and f be given. Let $G_f = \{(\bar{\varphi}_f(w), w) | w \in C\}$ be the graph of $\bar{\varphi}_f$ over C; it is a compact subset of R_{++}^{L+K-1}. Take a bounded open subset $\mathcal{V} \subset R_{++}^{L+K-1}$ such that $\bar{\mathcal{V}} \subset R_{++}^{L+K-1}$ and $G_f \subset \mathcal{V}$. For each $q = (p, w)$ in G_f, the matrices $A_f(q) = (a_\ell^i)$ and $B_f(q) = (b_\ell^k)$ are obtained by solving for every ℓ the following equations in $a_\ell \in R_{++}^L$, $b_\ell \in R_{++}^K$ and $\lambda_\ell > 0$:

$$\begin{aligned} &a_\ell^\ell = 0 \\ &Df_\ell(a_{-\ell}, b_\ell) = \lambda_\ell q \\ &f_\ell(a_{-\ell}, b_\ell) = 1 \end{aligned}$$

which are the optimality conditions in problem (7). Applying the implicit function theorem, with the help of Condition 3, we find that the solution $(a_{-\ell}, b_\ell)$ and the Lagrange multiplier λ_ℓ depend smoothly on q. It follows that there is a bounded open subset $\mathcal{U}_\ell$ the closure of which, $\bar{\mathcal{U}}_\ell$, is contained in R_{++}^{K+L-1}, such that $(A_f(q), B_f(q)) \in \mathcal{U}_\ell$ for every $q \in \bar{\mathcal{V}}$.

From now on we shall work in the Banach space $C^r(\bar{\mathcal{U}}_\ell)$. Applying the implicit function again, this time in $C^r(\bar{\mathcal{U}}_\ell)$, we find that there is some $\varepsilon_\ell > 0$ such that whenever $\|g_\ell - f_\ell\|_{C^r(\bar{\mathcal{U}}_\ell)} < \varepsilon_\ell$, and $q = (p, w) \in \bar{\mathcal{V}}$, then the solution $(a_{-\ell}, b_\ell, \lambda_\ell)$ of the system:

$$\begin{aligned} &a_\ell^\ell = 0 \\ &Dg_\ell(a_{-\ell}, b_\ell) = \lambda_\ell q \\ &g_\ell(a_{-\ell}, b_\ell) = 1 \end{aligned}$$

has the property that $(a_{-\ell}, b_\ell) \in \mathcal{U}_\ell$, and depends smoothly on g_ℓ (in the $C^r(\bar{\mathcal{U}}_\ell)$-norm) and q. Set:

$$B(f_\ell, \varepsilon) = \{g_\ell \in C^r(\bar{\mathcal{U}}_\ell) | \|g_\ell - f_\ell\|_{C^r(\bar{\mathcal{U}}_\ell)} < \varepsilon\}$$

$$B(f, \varepsilon) = \prod_{\ell=1}^{L} B(f_\ell, \varepsilon)$$

For fixed q, we can find the derivative with respect to g_ℓ as follows. Linearizing the system at $(a_{-\ell}, b_\ell, \lambda_\ell)$, and denoting by $(\alpha_{-\ell}, \beta_\ell, \mu_\ell)$ a tangent vector at that point, we relate them to the tangent vector h_ℓ at g_ℓ by:

$$[D^2 f_\ell(a_{-\ell}, b_\ell)](\alpha_{-\ell}, \beta_\ell) - \mu_\ell(p, w) = -Dh^\ell(a_{-\ell}, b_\ell) \tag{47}$$

$$[Df_\ell(a_{-\ell}, b_\ell)]'(\alpha_{-\ell}, \beta_\ell) = -h^\ell(a_{-\ell}, b_\ell) \tag{48}$$

where $h^\ell \in C^r(\bar{\mathcal{U}}_\ell)$, and $(\alpha_{-\ell}, \beta_\ell, \lambda_\ell) \in R^J \times R$. For any given h^ℓ, this is a system of $(L+K)$ equations with $(L+K)$ unknowns which is always uniquely solvable by Condition 3. By the implicit function theorem, the map $g_\ell \to (a_{-\ell}, b_\ell)$ from $B(f_\ell, \varepsilon_\ell)$ to R^J is C^{r-1} and its derivative is onto.

We now go back to $\bar{\varphi}$. Recall that, for $w \in W_1$ and $g \in \mathcal{F}^r$, we define $\varphi_g(w)$ as the unique p which solves the equation:

$$p = wB_g(p, w)(I - A_g(p, w))^{-1}$$

so that $\Phi : (g, w) \to \varphi_g(w)$ maps $B(f, \varepsilon) \times C$ into R_{++}^L. Using the implicit function theorem again, we see that φ_g is C^{r-1}. We shall henceforth assume that ε has been chosen so small that, for every $g \in B(f, \varepsilon)$, we have $G_g \subset \mathcal{V}$, where G_g is the graph of φ_g over C. By the envelope theorem, the derivative $D\varphi_f(f, w) : (h, \omega) \to \pi$, sending tangent vectors (h, ω) at (f, w) into tangent vectors π at p, is given by:

$$\pi = \omega B(I - A)^{-1} + \omega[(D_f B)h](I - A)^{-1} - \omega B(I - A)^{-1}[(D_f A)h](I - A)^{-1} \tag{49}$$

where $(D_f A, D_f B)$ denotes the derivative at $g = f$ of the map

$$g \to (A_g(\varphi_f(w), w), B_g(\varphi_f(w), w))$$

which is defined by relations (47) and (48) and has been shown to be onto.

Recall that $\bar{\varphi}_f$ is restricted to the first $2K$ components of φ_f. We claim that $D\bar{\varphi}_f(f, w)$ is onto for every (f, w). Indeed, take any $\pi \in R^{2K}$ and any $\omega \neq 0$. Pick some $X \in \mathcal{L}(R^{2K}, R^K)$ such that $\pi - \omega B(I - A)^{-1} = \omega X(I - A)^{-1}$. Since $(D_f A, D_f B)$ is onto, we can find some h such that $(D_f A)h = 0$ and $(D_f B)h = X$. Plugging this into the right-hand side of (49) gives us π, as announced.

Now go back to formula (46) defining Ψ. Set:

$$\Delta(f, \varepsilon, C) = \{(g, w^1, w^2, \lambda) \in B(f, \varepsilon) \times C \times C \times R_{++} | w^1 \neq w^2\}$$

and denote by $\bar{\Psi}^C$ the restriction of Ψ to $\Delta(f, \varepsilon, C)$. Since $w^1 \neq w^2$ in W_1, then $q^1 = (\varphi_f(w^1), w^1)$ and $q^2 = (\varphi_f(w^2), w^2)$ cannot be collinear. Plugging q^1 and q^2 in problem (7), we find different solutions. This means that the points $(a_{-\ell}(q^1), b_\ell(q^1))$ and $(a_{-\ell}(q^2), b_\ell(q^2))$ are distinct, so that the values of the function h and its derivatives at $(a_{-\ell}(q^1), b_\ell(q^1))$ and $(a_{-\ell}(q^2), b_\ell(q^2))$ can be chosen independently. Arguing as we did to prove that that $D\bar{\varphi}_f(f, w)$ is onto, we find that $D\bar{\Psi}^C$ is onto. In particular, it has to be onto at every point where $\bar{\Psi}^C = 0$, which means that $\bar{\Psi}^C$ is transversal to the origin.

By Thom's transversality theorem, generically with respect to $g \in B(f, \varepsilon)$, the partial map $\bar{\Psi}_g^C$ from Δ^C to R^L is transversal to the origin. Note that $g \to \bar{\Psi}_g$ is just the restriction of $g \to \Psi_g^C$ to $B(f, \varepsilon)$. It follows that every point $f \in \mathcal{F}^r$ has a neighbourhood $B^r(f, \varepsilon)$ where, generically with respect to $g \in B^r(f, \varepsilon)$, the map Ψ_g^C is transversal to the origin. □

We now derive Proposition 31 by applying another lemma. Recall first that a Lindelöf space is a topological space such that every covering by open subspaces has a countable subcover. Separable metric spaces (i.e. metric spaces which contain a countable dense subset) are Lindelöf, and every open subset of a Lindelöf space is Lindelöf; in particular, $\mathcal{F}^r$ is Lindelöf.

Lemma 33. *Suppose the space Θ is Baire and Lindelöf and there is a property $P(\theta)$ such that every $\theta \in \Theta$ has an open neighbourhood $\mathcal{V}(\theta)$ on which $P(\theta)$ is generic. Then $P(\theta)$ i* s generic on Θ.

Proof. By the Lindelöf property, take a countable subcover $\mathcal{V}_n$, $n \in N$. Let A_n be the set of points $\theta \in \mathcal{V}_n$ where $P(\theta)$ is true, and denote by B_n the complement of $\overline{(\mathcal{V}_n)}$, the closure of $\mathcal{V}_n$, in Θ. Set $C = \cap_n (A_n \cup B_n)$. By assumption, we have $A_n \supset \cap_k A_{n,k}$, where the $A_{n,k}$, $k \in N$, are open and dense in $\mathcal{V}_n$. Set $B_{n,k} = A_{n,k} \cup B_n$, clearly an open and dense subset in Θ, and C contains the intersection $\cap_{n,k} B_{n,k}$. Since the $\mathcal{V}_n$, $n \in N$, cover Θ, every point $\theta \in C$ must belong to some $\mathcal{V}_n$, and therefore cannot belong to its complement B_n. It follows that it belongs to A_n, so that $P(\theta)$ is true on C, as announced. □

Combining Lemmas 32 and 33, we find that, generically in $\mathcal{F}^r$, the partial map $\Psi_f^C : D^C \to R^{2K}$ will be transversal to the origin in R^{2K}. Taking a family of compact subsets $C_n \subset W_1$, such that $C_n \subset C_{n+1}$ and $\cup C_n = W_1$, and denote by $P_n(f)$ the property:

$$P_n(f) = \{\Psi_f^{C_n} \text{ is transversal to } 0 \in R^{2K}.\}$$

Since $P_n(f)$ is generic in $\mathcal{F}^r$, so is $\wedge_n P_n(f)$. This means that Ψ_f is transversal to the origin, and Proposition 31 is proved. So the set $(\Psi_f)^{-1}(0)$ either is a submanifold of codimension $2K$ in Δ (and hence of dimension $2K - 1 - L$) or is empty. Since Δ has dimension $2K - 1$, the latter has to be the case, and Theorem 24 follows.

Corollary 25 is an immediate consequence.

If $w^1 \neq w^2 \in R_{++}^K$ are such that $\varphi(w^1) = \varphi(w^2)$, since φ is positively homogeneous of degree one, we can find some positive λ_1 and λ_2 such that $w^1/\lambda_1 \in W_1$ and $w^2/\lambda_2 \in W_1$, so that:

$$\frac{\lambda_1}{\lambda_2}\varphi(w^1/\lambda_1) = \varphi(w^2/\lambda_2)$$

yielding a point $(w^1/\lambda_1, w^2/\lambda_2, \lambda_1/\lambda_2) \in (\Psi_f)^{-1}(0)$. In other words, the cone

$$\{(w^1, w^2) \in R^{2K} | w^1 \neq w^2, \varphi(w^1) = \varphi(w^2)\}$$

is generated by $(\Psi_f)^{-1}(0)$. If Ψ_f is transversal to the origin, this cone is a submanifold with one more dimension than $(\Psi_f)^{-1}(0)$, namely $2K - L$, and Theorem 26 follows.

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