J. DIFFERENTIAL GEOMETRY
13 (1978) 287-301

THE HOPF-RINOW THEOREM IN INFINITE DIMENSION

IVAR EKELAND

I. Statement of results

We begin by reviewing some essential features. By a *Riemannian manifold* M we understand a connected C^∞-manifold modelled on some Hilbert space H, such that the tangent space $TM_p \simeq H$ carries a scalar product $\langle \cdot, \cdot \rangle_p$ which is C^∞ in $p \in M$ and defines on TM_p a norm $\|\cdot\|_p$ equivalent to the original norm of H.

If p and q are two points in M, a *path* from p to q is a continuous map $c\colon [0, 1] \to M$ such that $c(0) = p$ and $c(1) = q$. The set of all piecewise C^∞ paths from p to q will be denoted by $\mathscr{C}_p^q$. If $c \in \mathscr{C}_p^q$ is such a path, its *length* $L_p^q(c)$ is the real number defined by

$$L_p^q(c) = \int_0^1 \|\dot{c}(t)\|_{c(t)} dt . \tag{1.1}$$

The *geodesic distance* d on M is defined by

$$d(p, q) = \inf\{L_p^q(c) \mid c \in \mathscr{C}_p^q\} , \qquad \forall p, q \in M . \tag{1.2}$$

It is compatible with the manifold topology of M. Any path $c \in \mathscr{C}_p^q$ such that $d(p, q) = L_p^q(c)$ and the speed $\|\dot{c}\|_c$ is constant will be called a *minimal geodesic*; it must be C^∞ and satisfy the equation (where ∇ denotes the Levi-Civita connection)

$$\nabla_{\dot{c}(t)}\dot{c}(t) = 0 , \tag{1.3}$$

which means that $\dot{c}(t)$ is obtained from $\dot{c}(0) \in TM_p$ by parallel translation along c. Conversely, any solution c of (1.3) is called a geodesic. The manifold M will often be assumed to be complete for the metric d; this will imply that solutions of (1.3) are defined for all $t \in \boldsymbol{R}$, i.e., that geodesics can be indefinitely extended.

Throughout this paper, for $\delta > 0$ and $p \in M$, we shall use the following notations:

$$B_p^\delta = \{\xi \in TM_p \mid \|\xi\|_p < \delta\} , \qquad S_p^\delta = \{\xi \in TM_p \mid \|\xi\|_p = \delta\} , \tag{1.4}$$

Communicated by R. S. Palais, November 10, 1976. Sponsored by the United States Army under Contract No. DAAG29-75-C-0024.

(1.5) $\mathscr{B}_p^\delta = \{m \in M \mid d(p, m) < \delta\}$, $\mathscr{S}_p^\delta = \{m \in M \mid d(p, m) = \delta\}$.

Whenever the solution of (1.3) with the initial condition $\dot{c}(0) = \xi \in TM_p$ exists up to $t = 1$, we set $\exp_p \xi = c(1)$, and call $\exp_p$ the *exponential map*. If the Riemannian manifold M is complete, $\exp_p \xi$ is defined for all $\xi \in TM_p$. Even if it is not, by the usual theorems on differential equations (e.g., [5, Th. IV. 1]), there is a neighborhood $\mathscr{U}$ of $(0, p)$ in TM such that the map $(\xi, m) \to \exp_m \xi$ is well-defined and C^∞ on $\mathscr{U}$. Now consider the map $(\xi, m) \to (\exp_m \xi, m)$ from $\mathscr{U}$ to $M \times M$. Its tangent map at $(0, p)$ is easily seen to be an isomorphism, so that we can apply the implicit function theorem. It follows that we can find $\delta_1 > 0$ with the property that, for all $\delta \in]0, \delta_1[$, there exists an $\eta > 0$ such that, whenever $m \in \mathscr{B}_p^\eta$, we have the inclusion $\mathscr{B}_m^\delta \supset \mathscr{B}_p^\eta$, and the map $\exp_m : B_m^\delta \to \mathscr{B}_m^\delta$ is an isomorphism.

Note in particular that any two points in $\mathscr{B}_p^\eta$ can be joined by a unique minimal geodesic, depending smoothly on the endpoints; i.e., whenever q and m belong to $\mathscr{B}_p^\eta$, there is a single $\xi \in TM_q$ such that $m = \exp_q \xi$, and the map $(m,q) \to \xi$ is C^∞.

We define $\Delta(p)$ as the supremum of all $\eta > 0$ with the property that any two points in $\mathscr{B}_p^\eta$ can be joined by a unique minimal geodesic, depending smoothly on the end points. We have just shown that $\Delta(p) > 0$. It follows from the definition that, for all $\delta \in]0, \Delta(p)[$, the exponential map is a C^∞ diffeomorphism of B_p^δ onto $\mathscr{B}_p^\delta$, and of S_p^δ onto $\mathscr{S}_p^\delta$:

$$d(p, m) < \Delta(p) \Rightarrow \exists \xi : m = \exp_p \xi \text{ and } \|\xi\|_p = d(p, m) . \tag{1.6}$$

The Hopf-Rinow theorem [7] states that any two points on a complete finite-dimensional Riemannian manifold can be joined by a minimal geodesic. This is no longer true in the infinite-dimensional case as observed by Grossman [4] and MacAlpin [6], who construct in Hilbert space an infinite-dimensional ellipsoid, the great axis points of which cannot be joined by a minimal geodesic. Recently, Atkin [1] has modified the Grossman counterexample to construct a complete infinite-dimensional Riemannian manifold M, and give two points on M which cannot be joined by any geodesic at all. In other words, the exponential map need not be surjective in the infinite-dimensional case.

In a preceding paper [2], the author proved that any two points can be joined by a path which is almost a minimal geodesic.

Theorem A. *Let M be a complete* (*infinite-dimensional*) *Riemannian manifold, and take two points p, q on M. For every $\varepsilon > 0$, there exist a C^∞ path c from p to q and a vector $\xi \in TM_p$ such that*

$$\int_0^1 \|\dot{c}(t)\|_{c(t)}\, dt \leq \varepsilon + d(p, q) , \tag{1.7}$$

$$\int_0^1 \|\dot{c}(t) - \xi(t)\|^2_{c(t)}\, dt \leq \varepsilon , \tag{1.8}$$

where $\xi(t) \in TM_{c(t)}$ is obtained from ξ by parallel translation along c.

In this paper, we shall prove that almost all points can be joined to a prescribed endpoint by a unique minimal geodesic. Recall that a G_δ subset is a countable intersection of open subsets.

Theorem B. *Let M be a complete Riemannian manifold, and take a point q on M. The set T of points $p \in M$ such that there exists a unique minimal geodesic from p to q contains a dense G_δ.*

Since M is a complete metric space, the Baire category theorem holds on M, so that a dense G_δ subset of M is very large indeed; for instance, a countable intersection of dense G_δ subsets is still a dense G_δ subset, and hence nonempty.

Note that the "uniqueness" part is of interest even when M is finite dimensional. In this case, $M \setminus T$ is the set of points $p \in M$ such that there exist at least two minimizing geodesics from p to q, and is known as the *cut locus* of q. Theorem B thus implies that the cut locus of any point in a complete finite-dimensional Riemannian manifold is included in a countable union of closed subsets with empty interior. This is a known fact, although the usual proof is different, relying on transversality arguments applied to the exponential map from q. In the infinite-dimensional case, however, even the "existence" part of Theorem B is new, settling a question raised in [1] and [2].

The proofs of Theorem A and B rely on special versions of Theorem 1.1 of [2], which is rephrased here for the reader's convenience (taking $\lambda = \sqrt{\varepsilon}$ in the original statement):

Theorem 1.1. *Let V be a complete metric space, and $F: V \to \boldsymbol{R}$ a lower semicontinuous function such that* $\inf F \neq \pm\infty$. *For every* $\varepsilon > 0$, *there exists some point $u \in V$* such that

$$F(u) \leq \varepsilon + \inf F , \tag{1.9}$$

$$F(v) \geq F(u) - \varepsilon d(u, v) , \qquad \forall v \in V . \tag{1.10}$$

The proof of Theorem A relies on a "smooth, Riemannian" version of Theorem 1.1, which was ([2])

Theorem A′. *Let M be a complete Riemannian manifold, and $f: M \to \boldsymbol{R}$ a nonnegative C^1 function. Then for every $\varepsilon > 0$, there exists some point $p \in M$ such that*

$$f(p) \leq \varepsilon + \inf f , \tag{1.11}$$

$$\|\operatorname{grad} f(p)\|_p \leq \varepsilon . \tag{1.12}$$

Similarly, the proof of Theorem B will rely on a "local, Riemannian" version of Theorem 1.1. In [3], such a result was proved in the framework of

Banach spaces with differentiable norms, and it is of course no trouble at all to restate it in framework of Riemannian manifolds. We begin by a definition:

Definition 1.2. Let M be a Riemannian manifold, and f a real-valued function on M. We shall say that f is *locally ε-supported* at $p \in M$ iff there exist an open neighborhood $\mathscr{U}$ of p and a C^∞ function $g: \mathscr{U} \to \boldsymbol{R}$ such that $g(p) = 0$ and

$$(1.13) \qquad f(m) - f(p) \geq g(m) - \varepsilon d(m. p)\,, \qquad \forall m \in \mathscr{U}\,.$$

Taking the local chart defined by the exponential map, we get the following characterization.

Proposition 1.3. *If f is locally ε-supported at $p \in M$, for every $\varepsilon' > \varepsilon$ there exist $\eta' > 0$ and $\zeta' \in TM_p$ such that*

$$(1.14) \qquad f(\exp_p \xi) - f(p) - \langle \xi, \zeta' \rangle_p \geq -\varepsilon' \|\xi\|_p\,, \qquad \forall \xi \in B_p^{\eta'}\,.$$

Conversely, if formula (1.14) *holds with $\varepsilon' = \varepsilon$, then f is locally ε-supported at p.*

Proof. Let us first assume formula (1.14) holds with $\varepsilon' = \varepsilon$, for some $\eta' = \eta > 0$ and some $\zeta' = \zeta \in TM_p$. We can always assume that $\eta < \Delta(p)$, so that $\exp_p^{-1}$ is a well-defined C^∞ map from $\mathscr{B}_p^\eta$ onto B_p^η, and formula (1.6) holds. Writing all this into (1.14), we get

$$(1.15) \qquad f(m) - f(p) - \langle \exp_p^{-1} m, \zeta \rangle_p \geq -\varepsilon d(p, m)\,, \qquad \forall m \in \mathscr{B}_p^\eta\,,$$

which coincides with formula (1.13) if we define $g: \mathscr{B}_p^\eta \to \boldsymbol{R}$ by

$$(1.16) \qquad g(m) = \langle \exp_p^{-1} m, \zeta \rangle_p\,.$$

There remains to prove the first part of Proposition 1.3. Assume condition (1.13) is satisfied, and let $\varepsilon' > \varepsilon$ be given. Choose $\eta \in]\in 0, \Delta(p)[$ so small that $\mathscr{B}_p^\eta \subset \mathscr{U}$. Taking formula (1.6) into account, we rewrite (1.13) as

$$(1.17) \qquad f(\exp_p \xi) - f(p) \geq g(\exp_p \xi) - \varepsilon \|\xi\|_p\,, \qquad \forall \xi \in B_p^\eta\,.$$

But the function $g \circ \exp_p: B_p^\eta \to \boldsymbol{R}$ is differentiable at zero, so that there exist $\eta' \in]0, \eta[$ and $\zeta' \in TM_p$ with (recall that $g(p) = 0$)

$$(1.18) \qquad \|\xi\|_p \leq \eta' \Rightarrow |g(\exp_p \xi) - \langle \xi, \zeta' \rangle_p| \leq (\varepsilon' - \varepsilon)\|\xi\|_p\,.$$

Formulas (1.17) and (1.18) together yield (1.14). q.e.d.

It is clear from the definition that if f is Frechet-differentiable at p, then both f and $-f$ are locally ε-supported at p for every $\varepsilon > 0$. The converse is proved in [3]. So Definition 1.2 can be looked upon as a very weak differentiability property. Its main interest is that it holds for all points of a dense (not G_δ) subset of M:

Theorem B′. *Let M be a Riemannian manifold, and f a lower semi-continuous* *function on M. For every $\varepsilon > 0$, the set of all points $p \in M$ at which f is locally ε-supported is dense in M.*

Proof. Let there be given a point $q \in M$ and a neighborhood $\mathscr{W}$ of q. We have to find some point $p \in \mathscr{W}$ where f is locally ε-supported.

Choose $\delta \in]0, \Delta(q)[$ so small that $B_q^\delta \subset \mathscr{W}$ and f is bounded from below on $\mathscr{B}_q^\delta$ (because of the lower semi-continuity):

$$\inf\{f(m) \mid m \in \mathscr{B}_q^\delta\} \neq -\infty \ . \tag{1.19}$$

By Lemma 1.4 below, we can assume that the closure $\bar{\mathscr{B}}_q^\delta$ is complete in the induced d-metric.

Define a function $\psi: \bar{\mathscr{B}}_q^\delta \to \bar{\boldsymbol{R}}$ by

$$\psi(m) = \begin{cases} [\delta^2 - \|\exp_q^{-1} m\|_q^2]^{-1} & \text{if } m \ \mathscr{B}_q^\delta \ , \\ +\infty & \text{if } m \in \mathscr{S}_q^\delta \ . \end{cases} \tag{1.20}$$

Clearly, ψ is lower semi-continuous on $\bar{\mathscr{B}}_q^\delta$ and smooth on $\mathscr{B}_q^\delta$. We now set $\varphi = \psi + f$. This is a lower semi-continuous function, bounded from below, on the complete metric space $\bar{\mathscr{B}}_q^\delta$. By Theorem 1.1, there is some point $p \in \bar{\mathscr{B}}_q^\delta$ such that

$$\varphi(p) \leq \inf\{\varphi(m) \mid m \in \bar{\mathscr{B}}_q^\delta\} + \varepsilon \ , \tag{1.21}$$

$$\varphi(m) \geq \varphi(p) - \varepsilon d(m, p) \ , \qquad \forall m \in \mathscr{B}_q^\delta \ . \tag{1.22}$$

By formula (1.21), $\varphi(p)$ is finite, so $p \in \mathscr{B}_q^\delta \subset \mathscr{W}$. Writing $\varphi = \psi + f$ into formula (1.22), we get

$$f(m) - f(p) \geq \psi(p) - \psi(m) - \varepsilon d(m, p) \ , \qquad \forall m \in \mathscr{B}_q^\delta \ . \tag{1.23}$$

But this is exactly Definition 1.3, with $g(m) = \psi(p) - \psi(m)$, so f is locally ε-supported at p, and the proof is complete. q.e.d.

Note that we did not assume the Riemannian manifold M to be complete. This is because of

Lemma 1.4. *Let M be a Riemannian manifold. Then every point $p \in M$ has a neighborhood which is complete in the induced d-metric.*

Proof. Choose $\delta \in]0, \Delta(p)[$ so small that all the maps $T_q \exp_p^{-1}$ are norm-bounded in $\mathscr{L}(TM_q, TM_p)$ by some uniform constant k when $q \in \mathscr{B}_p^\delta$. Take $\gamma \in]0, \delta(1 + k)^{-1}[$. We claim that $\bar{\mathscr{B}}_p^\gamma$ is complete.

Let us first note that, for any two points m and q in $\bar{\mathscr{B}}_p^\gamma$, we have, travelling along the minimal geodesics from m to p and from p to q,

$$d(m, q) \leq d(m, p) + d(p, q) \leq 2\gamma \ . \tag{1.24}$$

Let us now take a path $c \in \mathscr{C}_m^q$ which is not contained in $\mathscr{B}_p^\delta$: Denoting by σ

and τ the first and last moments in $]0, 1[$ when $d(p, c(t)) \geq \delta$, we have the inequality:

$$L_m^q(c) \geq \int_0^\sigma \|\dot{c}(t)\|_{c(t)}\, dt + \int_\tau^1 \|\dot{c}(t)\|_{c(t)}\, dr \,. \tag{1.25}$$

Setting $\xi(t) = \exp_p^{-1} c(t) \in B_p^\delta$ for $0 \leq t < \sigma$ and $\tau < t \leq 1$, we get

$$\begin{aligned} \int_0^\sigma \|\dot{c}(t)\|_{c(t)}\, dt &\geq k^{-1} \int_0^\sigma \|\dot{\xi}(t)\|_p\, dt \\ &\geq k^{-1}(\|\xi(\sigma)\|_p - \|\xi(0)\|_p) \geq k^{-1}(\delta - \gamma) \,, \end{aligned} \tag{1.26}$$

and similarly,

$$\int_\tau^1 \|\dot{c}(t)\|_{c(t)}\, dt \geq k^{-1}(\delta - \gamma) \,. \tag{1.27}$$

Writing formulas (1.26) and (1.27) into (1.25) yields $L_m^q(c) \geq 2k^{-1}(\delta - \gamma)$. Taking into account the assumption $\delta > (1 + k)\gamma$ we finally get

$$L_m^q(c) > 2\gamma \,. \tag{1.28}$$

It follows that, whenever m and q belong to $\bar{\mathscr{B}}_p^\gamma$, any path $c \in \mathscr{C}_m^q$ with length $\leq 2\gamma$ must be contained in $\mathscr{B}_p^\delta$. Hence

$$d(m, q) = \inf\{L(c) \mid c \in \mathscr{C}_m^q,\ c(t) \in \mathscr{B}_p^\delta,\ \forall t \in [0, 1]\} \,. \tag{1.29}$$

Setting $\xi(t) = \exp_p^{-1} c(t) \in B_p^\delta$, we get

$$L(c) \geq k^{-1} \int_0^1 \|\dot{\xi}(t)\|_p\, dt \geq k^{-1} \|\xi(1) - \xi(0)\|_p \,. \tag{1.30}$$

Writing this into formula (1.29) yields

$$d(m, q) \geq k^{-1} \|\exp_p^{-1} m - \exp_p^{-1} q\|_p \,, \qquad \forall m, q \in \bar{\mathscr{B}}_p^\gamma \,. \tag{1.31}$$

It follows from this estimation that if q_n, $n \in N$, is a Cauchy sequence in $\bar{\mathscr{B}}^\gamma$, then $\exp_p^{-1} q_n$ will be a Cauchy sequence in $\bar{B}_p^\gamma$, and hence will converge to some $\xi \in \bar{B}_p^\gamma$, so that q_n will converge to $\exp_p \xi \in \bar{\mathscr{B}}_p^\gamma$. Hence $\bar{\mathscr{B}}_p^\gamma$ is complete, and so is the proof.

II. Proof of Theorem B

From now on, we are given a complete Riemannian manifold M and some point $q \in M$. We shall denote by d_q the function $m \to d(q, m)$ on M.

For any $p \neq q$, we set $D(p) = \inf\{\Delta(p), d(q, p)\} > 0$. For any $\delta \in]0, D(p)[$

and any path $c \in \mathscr{C}_p^q$, we denote by $T_p^\delta(c)$ the set of all moments when c crosses $\mathscr{S}_p^\delta$:

$$T_p^\delta(c) = \{t \in [0, 1] \mid d(c(t), p) = \delta\} . \tag{2.1}$$

With any $\alpha > 0$, we associate the nonempty closed subset $C_p^\delta(\alpha)$ of S_p^δ defined by

$$C_p^\delta(\alpha) = \{c(t) \mid c \in \mathscr{C}_p^q, L_p^q(c) \leq d(p, q) + \alpha, t \in T_p^\delta(c)\} . \tag{2.2}$$

Recall that the diameter of $C_p^\delta(\alpha)$, denoted by diam $C_p^\delta(\alpha)$, is the supremum of the distance between two points in $C_p^\delta(\alpha)$.

The proof of Theorem B goes through five lemmas.

Lemma 2.1. *Assume d_q is locally ε-supported at $p \in M \setminus \{q\}$ for some $\varepsilon > 0$. Then, for any $\theta > 4\sqrt{\varepsilon}$,*

$$\exists \eta \in]0, D(p)[: \forall \delta \in]0, \eta[, \exists \alpha > 0 : \text{diam } C_p^\delta(\alpha) \leq \theta\delta . \tag{2.3}$$

Proof. Take $\varepsilon' > \varepsilon$ and $\beta > 0$ so small that

$$\theta > 4(\varepsilon' + \beta/2)^{1/2} . \tag{2.4}$$

By Proposition 1.3 there exist $\eta \in]0, D(p)[$ and $\zeta \in TM_p$ such that

$$\|\xi\|_p \leq \eta \Rightarrow d(q, \exp_p \xi) \geq d(q, p) + \langle \xi, \zeta \rangle_p - \varepsilon' \|\xi\|_p . \tag{2.5}$$

A first consequence of this is as follows. Applying the triangle inequality,

$$d(q, \exp_p \xi) \leq d(q, p) + d(p, \exp_p \xi) , \tag{2.6}$$

and taking formula (1.6) into account, we get

$$\|\xi_p\| \leq \eta \Rightarrow (1 + \varepsilon')\|\xi\|_p \geq \langle \xi, \zeta \rangle_p , \tag{2.7}$$

which means that

$$\|\zeta\|_p \leq 1 + \varepsilon' . \tag{2.8}$$

We now take $\delta \in]0, \eta[$, any path $c \in \mathscr{C}_p^q$ with $L(c) \leq d(q, p) + \beta\delta$, and any $s \in T_p^\delta(c)$. It follows from the definitions that

$$d(q, c(s)) \leq L(c) - \int_0^s \|\dot{c}(t)\|_{c(t)}\, dt \leq d(q, p) + \beta\delta - d(p, c(s)) . \tag{2.9}$$

Writing this inequality into formula (2.5), and setting $\xi(s) = \exp_p^{-1} c(s) \in S_p^\delta$, we get

$$(2.10)\qquad -(1-\varepsilon')\|\xi(s)\|_p + \beta\delta \geq \langle \xi(s), \zeta\rangle_p \,.$$

Dividing by δ throughout, this becomes

$$(2.11)\qquad \langle \zeta, \xi(s)/\delta\rangle_p \leq -1 + \varepsilon' + \beta \,.$$

Taking formula (2.8) into account, we get

$$(2.12)\qquad \langle -\zeta/\|\zeta\|_p, \xi(s)/\delta\rangle_p \geq (1-\varepsilon'-\beta)(1+\varepsilon')^{-1} \,.$$

As both $-\zeta/\|\zeta\|$ and $\xi(s)/\delta$ are unit vectors, this implies that

$$(2.13)\qquad \|\xi(s)/\delta + \zeta/\|\zeta\|_p\|_p^2 \leq 2(2\varepsilon'+\beta)(1+\varepsilon')^{-1} \,.$$

If $c' \in \mathscr{C}_p^q$ is any other path with $L(c') \leq d(p, q) + \beta\delta$, and if $s' \in T_p^\delta(c')$, we also will have, setting $\xi'(s') = \exp_p^{-1} c'(s')$,

$$(2.14)\qquad \|\xi'(s')/\delta + \zeta/\|\zeta\|_p\|_p^2 \leq 2(2\varepsilon'+\beta)(1+\varepsilon')^{-1} \,.$$

Comparing formulas (2.13) and (2.14), we get

$$(2.15)\qquad \|\xi(s) - \xi'(s')\|_p \leq 2\sqrt{2}(2\varepsilon'+\beta)^{1/2}(1+\varepsilon')^{-1/2}\delta \,.$$

Taking inequality (2.4) into account, this becomes

$$(2.16)\qquad \|\xi(s) - \xi'(s')\|_p \leq \theta\delta \,,$$

which is the desired result, since $\xi(s)$ and $\xi'(s')$ are arbitrary points in $C_p^\delta(\beta\eta)$.

q.e.d.

We now introduce the subset R_θ of M defined by

$$(2.17)\qquad R_\theta = \{p \neq q \mid \exists\delta \in]0, D(p)[, \exists\alpha > 0 \colon \operatorname{diam} C_p^\delta(\alpha) \leq \theta\delta\} \,.$$

Lemma 2.1 implies that if d_q is ε-supported at $p \neq q$, then p belongs to R_θ for all $\theta > 4\sqrt{\varepsilon}$. More precisely,

Lemma 2.2. *Assume d_q is locally ε-supported at $p \in M\backslash\{q\}$ for some $\varepsilon > 0$. Then p belongs to the interior of R_θ for all $\theta > 4\sqrt{\varepsilon}$.*

Proof. Choose $k \in]1, 2[$ and $\theta' > 4\sqrt{\varepsilon}$ such that $k^2(2-k)^{-1}\theta' = \theta$. Now take any $\delta_1 \in]0, D(p)[$. It follows from the definition of $D(p)$ that the map $\varphi_m = \exp_m^{-1} \circ \exp_p$ is well-defined from $B_p^{\delta_1}$ into TM_m, for all $m \in \mathscr{B}_p^{\delta_1}$. Note that $\varphi_m(\xi)$ is C^∞ in (m, ξ) and that φ_p is the identity on $B_p^{\delta_1}$. It follows that $\delta_2 \in]0, \delta_1[$ can be found so that

$$(2.18)\qquad \|\xi\|_p \leq \delta_2 \quad \text{and} \quad d(m,p) \leq \delta_2 \Rightarrow k^{-1} \leq \|T_\xi\varphi_m\| \leq k \,.$$

Set $\delta_3 = \delta_2/3$. For any $m \in \mathscr{B}_p^{\delta_3}$ and any two points ξ and ξ' in $B_p^{\delta_3}$, the inverse image by φ_m of the line segment between $\varphi_m(\xi)$ and $\varphi_m(\xi')$ lies entirely within

$B_p^{3\delta_3}$, so that estimation (2.18) holds all along. It follows easily that whenever m, $\exp_p \xi$ and $\exp_p \xi'$ belong to $\mathscr{B}_p^{\delta_3}$, we have the inequality:

$$k^{-1}\|\xi - \xi'\|_p \leq \|\varphi_m(\xi) - \varphi_m(\xi')\|_m \leq k\|\xi - \xi'\|_p . \tag{2.19}$$

By Lemma 2.1, we can choose $\delta \in]0, \delta_3[$ and $\alpha > 0$ such that

$$\operatorname{diam} C_p^{3\delta/4}(\alpha) \leq 3\delta\theta'/4 . \tag{2.20}$$

Set $\eta = \inf\{\delta/4, \alpha/3, 3\delta(1 - k^{-1})/4\}$. We claim that $\mathscr{B}_p^\eta \subset R_\theta$.

First of all, we notice that $\mathscr{B}_p^{3\delta/4} \subset \mathscr{B}_m^\delta \subset \mathscr{B}_p^{5\delta/4}$ whenever $m \in \mathscr{B}_p^{\delta/4}$, by the triangle inequality,. Since $5\delta/4 < \delta_1 < D(p)$, any two points in $\mathscr{B}_m^\delta$ can be joined by a minimal geodesic depending smoothly on the end points, so $\eta < D(m)$ for all $m \in \mathscr{B}_p^{\delta/4}$.

Take any $m \in \mathscr{B}_p^\eta$, any path $c \in \mathscr{C}_m^q$ such that $L_m^q(c) \leq d(m, q) + \alpha/3$, and any time $s \in T_m^\delta(c)$. Set $\xi_m = \exp_m^{-1} c(s) \in S_m^\delta$. We define a new path $\bar{c} \in \mathscr{C}_m^q$ by

$$\bar{c}(t) = \begin{cases} c(2t(1 - s) + 2s - 1) & \text{for } \frac{1}{2} \leq t \leq 1 , \\ \exp_m 2t\xi(s) & \text{for } 0 \leq t \leq \frac{1}{2} . \end{cases} \tag{2.21}$$

Clearly, $\bar{c}$ is obtained from c by cutting short between m and $c(s)$, so that $L_m^q(\bar{c}) \leq L_m^q(c)$. We now go one step further to build a path $\hat{c} \in \mathscr{C}_p^q$; set $\mu = \exp_p^{-1} m$, and define

$$\hat{c}(t) = \begin{cases} \bar{c}(2t - 1) & \text{for } \frac{1}{2} \leq t \leq 1 , \\ \exp_p 2t\mu & \text{for } 0 \leq t \leq \frac{1}{2} . \end{cases} \tag{2.22}$$

Clearly,

$$L_p^q(\hat{c}) = L_m^q(\bar{c}) + d(m, p) \leq d(m, q) + \tfrac{1}{3}\alpha + d(m, p) . \tag{2.23}$$

Using the triangle inequality we get

$$L_p^q(\hat{c}) \leq d(p, q) + d(m, p) + \tfrac{1}{3}\alpha + d(m, p) \leq d(p, q) + \alpha . \tag{2.24}$$

Take $\sigma \in T_p^{3\delta/4}(\hat{c})$, and set $\zeta_p = \exp_p^{-1} c(\sigma) \in S_p^{3\delta/4}$. If $c' \in \mathscr{C}_m^q$ is another path such that $L_m^q(c') \leq d(m, q) + \alpha/3$, we define $\xi'_m \in S_m^\delta$, $\bar{c}' \in \mathscr{C}_m^q$, $\hat{c}' \in \mathscr{C}_p^q$, $\zeta'_p \in S_p^{3\delta/4}$ in the same way, and we still have $L_p^q(\hat{c}') \leq d(p, q) + \alpha$. It follows from formula (2.20) that

$$\|\zeta_p - \zeta'_p\|_p \leq 3\delta\theta'/4 . \tag{2.25}$$

Using estimation (2.18), this implies that

$$\|\varphi_m(\zeta_p) - \varphi_m(\zeta'_p)\|_m \leq 3\delta k\theta'/4 . \tag{2.26}$$

The same estimation applied to $\|\zeta_p\|_p = 3\delta/4$ yields

$$\|\varphi_m(\zeta_p) - \exp_m^{-1}(p)\|_m \geq 3\delta k^{-1}/4 \ , \tag{2.27}$$

and since $\|\exp_m^{-1}(p)\|_m = d(m, p)$, this yields

$$\|\varphi_m(\zeta_p)\|_m \geq 3\delta(2k^{-1} - 1)/4 \ , \tag{2.28}$$

and likewise,

$$\|\varphi_m(\zeta'_p)\|_m \geq 3\delta(2k^{-1} - 1)/4 \ . \tag{2.29}$$

It follows from the construction that

$$\xi_m = \delta\varphi_m(\zeta_p)/\|\varphi_m(\zeta_p)\|_m \ , \tag{2.30}$$

$$\xi'_m = \delta\varphi_m(\zeta'_p)/\|\varphi_m(\zeta'_p)\|_m \ , \tag{2.31}$$

and hence

$$\|\xi_m - \xi'_m\|_m \leq \delta \|\varphi_m(\zeta_p) - \varphi_m(\zeta'_p)\|_m/\inf\{\varphi_m(\zeta_p), \varphi_m(\zeta'_p)\} \ . \tag{2.32}$$

Using inequalities (2.26) and (2.28), this yields

$$\|\xi_m - \xi'_m\|_m \leq \delta k^2(2 - k)^{-1}\theta' \ , \tag{2.33}$$

which is the desired result, since ξ_m and ξ'_m are two arbitrary points in $C_m^{\delta(\alpha)}$.

q.e.d.

Take $p \neq q$ and $\delta \in \,]0, D(p)[$. We shall say that a path $c \in \mathscr{C}_q^p$ is *geodesic inside* $\mathscr{B}_p^\delta$ if there exists $\xi \in TM_p$ such that $c(t) = \exp_p t\xi$ for $0 \leq t \leq \delta/\|\xi\|_p$. We denote by R_∞ the set of points $p \in M\backslash\{q\}$ such that there exists an increasing sequence $\rho_n \to D(p)$ with the following property: for any sequence $c_n \in \mathscr{C}_p^q$ such that $L_p^q(c_n) \to d(p, q)$ and c_n is geodesic inside $\mathscr{B}_p^{\rho_n}$, the sequence $\dot{c}_n(0)/\|\dot{c}_n(0)\|_p$ converges in S_p^1. We first connect this up by proving that $R_\infty \supset \bigcap_{\theta>0} R_\theta$.

Lemma 2.3. *Assume p belongs to R_θ for every $\theta > 0$. Then p belongs to R_∞.*

Proof. By the assumption on p, there exists a sequence δ_n in $]0, D(p)[$ and a decreasing sequence $\alpha_n > 0$ converging to zero such that

$$\operatorname{diam} C_p^{\delta_n}(\alpha_n) \leq n^{-1}\delta_n \ . \tag{2.34}$$

Let ρ_n be an increasing sequence such that: $\delta_n \leq \rho_n < D(p)$ and $\rho_n \to D(p)$. Now let c_n be any sequence in $\mathscr{C}_p^q$ such that $L_p^q(c_n) \to d(p, q)$ and c_n is geodesic inside $\mathscr{B}_p^{\rho_n}$. By readjusting the time parameter if necessary, we may assume that there exists $\xi_n \in S_p^1$ with $c_n(t) = \exp_p t\xi_n$ for $0 \leq t \leq \rho_n$, so that $\xi_n = \dot{c}_n(0)/\|\dot{c}_n(0)\|_p$. Note that whenever $k \geq n$, we have $\delta_n \leq \rho_n \leq \rho_k$, so that c_k intersects $\mathscr{S}_n^{\delta_n}$ at $\exp_p \delta_n\xi_k$.

For any prescribed n, there is an $N \geq n$ such that $L_p^q(c_k) \leq d(p, q) + \alpha_n$ for all $k \geq N$. Taking any $l \geq k \geq N$, we find by the preceding remark that both $\delta_n \xi_l$ and $\delta_n \xi_k$ belong to $c_p^{\delta_n}(\alpha_n)$. By formula (2.34), this boils down to

$$(2.35) \qquad \forall n, \exists N: \forall l \geq k \geq N, \qquad \|\xi_k - \xi_l\|_p \leq n^{-1}.$$

so that the sequence ξ_n is Cauchy, and hence converges in S_p^1. q.e.d.

Take $p \in M \backslash \{q\}$ and $\delta \in]0, D(p)[$. Recall that the distance from q to $\mathscr{S}_p^\delta$ is defined as

$$(2.36) \qquad d(q, \mathscr{S}_p^\delta) = \inf \{d(q, m) \mid m \in \mathscr{S}_p^\delta\}.$$

It follows easily from the definitions that

$$(2.37) \qquad d(q, \mathscr{S}_p^\delta) = d(q, p) - \delta.$$

A nearest point to q in $\mathscr{S}_p^\delta$ is a point $m \in \mathscr{S}_p^\delta$ such that $d(q, m) = d(q, \mathscr{S}_p^\delta)$. Such points do not always exist in the infinite-dimensional case, and even in the finite-dimensional case they need not be unique. So one of the main interests of R_∞ lies in the following.

Lemma 2.4. *Assume p belongs to* R_∞. *Then there exists* $\mu_p \in S_p^1$ *such that for all* $\delta \in]0, D(p)[$, $\exp_p \delta\mu_p$ *is a nearest point to q in* $\mathscr{S}_p^\delta$. *This point* μ_p *is the common limit of all sequences* $\dot{c}_n(0)/\|\dot{c}_n(0)\|_p$ *for* $c_n \in \mathscr{C}_p^q$ *geodesic inside* $\mathscr{B}_p^{\rho_n}$ *and* $L_p^q(c_n) \to d(p, q)$.

Proof. Take a sequence c_n in $\mathscr{C}_p^q$ such that c_n is geodesic inside $\mathscr{B}_p^{\rho_n}$ and $L_p^q(c_n) \to d(q,p)$. We know that $\dot{c}_n(0)/\|\dot{c}_n(0)\|_p$ converges to some $\bar{\xi}$ in S_p^1. Now this limit $\bar{\xi}$ cannot depend on the particular sequence chosen. For if c_n' were another, with $\dot{c}_n'(0)/\|\dot{c}_n'(0)\|_p$ converging to $\bar{\xi}'$, we could define a third sequence c_n'' with the same properties by setting alternatively $c_n'' = c_n$ if n is odd and $c_n'' = c_n'$ if n is even, and $\dot{c}_n''(0)/\|\dot{c}_n''(0)\|_p$ would still have to converge since $p \in R_\infty$. So $\bar{\xi} = \bar{\xi}'$, and we denote by μ_p this common value.

Let c_n be any sequence in $\mathscr{C}_p^q$ such that $L_p^q(c_n) \to d(p, q)$. Take $s_n \in T_p^{\rho_n}(c_n)$, and set $c_n(s_n) = \exp_p \rho_n \xi_n$, with $\xi_n \in S_p^1$. We replace c_n by the shortcut $\bar{c}_n$ constructed as follows:

$$(2.38) \qquad \bar{c}_n(t) = \begin{cases} c_n(2t(1 - s_n) + 2s_n - 1) & \text{for } \frac{1}{2} \leq t \leq 1, \\ \exp_p 2t\rho_n\xi_n & \text{for } 0 \leq t \leq \frac{1}{2}. \end{cases}$$

We have $L_p^q(c_n) \geq L_p^q(\bar{c}_n) \geq d(p, q)$, so $L_p^q(\bar{c}_n)$ must converge to $d(p, q)$. Moreover, $\bar{c}_n$ obviously is geodesic inside $\mathscr{B}_p^{\rho_n}$. If follows that the sequence $\xi_n = \dot{\bar{c}}_n(0)/\|\dot{\bar{c}}_n(0)\|$ converges to μ_p in S_p^1.

Now take any $\delta \in]0, D(p)[$. There is an N so large that $\rho_n \geq \delta$ whenever $n \geq N$; then $\bar{c}_n$ intersects $\mathscr{S}_p^\delta$ at $\exp_p \delta\xi_n$, which converges to $\exp_p \delta\mu_p$. We have

(2.39) $$d(q, \exp_p \delta\xi_n) \leq L_p^q(\bar{c}_n) - \delta .$$

Letting n go to infinity yields

(2.40) $$d(q, \exp_p \delta\mu_p) \leq d(p, q) - \delta .$$

By formula (2.37), this means precisely that $\exp_p \delta\mu_p$ is a nearest point to q in $\mathscr{S}_p^\delta$. q.e.d.

Another useful property of R_∞ is the following.

Lemma 2.5. *Assume p belongs to R_∞. Then so does* $\exp_p \delta\mu_p$, *whenever* $\delta \in]0, D(p)[$.

Proof. Set $m = \exp_p \delta\mu_p$, with $0 < \delta < D(p)$; we have to prove that $m \in R_\infty$. Note first that $D(m) \geq D(p) - \delta$; indeed, whenever $\eta < D(p) - \delta$, there is some $\delta' < D(p)$ with $\mathscr{B}_m^\eta \subset \mathscr{B}_p^{\delta'}$, so any two points in $\mathscr{B}_p^{\delta'}$—and hence in $\mathscr{B}_m^\eta$—can be joined by a unique miminal geodesic, depending smoothly on the end-points. Let ρ_n, $0 < \rho_n < D(p)$, $\rho_n \to D(p)$, be the increasing sequence characteristic of $p \in R_\infty$. By the preceding remark, it is possible to choose an increasing sequence δ_n, $0 < \sigma_n < D(m)$, $\delta_n \to D(m)$, such that

(2.41) $$\sigma_n \geq \rho_n - \delta , \qquad \forall n \in N .$$

Let c_n be a sequence in $\mathscr{C}_m^q$ such that $L_m^q(c_n) \to d(m, q)$ and c_n is geodesic inside $\mathscr{B}_m^{\sigma_n}$. We can write, by readjusting the time parameter if necessary, $c_n(t) = \exp_m t\zeta_n$ for $0 \leq t \leq \sigma_n$, with $\|\zeta_n\|_m = 1$. Now set $\bar{m} = \exp_p D(p)\mu_p \in M$. We claim that the sequence $c_n(\rho_n - \delta) \in M$ converges to $\bar{m}$. From inequality (2.41) it will follow that the sequence ζ_n converges in S_m^1, and Lemma 2.5 will be proved.

Since $\rho_n \to D(p)$, we can choose N_1 so large that $\rho_n \geq \delta$ whenever $n \geq N_1$, so that the sequence $c_n(\rho_n - \delta)$ is well-defined, starting at N_1.

Let $\varepsilon > 0$ be given. We have seen in Lemma 2.4 that m is a nearest point to q in $\mathscr{S}_p^\delta$. It follows from this and formula (2.37) that

(2.42) $$L_m^q(c_n) + \delta \to d(p, q) .$$

Take $s_n \in T_p^{\rho_n}(c_n)$, and set $\xi_n = \rho_n^{-1} \exp_p^{-1} c_n(s_n)$. Define a new path $\bar{c}_n \in \mathscr{C}_p^q$ by

(2.43) $$\bar{c}_n(t) = \begin{cases} c_n(2t(1 - s_n) + 2s_n - 1) & \text{for } \frac{1}{2} \leq t \leq 1 , \\ \exp_p 2t\rho_n\xi_n & \text{for } 0 \leq t \leq \frac{1}{2} . \end{cases}$$

Let us do some elementary computations:

(2.44) $$L_p^q(\bar{c}_n) \leq d(p, m) + d(m, c_n(s_n)) + \int_{s_n}^1 \|\dot{c}_n(t)\|_{c_n(t)}\, dt$$

$$\leq \delta + \int_0^{s_n} \|\dot{c}_n(t)\|_{c_n(t)}\,dt + \int_{s_n}^1 \|\dot{c}_n(t)\|_{c_n(t)}\,dt = \delta + L_m^q(c_n)\,.$$

It follows from this and formula (2.42) that $L_p^q(\bar{c}_n) \to d(p, q)$. Moreover, $\bar{c}_n$ is clearly geodesic inside $\mathscr{B}_p^{\rho_n}$. Using Lemma 2.4, we conclude that the sequence ξ_n converges to μ_p in S_p^1. Recalling that $\rho_n \to D(p)$, we see that $\bar{c}_n(\frac{1}{2}) = \exp_p \rho_n \xi_n$ converges to $\bar{m}$ in M. Since $\bar{c}_n(1/2) = c_n(s_n)$, we get

$$d(\bar{m}, c_n(s_n)) \to 0\,. \tag{2.45}$$

We know that

$$d(m, c_n(s_n)) \geq d(p, c_n(s_n)) - \delta = \rho_n - \delta\,. \tag{2.46}$$

Hence $\rho_n - \delta \leq s_n$. Let us do some elementary computations again:

$$\begin{aligned} & d(c_n(s_n), c_n(\rho_n - \delta)) \\ & \quad \leq \int_{s_n}^{\rho_n - \delta} \|\dot{c}_n(t)\|_{c(t)}\,dt \\ & \quad = L_m^q(c_n) - \int_0^{\rho_n - \delta} \|\dot{c}_n(t)\|_{c(t)}\,dt - \int_{s_n}^1 \|\dot{c}_n(t)\|_{c(t)}\,dt\,. \end{aligned} \tag{2.47}$$

Using inequality (2.41) and the fact that c_n is geodesic inside $\mathscr{B}_m^{\sigma_n}$, we reduce (2.47) to

$$d(c_n(s_n), c_n(\rho_n - \delta)) \leq L_m^q(c_n) - (\rho_n - \delta) - d(q, c_n(s_n))\,. \tag{2.48}$$

But $c_n\,(s_n) \in \mathscr{S}_p^{\rho_n}$; using formula (2.37) yields

$$d(q, c_n(s_n)) \geq d(q, \mathscr{S}_p^{\rho_n}) = d(q, p) - \rho_n\,. \tag{2.49}$$

Writing this into formula (2.48), we get

$$d(c_n(s_n), c_n(\rho_n - \delta)) \leq L_m^q(c_n) + \delta - d(q, p)\,. \tag{2.50}$$

Letting n go to infinity, we have by formula (2.42)

$$d(c_n(s_n), c_n(\rho_n - \delta)) \to 0\,. \tag{2.51}$$

Adding (2.45) and (2.51) yields the desired result. q.e.d.

The hard part of the proof is over now, and the remainder is soft analysis. For all $n \in N$, define Ω_n as the interior of $R_{1/n}$. By construction, Ω_n is an open subset of M. By Lemma 2.2, for any $\varepsilon \in\,]0, 1/16n^2[$, it contains the set T_ε of all points $p \neq q$ at which d_q is ε-supported. By Theorem B′, the set T_ε is dense in M. So Ω_n, $n \in N$, is a sequence of open dense subsets of M. Since M is a complete metric space, the Baire category theorem holds, and the intersection

$R = \bigcap_n \Omega_n$ is a dense G_δ subset of M. Note that $R_\theta \subset R_{\theta'}$, whenever $0 < \theta < \theta'$, so that $R \subset \bigcap_\theta R_\theta$. It follows by Lemma 2.3 that $R \subset R_\infty$. We claim that from every $p \in R_\infty$ there is a single minimal geodesic to q.

The proof now mimics the classical argument for Hopf-Rinow in the finite-dimensional case (see [7] for instance). Take any $p \in R_\infty$ and set $\rho = d(p, q)$. Take $\delta \in]0, D(p)[$. By Lemma 2.4, there is in $\mathscr{S}_p^\delta$ a nearest point $\exp_p \delta\mu_p$ to q. Define a C^∞ path c by $c(t) = \exp_p t\rho\mu_p$. We claim that, for all $t \in]0, 1[$,

$$c(t) \in R_\infty \quad \text{and} \quad d(q, c(t)) = \rho(1 - t)\ . \tag{2.52}$$

Letting $t \to 1$, since d_q is continuous, this yields $d(q, c(1)) = 0$, so $c(1) = q$ and c is actually a geodesic from p to q. By construction, its length is $\rho = d(p, q)$, so c is minimal. If $c' \in \mathscr{C}_p^q$ is another minimal geodesic, then the constant sequence in $\mathscr{C}_p^q$ defined by $c_n = c'$ for all n satisfies $L_p^q(c_n) \to d(p, q)$ and certainly is geodesic inside $\mathscr{B}_p^{\rho n}$. By Lemma 2.4, $\dot{c}'(0)/\delta$ has to be μ_p, so c' coincides with c.

So we are left with proving formula (2.52) for all t in $[0, 1[$. By Lemmas 2.4 and 2.5, used in conjunction with formula (2.37), it is true for all t in $[0, D(p)[$. Let us denote by $\bar{t}$ the supremum of all $s \in]0, 1[$ such that formula (2.52) holds on $[0, s[$, and assume that $\bar{t} < 1$. We will derive a contradiction. Indeed, set $\bar{p} = c(\bar{t})$, and take any $\delta \in]0, D(\bar{p})[$. We already know that $\bar{t} > D(p) > 0$. Since c is smooth, there exists a time $s \in]0, \bar{t}[$ such that $d(\bar{p}, c(s)) < \delta/4$. Set $c(s) = m$.

Now, since $\delta < D(\bar{p})$, any two points in $\mathscr{B}_{\bar{p}}^\delta$ can be joined by a unique minimal geodesic, depending smoothly on the endpoints. By the triangle inequality, $\mathscr{B}_m^{\delta/2} \subset \mathscr{B}_{\bar{p}}^\delta$, so that $\delta/2 < D(m)$. By assumption, formula (2.52) is satisfied on $[0, \bar{t})$. It follows that $d(q, m) = \rho(1 - s)$, and $m \in R_\infty$. By Lemma 2.4, $\exp_m \delta\mu_m/2$ is a nearest point to q in $\mathscr{S}_m^{\delta/2}$, and we have, by formula (2.37),

$$d(q, \exp_m \delta\mu_m/2) = \rho(1 - s) - \delta/2\ . \tag{2.53}$$

By the triangle inequality,

$$d(p, \exp_m \delta\mu_m/2) \geq d(p, q) - d(q, \exp_m \delta\mu_m/2) = \rho s + \delta/2\ . \tag{2.54}$$

Set $\rho s(\rho s + \delta/2)^{-1} = \alpha$. We define a path c' from p to $\exp_m \delta\mu_m/2$ by

$$c'(t) = \begin{cases} \exp_p [t\rho s\mu_p/\alpha] & \text{for } 0 \leq t \leq \alpha\ , \\ \exp_m [\frac{1}{2}(t - \alpha)\delta\mu_m/(1 - \alpha)] & \text{for } \alpha \leq t \leq 1\ . \end{cases} \tag{2.55}$$

The length of c' is precisely $\rho s + \delta/2$, and $\|\dot{c}'\|_c$ is constant. It follows by inequality (2.54) that c' is a minimizing geodesic from p to $\exp_m \delta\mu_m/2$. Since $c'(t) = c(ts/\alpha)$ for $0 \leq t \leq \alpha$, and c also is a geodesic, we get

$$c'(t) = c(ts/\alpha) \qquad \text{for } 0 \leq t \leq 1\ . \tag{2.55}$$

It follows that

$$(2.56) \qquad c(t) = \exp_m \left[\frac{\alpha(t-s)}{2s(1-\alpha)} \delta\mu_m \right] \qquad \text{for } s \leq t \leq s/\alpha .$$

By Lemmas 2.4 and 2.5, always taking equality (2.37) into account, formula (2.52) holds for $s \leq t \leq s/\alpha$. But $s/\alpha = s + \frac{1}{2}\delta/\rho$. Since $d(c(\bar{t}), c(s)) < \delta/4$, and the speed along c has constant magnitude ρ, we have $\bar{t} - s < \frac{1}{4}\delta/\rho$. So $\bar{t} - s < \delta/4$. Hence $\bar{t} < s/\alpha$, the desired contradiction.

References

[1] C. J. Atkin, *The Hopf-Rinow theorem is false in infinite dimensions*, Bull. London Math. Soc. **7** (1975) 261–266.
[2] I. Ekeland, *On the variational principle*, J. Math. Anal. Appl. **47** (1974) 324–353.
[3] I. Ekeland & G. Lebourg, *Generic Frechet-differentiability, and perturbed optimization problems in Banach spaces*, to appear in Trans. Amer. Math. Soc. **224** (1976) 193–216.
[4] N. Grossman, *Hilbert manifolds without epiconjugate points*, Proc. Amer. Math. Soc. **16** (1965) 1365–1371.
[5] S. Lang, *Introduction to differentiable manifolds*, Wiley-Interscience, New York, 1962.
[6] J. MacAlpin, Ph. D. Thesis, Columbia University, 1965.
[7] J. Milnor, *Morse theory*, Annals of Math. Studies, No. 51, Princeton University Press, Princeton, 1963.

UNIVERSITY OF PARIS-DAUPHINE