ELSEVIER

Journal of Mathematical Economics 31 (1999) 111–129

JOURNAL OF Mathematical ECONOMICS

Disaggregation of excess demand functions in incomplete markets [1]

Pierre-André Chiappori [a,*], Ivar Ekeland [b]

[a] *Department of Economics, University of Chicago, 5801 Ellis Avenue, Chicago, 60637 IL, USA*
[b] *CEREMADE et Institut de Finance, Université Paris-Dauphine, Paris, France*

Received 14 May 1998; accepted 29 June 1998

Abstract

We are interested in general equilibrium incomplete markets, where the number of consumers is N, the number of goods is L, and the dimension of the space of admissible trades is K (the case of complete markets being then $K=(L-1)$). We prove that, if $N \geq K$, any non-vanishing analytic function satisfying the natural extension of the Walras law is, locally at least, the excess demand function of such a market. To be precise, consider a map $\theta \to \Phi(\theta)$ associating with a T-dimensional parameter θ a K-dimensional linear subspace $\Phi(\theta)$ of $\mathbf{R}^L$, representing the set of market transactions allowed by θ. Given parameter values $\bar{\theta}_1, \ldots, \bar{\theta}_T$, and a non-vanishing analytic function Z defined on some neighbourhood of $\bar{\theta}$ with values in $\mathbf{R}^L$, with $X(\theta) \in \Phi(\theta) \forall \theta$, then there exist concave utility functions U_n, $1 \leq n \leq N$ and individual endowments $\omega_1, \ldots, \omega_N$, such that the corresponding aggregate excess demand function coincides with Z on a (possibly smaller) neighbourhood of $\bar{\theta}$. If Z vanishes at $\bar{\theta}$, the disaggregation is still possible, but requires $(K+1)$ agents.

Keywords: Incomplete markets; Excess demand function; Disaggregation

* Corresponding author.

[1] The authors wish to thank Andreu Mas-Collel (Mas-Colell) for suggesting the problem to them, and for giving them much of his time and experience. They also wish to thank Vladimir Arnold, Darrell Duffie, Heracles Polemarchakis, and the participants in seminars at CORE, Paris, Toulouse and Chicago where this result was presented. Financial support from the NSF (grant SBR 9729559) is gratefully acknowledged. Errors are ours.

PII: S0304-4068(98)00059-7

1. Setting up the problem

1.1. Introduction

Hugo Sonnenschein was the first to ask whether a aggregate excess demand function, arising from the aggregation of individual excess demands from utility-maximizing agents, had some identifiable property. It is well-known by now that the answer is negative if the number of agents is greater than or equal to the number of goods in the economy. This result is essentially due to Sonnenschein, Mantel and Debreu, and has spawned a lot of interest and further work (see Shafer and Sonnenschein, 1982 for a review of the literature up to 1982).

All these papers deal with excess demand functions for complete markets. A negative result of the same kind has been recently proved for demand functions (see Chiappori and Ekeland, 1996 and Chiappori and Ekeland, forthcoming), still within the framework of complete markets. By contrast, the case of incomplete markets has been much less studied. For the convenience of the reader, we will recall here enough of this theory to motivate the mathematical model and state the disaggregation problem.

Consider first an exchange economy with N agents and L goods. Each agent n is characterized by his utility function U_n (assumed to be smooth and strictly concave) and his initial endowment ω_n. He then solves the optimization problem:

$$\text{Maximize } U_n(x) \tag{1}$$

$$p(x-\omega_n) \le 0 \tag{2}$$

$$x \ge 0 \tag{3}$$

the solution of which, $x_n(p)$, is his demand function. Summing all individual demands, we get the aggregate demand function:

$$\boldsymbol{X}(p) = \sum_{n=1}^{N} x_n(p) \tag{4}$$

Similarly, the excess demand function of agent n is given by:

$$z_n(p) = x_n(p) - \omega_n \tag{5}$$

and the aggregate excess demand function by:

$$Z_n(p) = \sum_n x_n(p) - \sum_n \omega_n \tag{6}$$

Note that the excess demand functions z_n and Z are all positively homogeneous of degree zero.

It is the famous theorem of Mantel and Debreu, 1974 that, provided $n \ge L$, any continuous map Z from a compact subset with the interior of $\mathbf{R}^L_+$ into $\mathbf{R}^L$, satisfying the natural homogeneity condition, and Walras Law is an aggregate excess demand function (see Shafer and Sonnenschein, 1982 again for a precise statement and a review of the literature). Geanakoplos and Polemarchakis (1980) proved that, if $N = L - 1$, while the preceding result may fail globally, it will still hold locally, at least outside an equilibrium given any $\bar{\boldsymbol{p}}$ in the interior of $\boldsymbol{R}_{+}^{L}$ and a positively homogeneous map Z defined from some neighbourhood of $\bar{\boldsymbol{p}}$ into $\mathbf{R}^{L}$, there is a (possibly smaller) neighbourhood of $\bar{\boldsymbol{p}}$ on which Z decomposes as a sum (6), where each of the $x_n(p)$ is an individual demand, and the ω_n are suitably chosen individual endowments.

In later developments, Chiappori and Ekeland (1996) and Chiappori and Ekeland (forthcoming) showed that such a local result extends to demand functions (while the global result does not): if $N \geq L$, then every analytic map $\boldsymbol{X}$ defined from some neighbourhood of $\bar{\boldsymbol{p}}$ into $\mathbf{R}^{L}$ decomposes, possibly on a smaller neighbourhood, as a sum (4), where each of the $x_n(p)$ is an individual demand. To do so, they introduced techniques from differential geometry, which are basically due to Elie Cartan (see Cartan, 1945), and for which a modern reference is Bryant et al. (1991). These techniques will be extensively used in the present paper.

1.2. Incomplete markets

All previous papers deal with complete markets. The market is *incomplete* if certain trades are not possible. Such situations arise naturally in the framework of risk-trading. Consider for instance a two-period model. In period 0, agent n receives an endowment ω_n^0. He then trades in the L' goods present in the economy, and in M risky assets; the price of good l is p_l^0, and the price of asset m is q_m^0. In period 1 one of S possible states of nature will be realized. If state s occurs, the prevailing price system then becomes p^s, agent n receives an endowment ω_n^s, and each unit of asset m yields to its owner a dividend d_m^s.

In the first period, agent n buys a bundle x^0 of goods for immediate consumption and invests in a portfolio y of assets; in the second period, he finds which state of nature s obtains, and then buys a new bundle of goods x^s. His overall utility ex ante is $U_n(x^0, x^1, \dots, x^S)$. With perfect foresight, he is led to solve the following optimization problem:

$$\text{Maximize } U_n(x^0, x^1, \dots, x^S) \tag{7}$$

$$\boldsymbol{p}^0\left(x^0 - \omega_n^0\right) \leq -q^0 y \tag{8}$$

$$p^s\left(x^s - \omega_n^s\right) \leq d^s y, 1 \leq s \leq S \tag{9}$$

$$x^{s'} \geq 0 \; s' = 0, \dots S \tag{10}$$

Let us assume that, in relations (8) and (9), the constraints are effective, which means that the inequalities are in fact equalities. The constraints (8), (9) can then be rewritten as follows:

$$\mathscr{L}_p(z) \in E_q \tag{11}$$

where $z=(x^0-\omega_n^0,\ x^1-\omega_n\ldots,\ x^S-\omega_n^S)$ is the excess demand, the linear space E_q is defined by

$$E_{d,q}=\{(-q^0y,d^1y,\ldots,d^Sy)|y\in\boldsymbol{R}^M\} \tag{12}$$

and the linear map $\mathscr{L}_p$ from $\mathbf{R}^{L'(S+1)}$ to $\boldsymbol{R}^{(S+1)}$ is given by

$$\mathscr{L}_p(z)=(p^0z^0,p^1z^1,\ldots,p^Sz^S) \tag{13}$$

Denote by $x_n(d,\ q,\ p)$ the demand function of consumer n, and by $z_n(d,\ q,\ p)$ the corresponding excess demand. The latter satisfies the linear constraints (11), and by Walras' law, so must the aggregate excess demand $Z(d,\ q,\ p)$:

$$Z(d,q,p)=\sum_n z_n(d,q,p) \tag{14}$$

$$\mathscr{L}_p(Z(d,q,p)\in E_{d,q} \tag{15}$$

From now on, we will assume that the positivity constraints (3) or (10) are slack for all consumers. The optimization problem of agent n can then be rewritten as follows:

$$\text{Maximize } U_n(z^0+\omega_n^0,\ldots,z^s+\omega_n^s) \tag{16}$$

$$z\in\Phi(d,q,p) \tag{17}$$

where

$$\Phi(d,q,p)=(\mathscr{L}_p)^{-1}(E_{d,q}) \tag{18}$$

It follows from Eq. (12) that the space $E_{d,q}$ has dimension M at most. If it has exactly dimension M, then its codimension in $\mathbf{R}^{L'(S+1)}$ is $(S+1-M)$, and if in addition $\mathscr{L}_p$ has full rank, then $\Phi(d,\ q,\ p)$ is a linear subspace of codimension $(S+1-M)$ in $\mathbf{R}^{L'(S+1)}$, so that the relation (17) boils down to $(S+1-M)$ linearly independent equations in z. This is the generic case; on the other hand, great technical difficulty is caused by the fact that the codimension of $\Phi(d,\ q,\ p)$ may jump above $(S+1-M)$ for certain values of $(d,\ q,\ p)$, either because the dimension of $E_{d,q}$ falls below M or because $\mathscr{L}_p$ degenerates. We shall assume that we are in the generic case.

If $M=S$, that is, there are enough contingent claims to cover all possible states of the world in the second period, the market is complete, and $\Phi(q,\ p)$ has codimension 1, so it is a hyperplane. Formally, this is exactly the same situation as the first one we described (see Eq. (2)). On the other hand, if $M<S$, the market is incomplete, and $\Phi(q,\ p)$ is a linear subspace of smaller dimension.

One generally chooses d to be some function of p, say $d_m^s=\mathbf{D}_m^s(p)$, the two polar cases being, the following.

- The constant case (nominal assets): the matrix **D** is independent of q. For instance, $d_m^s=1$ if $m=s$ and 0 otherwise. Each unit of security m pays one unit of account if state m occurs. If $M=S$, perfect coverage is provided, and the market is complete.

· The linear case (real assets): the matrix **D** depends linearly on p. For instance, $d_m^s = \sum_l p^{s,l} a_{m,l}^s$ if $m = s$ and 0 otherwise. Each unit of security m yields the commodity bundle a_m^s if state s occurs, and 0 otherwise.

Substituting $d = \mathbf{D}(p)$, we get a linear subspace $\Phi(q, p)$, depending on (q, p). In this particular framework, the disaggregation problem for excess demand in incomplete markets can now be stated as follows: given a smooth map Z:$(q, p) \to Z(q, p)$ such that $Z(q, p) \in \Phi(q, p)$ for every (q, p), find smooth maps $z_n(q, p)$ and $U_n(z)$ such that $Z(q, p) = \sum_{n=1}^{N} z_n(q, p)$ and $z_n(q, p)$ maximizes $(U_n(z_n + \omega_n)$ over $\Phi(q, p)$ for every p.

We refer to Duffie (1992) and MacGill and Quinzii (1997) for other examples and a review of the literature.

1.3. The disaggregation problem

We shall now state the disaggregation problem for incomplete markets in full generality, as it is given for instance in Mas-Colell. Consider a map $\theta \to \Phi(\theta)$ associating with a T-dimensional parameter θ a K-dimensional linear subspace $\Phi(\theta)$ of $\mathbf{R}^L$. In the preceding example, $\theta = (q, p)$, $T = L' + M$, $L = L'(S+1)$, $K = L'(S+1) - S + M - 1$. Let Z be a map of some neighbourhood of $\bar{\theta}$ into $\mathbf{R}^L$ such that

$$Z(\theta) \in \Phi(\theta) \; \forall \theta \tag{19}$$

Given N, the disaggregation problem consists in finding N concave utility functions U_n, defined on $\mathbf{R}^L$, and N maps $z_n(\theta)$, defined on $\mathbf{R}^T$ into $\mathbf{R}^L$ such that:

$$Z(\theta) = \sum_n z_n(\theta) \; \forall \theta \tag{20}$$

$$z_n(\theta) = \operatorname{ArgMax}\{U_n(z) \mid z \in \Phi(\theta)\} \tag{21}$$

Eq. (20) means that $Z(\theta)$ decomposes into a sum of individual demands, each of which, by Eq. (21), stems from utility maximization.

As stated, this is a global problem, insofar as the unknown functions U_n and maps z_n are to be defined on all of $\mathbf{R}^L$ and $\mathbf{R}^L$. In the following, we shall deal with a local version of this problem. The map Z will be given only on some neighbourhood of $\bar{\theta}$, while the z_n and U_n will be found on some neighbourhood of $\bar{\theta}$ and $z_n(\bar{\theta})$, respectively. Our main result states that, under a natural non-degeneracy assumption of the map Z at $\bar{\theta}$, the local disaggregation problem can always be solved with N agents provided $N \geq K$.

Before stating this result, let us spend some time explaining the non-degeneracy assumption. Each K-plane in $\mathbf{R}^L$ is entirely defined by $K(L-K)$ real parameters. We are given a map Φ associating with every value of the T-dimensional parameter θ a K-plane $\Phi(\theta)$. If $T > K(L-K)$, some of the parameters θ_t are redundant: $T - K(L-K)$ of them can be expressed in terms of the others. If $T < K(L-K)$, some of the parameters are missing: $K(L-K) - T$ new parameters $\sigma_{T+1}, \ldots, \sigma_{K(L-K)}$ can be defined, and the map $\Phi(\theta)$ extended to a map $\Phi(\theta, \sigma)$ in a non-trivial way.

To express these ideas formally, let us consider the set of all K-planes in $\mathbf{R}^L$, which we denote henceforth by $G^{K,L}$. It is a compact manifold of dimension $K(L-K)$, meaning, as we just stated, that positioning a K-plane in $\mathbf{R}^L$ requires $K(L-K)$ parameters. But there is something more to the manifold structure: we can differentiate, so that we can define non-degeneracy by the non-vanishing of the appropriate Jacobians. This yields the following.

Definition 1. Assume Φ is a C^1 map from a neighbourhood of $\bar{\theta}$ into $G^{K,L}$. We shall say that it is non-degenerate if the tangent map $T_{\bar{\theta}}\Phi$ is injective (one-to-one).

It follows from the definition that if Φ is called *non-degenerate* at $\bar{\theta}$, then it is non-degenerate on a neighbourhood of $\bar{\theta}$, and we must have $T < K(L-K)$. If $T = K(L-K)$, then Φ is a local diffeomorphism. If $T > K(L-K)$, then new parameters can be added and Φ can be extended to a local diffeomorphism, as we shall see in Section 1.4.

1.4. The main result

We can now state our main result.

Theorem 1. *Let U be some neighbourhood of* $\bar{\theta}$ *in* $\mathbf{R}^T$, *and let* Φ *be a non-degenerate analytic map of* $\mathscr{U}$ *into* $G^{K,L}$. *Assume that* $Z{:}\mathscr{U} \to G^{K,L}$ *is real analytic and that:*

$$Z(\theta) \in \Phi(\theta)\,\forall\theta \tag{22}$$

Let $(\bar{\mathbf{z}}_1, \ldots, \bar{\mathbf{z}}_N)$ *be a family of N vectors in* $\mathbf{R}^L$ *with rank at least K and such that:*

$$\sum_{n=1}^{N} \bar{\mathbf{z}}_n = Z(\bar{\theta}) \tag{23}$$

$$\bar{\mathbf{z}}_n \in \bar{\theta}\,\forall n \tag{24}$$

Then, there exists N real analytic maps $z_n(\theta)$, *defined in some neighbourhood of* $\bar{\theta}$, *and for each n a real analytic function* U_n, *defined in some neighbourhood of* $\bar{\mathbf{z}}_n$ *and concave in that neighbourhood, such that:*

$$z_n(\bar{\theta}) = \bar{\mathbf{z}}_n \tag{25}$$

$$\sum_{n=1}^{N} z_n(\theta) = Z(\theta)\,\forall\theta \tag{26}$$

$$\forall n,\theta\ z_n(\theta) = \operatorname{ArgMax}\{U_n(z)\,|\,z \in \Phi(\theta)\} \tag{27}$$

Some comments are in order.

First, since a family of N vectors is required to have rank K, we must have $N \geq K$: the number of agents must at least as large as the dimension of the space of feasible trades. In the case of complete markets, for instance, $K = L - 1$, and $(L - 1)$ agents are needed.

However, in the particular case where markets clear, that is $Z(\bar{\theta}) = 0$, Eq. (23) translates into a non-trivial linear relation between the $\bar{\mathbf{z}}_k$. For the family to have rank K, there must now be at least $(K + 1)$ of them. So, if $Z(\bar{\theta}) = 0$, the theorem requires in fact that $N \geq K + 1$.

Finally, there is an assumption of real analyticity, which means that the corresponding functions can be represented by convergent power series in some neighbourhood of every point. Although we have no counter-example, and would rather conjecture the theorem that holds in the C^{∞} case, its proof relies on an existence result for partial differential equations (the Cartan–Kähler theorem) which is known to be false in the C^{∞} case.

In the example we have given (risk-trading), we must take $L = L'(S + 1)$ and $K = L'(S + 1) - S + M - 1$. Theorem 1 will then hold, provided the map Φ is analytic and non-degenerate. If this is the case, the (local) disaggregation of the demand function will require at least $L'(S + 1) - S + M - 1$ agents, one more if the market is required to clear.

Let us point to related work. The recent papers of Bottazi and Hens (for the case when assets yield real returns) and Gottardi and Hens (1995) (for the case when assets yield nominal returns, as in the example given here), for instance, analyse the linearized situation: they show, for any given matrix $\mathbf{A}$ of suitable dimension satisfying the natural restrictions arising from homogeneity and the Walras law, how to build an incomplete market such that $\mathbf{A}$ is the Jacobian of its excess demand function at a prescribed point. This, of course, is a weaker result than the one we state here.

Note that the individual utility functions we find do not have the standard von Neumann–Morgenstern form. The question whether it is possible to solve disaggregation problems with expected utility models will be studied in a forthcoming paper.

We shall now proceed to the proof of Theorem 1. This proof draws heavily on the exterior differential calculus of Elie Cartan. An exposition can be found in Cartan himself (see Cartan, 1945 for instance), or in the excellent work of Bryant et al. (1991), which is much more appropriate for a modern reader. Other applications, and a description on the Cartan–Kähler theorem, can be found in Chiappori and Ekeland (1996).

2. Proof of Theorem 1

2.1. Recasting the problem

The proof splits in two unequal parts. We first reduce the given problem to a geometrical one, by introducing a new set of parameters instead of the given ones

$(\theta_1, \ldots, \theta_T)$. We then solve the geometric problem by using the heavy machinery of exterior differential calculus.

We begin by casting the given problem into a standard, geometrical form. Since Φ is non-degenerate, if $T = K(L-K)$, it is a local diffeomorphism. If $T < K(L-K)$, it follows from the implicit function theorem that $T - K(L-K)$ new parameters $(\sigma_{K(L-K)+1}, \ldots, \sigma_T)$ can be found so that (θ, σ) is a local coordinate system on $G^{K,L}$ around $\Phi(\bar{\theta})$. In other words, there is a local diffeomorphism Ψ of some neighbourhood of $((\bar{\theta}, 0)$ in $\mathbf{R}^{K(L-K)}$ onto some neighbourhood of $\Phi(\bar{\theta})$ in $G^{K,L}$ such that $\Psi(\theta,0) = \Phi(\theta)$ for all θ.

We then extend Z to an analytic map Y defined on a neighbourhood of $\Phi(\bar{\theta})$ in $G^{K,L}$ by setting:

$$Y(\theta,0) = Z(\theta) \tag{28}$$

and we then define:

$$\boldsymbol{X}(\theta,\sigma) = P_{\Psi(\theta,\sigma)} Y(\theta,\sigma) \tag{29}$$

where $P_{\Psi(\theta,\sigma)}$ is the orthogonal projection on $\Psi(\theta,\sigma)$ in $\mathbf{R}^L$ If $T = K(L-K)$, $\boldsymbol{X} = Z$.

We can make the notation a little bit easier, by writing:

$$\Psi(\theta,\sigma) = \Pi \tag{30}$$

and by using directly Π in $G^{K,L}$ as our new variable, instead of the local coordinates (θ,σ) (recall that Ψ is a local diffeomorphism). Eq. (29) then becomes:

$$\boldsymbol{X}(\Pi) = P_{\Pi} Y(\Pi) \tag{31}$$

We have thus defined an analytic map $\boldsymbol{X}$ of some neighbourhood $\mathscr{V} \subset G^{K,L}$ of $\bar{\Pi} = \overline{\Psi(\theta,\sigma)}$ into $\mathbf{R}^L$ with the properties that:

$$\boldsymbol{X}(\Pi) \in \Pi \, \forall \Pi \in \mathscr{V} \tag{32}$$

$$Z(\theta) = \boldsymbol{X}(\Phi(\theta)) \forall \theta \in \mathscr{U} \tag{33}$$

If one can decompose $\boldsymbol{X}$ into a sum $\sum_n x_n$, with $x_n(\Pi) \in \Pi$ for all $\Pi \in \mathscr{V}$, then, setting $z_n(\theta) = x_n(\Phi(\theta))$, we get a decomposition of Z such that $z_n(\theta) \in \Phi(\theta)$ for all θ. We are thus led to a purely geometric problem: given a smooth map $\boldsymbol{X}: \Pi \to \boldsymbol{X}(\Pi)$ such that $\boldsymbol{X}(\Pi) \in \Pi$ for every Π, find smooth maps $x_n: \Pi \to x_n(\Pi)$ and smooth concave functions U_n on $\mathbf{R}^L$ such that:

$$\boldsymbol{X}(\Pi) = \sum_{n=1}^{N} x_n(\Pi) \tag{34}$$

$$x_n(\Pi) = \operatorname{ArgMax}\{U_n(x) \mid x \in \boldsymbol{R}^L\} \forall \Pi \in \mathscr{V} \tag{35}$$

In other words, in every K-dimensional linear subspace Π in the $\mathbf{R}^L$, we are given a point $X(\Pi)$. We want to find N concave functions U_n such that, for every

K-plane Π, the vector $X(\Pi)$ is just the sum of the N points obtained by maximizing each U_n on Π.

2.2. Recasting the result

We solve the preceding problem locally. The result, given in Theorem 2 below, implies Theorem 1 by the preceding discussion

Theorem 2. *Assume that we are given an analytic map* $\boldsymbol{X}$ *from some neighbourhood of* $\overline{\Pi}$ *in* $G^{K,L}$ *into* $\mathbf{R}^L$ *such that:*

$$\boldsymbol{X}(\Pi) \in \Pi \;\forall \Pi \tag{36}$$

Take $\overline{\Pi} \in G^{K,L}$*, and let* $(\bar{\boldsymbol{x}}_1, \ldots, \bar{\boldsymbol{x}}_N)$ *be a family of N vectors in* $\mathbf{R}^L$ *with rank at least K and such that:*

$$\sum_{n=1}^{N} \bar{\boldsymbol{x}}_n = \boldsymbol{X}(\overline{\Pi}) \tag{37}$$

$$\bar{\boldsymbol{x}}_n \in \overline{\Pi} \;\forall n \tag{38}$$

Then, there exists N real analytic maps $x_n(\Pi)$*, defined in some neighbourhood of* $\overline{\Pi}$*, and for each n a real analytic function* U_n*, defined in some neighbourhood of* $\bar{\boldsymbol{x}}_n$ *and concave in that neighbourhood, such that:*

$$x_n(\overline{\Pi}) = \bar{\boldsymbol{x}}_n \tag{39}$$

$$\sum_{n=1}^{N} x_n(\Pi) = \boldsymbol{X}(\Pi) \;\forall \Pi \tag{40}$$

$$\forall n, \Pi \; x_n(\Pi) = \operatorname{ArgMax}\{U_n(x) \mid x \in \Pi\} \tag{41}$$

We now proceed to prove Theorem 2. Let us first choose a coordinate system π_k, $1 \leq k \leq K(L-K)$ for $G^{K,L}$ near $\overline{\Pi}$. In other words, we are given a set of real analytic maps ϕ_i from a neighbourhood $\mathscr{U}$ of $\overline{\pi}$ in $\mathbf{R}^{K(L-K)}$ into $\mathbf{R}^L$, with the property that the map $\Phi: U \to G^{K,L}$ defined by:

$$\Phi(\pi) = \left\{ x \,\middle|\, \sum_{l=1}^{L} \phi_{i,l}(\pi)\, x^l = 0 \;\forall i \right\} \tag{42}$$

is a diffeomorphism, with $\Phi(\overline{\pi}) = \overline{\Pi}$.

There is no loss of generality in assuming that the set of $\bar{p}_i = \phi_i(\overline{\pi})$ defining $\overline{\Pi}$ is orthogonal, and we can even choose them as the first $L-K$ vectors in a basis for $\mathbf{R}^L$. That is, we may assume that:

$$\begin{aligned} \phi_{i,l}(\overline{\pi}) &= 1 \text{ if } l = i \leq L-K \\ &= 0 \text{ otherwise} \end{aligned} \tag{43}$$

For the sake of simplicity, let us write $\boldsymbol{X}(\pi)$ instead of $\boldsymbol{X}(\Phi(\pi))$. The problem then is to find maps $x_n(\pi)$ and concave functions $U_n(x)$, such that:

$$x_n^l(\bar{\pi}) = \bar{\boldsymbol{x}}_n \ \forall n \tag{44}$$

$$\sum_{n=1}^{N} x_n^l(\pi) = \boldsymbol{X}^l(\pi) \ \forall l, \pi \in \mathscr{U} \tag{45}$$

$$\forall n, \pi \in \mathscr{U},\ x_n(\Pi) = \text{ArgMax}\left\{ U_n(x) \middle| \sum_{l=1}^{L} \phi_{i,l}(\pi)\, x^l = 0, \forall i \right\} \tag{46}$$

2.3. An exterior differential system

Taking into account the fact that U_n is required to be concave, Eq. (46) is equivalent to the existence of positive functions $\lambda_n^i(\pi)$, defined in $\mathscr{U}$, such that:

$$\sum_{i=1}^{L-K} \lambda_n^i \phi_{i,l} = \frac{\partial U_n}{\partial x^l} \tag{47}$$

with of course, for every n:

$$\sum_{l=1}^{L} \phi_{i,l}\, x_n^l = 0, \forall i, n \tag{48}$$

By the Poincaré lemma, Eq. (47) can be rewritten as follows

$$\sum_{i=1}^{N} \lambda_n^i \phi_{i,l} = y_{n,l} \tag{49}$$

with:

$$\sum_l \mathrm{d} y_{n,l} \wedge \mathrm{d} x_n^l = 0 \tag{50}$$

This is the first time we have written an exterior (or wedge) product in this paper. In the language of exterior differential calculus, Eq. (50) expresses the fact that the cross-derivatives $(\partial y_n)/(\partial x_m)$ and $(\partial y_m)/(\partial x_n)$ are equal. From now on, we shall use extensively the machinery of exterior differential calculus, as described in Bryant et al. (1991) (see Chiappori and Ekeland (1995) for a short review).

Summarizing, we are looking for functions x_n^l, $\lambda_n^i > 0$, $y_{n,l}$ of $(\pi_1, \ldots, \pi_K(L-K))$ satisfying Eqs. (44), (45), (48)–(50). Let us reformulate the problem as an exterior differential system.

Consider the space $\mathscr{E} = \mathbf{R}^{K(L-K)+NL+NL+N(L-K)}$ with the following coordinates:

$$\mathscr{E} = \left\{ \left(\pi_k, x_n^l, y_{n,l}, \lambda_n^i \right) \middle| 1 \le k \le K(L-K), \right.$$
$$\left. 1 \le n \le N, 1 \le l \le L, 1 \le i \le L-K \right\} \tag{51}$$

In the space $\mathscr{E}$, we define a subset $\mathscr{M}$ by the set of equations:

$$\sum_{n=1}^{N} x_n^l = \boldsymbol{X}^l(\pi)\ \forall l \tag{52}$$

$$\sum_{l=1}^{L} \phi_{i,l}(\pi)\, x_n^l = 0, \forall i, n \tag{53}$$

$$\sum_{i=1}^{L-K} \lambda_n^i \phi_{i,l}(\pi) = y_{n,l}, \forall n, l \tag{54}$$

Since $\boldsymbol{X}(\Pi) \in \Pi\, \forall \Pi$, we must also have

$$\sum_{l=1}^{L} \phi_{i,l}(\pi)\, \boldsymbol{X}^l(\pi) = 0, \forall i \tag{55}$$

so that the last $L-K$ equations in Eq. (53) follow from the $(N-1)(L-K)$ preceding ones. The remaining equations are clearly independent, so that $\mathscr{M}$ is an analytic submanifold of codimension $L+(N-1)(L-K)+NL$ in $\mathscr{E}$.

In $\mathscr{M}$ (and *not* in $\mathscr{E}$), we consider the exterior differential system:

$$\sum_{l} \mathrm{d}y_{n,l} \wedge \mathrm{d}x_n^l = 0\ \forall n \tag{56}$$

$$\mathrm{d}\pi_1 \wedge \ldots \wedge \mathrm{d}\pi_{K(L-K)} \neq 0 \tag{57}$$

From the definitions, the following lemma is evident.

Lemma 1. *Any integral manifold of the exterior differential system (56), (57) on $\mathscr{M}$ is the graph of functions $x_n^l(\pi)$, $y_{n,l}(\pi)$, $\lambda_n^i(\pi) > 0$, of $(\pi_1, \ldots, \pi_K(L-K))$ satisfying Eqs. (44), (45), (48)–(50). One can then find functions $U_n(x)$ satisfying Eq. (46) and such that:*

$$y_{n,l}(\pi) = \frac{\partial U_n}{\partial x^l}(x_n(\pi)) \tag{58}$$

We shall now apply the Cartan–Kähler theorem, as in Chiappori and Ekeland, and show that through any point $(\bar{\pi}_k, \bar{\boldsymbol{x}}_n^l, \bar{y}_{n,l}, \bar{\lambda}_n^i)$ in $\mathscr{M}$ there is an integral manifold. This will give us functions $x_n^l(\pi)$ and $U_n(x)$ satisfying Eqs. (44)–(46). In addition, we shall prove that the matrix of second derivatives

$$\frac{\partial^2 U_n}{\partial x_n^l \partial x_n^k}$$

is negative definite at $\bar{\boldsymbol{x}}_n$, which will prove that the function U_n is concave in some neighbourhood of $\bar{\boldsymbol{x}}_n$. Theorem 2 then follows.

Finding integral manifolds by the Cartan–Kähler theorem is a two-step procedure. One first shows that the linearized equations at the given point have a solution. One then has to compute the codimension of the space of solutions in two different ways to check that it gives the same result.

2.4. Solving the system: finding integral elements

The systems (56), (57) are clearly closed. To apply Cartan–Kähler, we have to find all the integral elements through $(\overline{\pi}_k, \bar{x}_n^l, \bar{y}_{n,l}, \overline{\lambda}_n^i)$ satisfying $d\pi_1 \wedge \ldots \wedge d\pi_{K(L-K)} \neq 0$. For this purpose, we differentiate Eqs. (52)–(54):

$$dX^l = \sum_n dx_n^l \tag{59}$$

$$0 = \sum_l \left(d\phi_{i,l} x_n^l + \phi_{i,l} dx_n^l \right) \tag{60}$$

$$dy_{n,l} = \sum_i \left(\phi_{i,l} d\lambda_n^i + \lambda_n^i d\phi_{i,l} \right) \tag{61}$$

and we substitute the equations:

$$dx_n^l = \sum_k \mathbf{A}_n^{l,k} d\pi_k \tag{62}$$

$$d\lambda_n^i = \sum_k \mathbf{B}_n^{i,k} d\pi_k \tag{63}$$

As for $dy_{n,l}$, by a celebrated lemma of Cartan, Eq. (56) holds if and only if there exists $L \times L$ matrices $\mathbf{C}_n$, $1 \leq n \leq N$, which are symmetric:

$$\mathbf{C}_{n,l,l'} = \mathbf{C}_{n,l',l} \tag{64}$$

and satisfy:

$$dy_{n,l} = \sum_{l'} \mathbf{C}_{n,l,l'} dx_n^{l'} \tag{65}$$

Substituting relations (62), (63) and (65) into Eqs. (59)–(61), we get the equations:

$$\sum_n \mathbf{A}_n^{l,k} = \frac{\partial X^l}{\partial \pi_k} \ \forall l,k \tag{66}$$

$$\sum_l \phi_{i,l}(\pi) \mathbf{A}_n^{l,k} = -\sum_l x_n^l \frac{\partial \phi_{i,l}}{\partial \pi_k}, \forall i,n,k \tag{67}$$

$$\sum_{l'} \mathbf{C}_{n,l,l'} \mathbf{A}_n^{l',k} - \sum_i \phi_{i,l} \mathbf{B}_n^{i,k} = \sum_i \lambda_n^i \frac{\partial \phi_{i,l}}{\partial \pi_k} \ \forall n,l,k \tag{68}$$

The intuitive significance of this set of linear equations is clear: they give the derivatives of the solutions we are looking for. More precisely, if $x_n^l(\pi)$, $y_{n,l}(\pi)$,

$\lambda_n^i(\pi) > 0$ and solves the non-linear system (52), (53), (54), and if there is a function U_n such that $y_{n,l} = (\partial U_n)/(\partial x_l)$, then the derivatives $\mathbf{A}_n^{l,k} = (\partial x_n^l)/(\partial \pi_k)$, $\mathbf{B}_n^{i,k} = (\partial \lambda_n^i)/(\partial \pi_k)$ and

$$\mathbf{C}_{n,l,l'} = \frac{\partial^2 U_n}{\partial x_n^l \partial x_n^{l'}}$$

must satisfy Eqs. (66)–(68).

The solutions of this set of linear equations are the so-called integral elements of the exterior differential system (56), (57). Our first task is to show that they exist, that is, that the linearized system can be solved. This is the content of the next Lemma, the proof of which will occupy the rest of this subsection.

Lemma 2. *The system of linear Eqs. (66)–(68) in* $\mathbf{A}_n^{l,k}$, $\mathbf{B}_n^{i,k}$ *and* $\mathbf{C}_{n,l,l'}$ *has a solution such that every matrix* $\mathbf{C}_n$ *is symmetric and negative definite.*

Let us rewrite Eqs. (66)–(68) in matrix form. With obvious notations, we have (we do not write the right-hand sides: they just obscure the argument)

$$\sum_n \mathbf{A}_n = \tag{69}$$

$$\phi_i \mathbf{A}_n = \tag{70}$$

$$\mathbf{C}_n \mathbf{A}_n - \sum_i \phi_i \left(\mathbf{B}_n^i\right) * = \tag{71}$$

By $(\mathbf{B}_n^i)*$, we denote the transpose of the vector $(\mathbf{B}_n^i)^k$, $1 \le k \le K(L-K)$, which is a linear functional on $\mathbf{R}^{K(L-K)}$. The second term on the left of Eq. (71) is a sum of rank-one operators, each of which has a range generated by one of the ϕ_i. Thus, Eq. (71) means simply that, up to a translation by a known vector, the range of $\mathbf{C}_n \mathbf{A}_n$ is contained in the linear span of the ϕ_i, $1 \le i \le L-K$. We lose no information by projecting on a complementary subspace.

As we noted in the beginning, the $\phi_i(\overline{\pi}) = \bar{\boldsymbol{p}}_i$ are the first $(L-K)$ basis vectors in $\mathbf{R}^L$. We can complete the basis with vectors $\bar{\boldsymbol{p}}_j$, $(L-K+1) \le j \le L$. Projecting Eq. (71) on the subspace generated by the $\bar{\boldsymbol{p}}_j$, and writing the result together with Eq. (70), we get:

$$\left(I_{(L-K)},0\right)\mathbf{A}_n = \tag{72}$$

$$(0,I_K)\mathbf{C}_n \mathbf{A}_n = \tag{73}$$

where $I_{(L-K)}$ and I_K denote the $(L-K)\times(L-K)$ and $K \times K$ identity matrices. Eq. (73) can be rewritten as follows:

$$\left(\mathbf{M}_n,\hat{\mathbf{C}}_n\right)\mathbf{A}_n = \tag{74}$$

where $\mathbf{C}_{\hat{n}}$, is the $K \times K$ matrix in the lower right corner of $\mathbf{C}_n$:

$$\mathbf{C}_n = \begin{pmatrix} \overline{\mathbf{C}_n} & \mathbf{M}_n * \\ \mathbf{M}_n & \hat{\mathbf{C}}_n \end{pmatrix} \tag{75}$$

It is now possible to write Eqs. (72) and (73) together:

$$\begin{pmatrix} I_{(L-K)} & 0 \\ \mathbf{M}_n & \hat{\mathbf{C}}_n \end{pmatrix} \mathbf{A}_n = \tag{76}$$

If $\mathbf{C}_n$, and hence $\hat{\mathbf{C}}_n$, is negative definite, the matrix on the left is invertible, and its inverse is easily computed to be:

$$\begin{pmatrix} I_{(L-K)} & 0 \\ \mathbf{M}_n & \hat{\mathbf{C}}_n \end{pmatrix}^{-1} = \begin{pmatrix} I_{(L-K)} & 0 \\ -\hat{\mathbf{C}}_n^{-1}\mathbf{M}_n & \hat{\mathbf{C}}_n^{-1} \end{pmatrix} \tag{77}$$

Now is the time to write the right-hand side of Eq. (76). It is:

$$\mathbf{D}_n = \begin{pmatrix} -\sum_l x_n^l(0) \dfrac{\partial \phi_{i,l}}{\partial \pi_k}(0), 1 \le i \le L-K \\ \sum_i \lambda_n^i(0) \dfrac{\partial \phi_{i,l}}{\partial \pi_k}(0), l \ge L-K+1 \end{pmatrix} \tag{78}$$

We now solve Eq. (76) and write the result in Eq. (69). We are left with a single equation in $\hat{\mathbf{C}}_n^{-1}$:

$$\sum_n \begin{pmatrix} I_{(L-K)} & 0 \\ -\hat{\mathbf{C}}_n^{-1}\mathbf{M}_n & \hat{\mathbf{C}}_n^{-1} \end{pmatrix} \mathbf{D}_n = \left(\frac{\partial \boldsymbol{X}^l}{\partial \pi_k}, 1 \le l \le L, 1 \le k \le K(L-K) \right) \tag{79}$$

We have to check that the first $L-K$ equations are automatically satisfied. This is done by differentiating Eq. (55) at $\pi = 0$. We get:

$$\sum_l \boldsymbol{X}^l \frac{\partial \phi_{i,l}}{\partial \pi_k} + \sum_l \phi_{i,l} \frac{\partial \boldsymbol{X}^l}{\partial \pi_k} = 0 \tag{80}$$

Recall from Eq. (43) that $\phi_{i,l}(0) = \delta_{i,l}$ (Kronecker symbol), so that the last term in the preceding equation reduces to $(\partial \boldsymbol{X}^l)/(\partial \pi_k)(0)$. Substituting in the first $L-K$ equations of Eq. (79) yields:

$$\begin{aligned} -\sum_{n,l} x_n^l(0) \frac{\partial \phi_{i,l}}{\partial \pi_k}(0) &= -\sum_l \boldsymbol{X}^l(0) \frac{\partial \phi_{i,l}}{\partial \pi_k}(0) \\ &= \frac{\partial \boldsymbol{X}^i}{\partial \pi_k}(0) \end{aligned} \tag{81}$$

as desired.

We are left with the last K equations in Eq. (79). Splitting $\mathbf{D}_n$ horizontally:

$$\mathbf{D}_n = \begin{pmatrix} \mathbf{D}_n^1 \\ \mathbf{D}_n^2 \end{pmatrix} \tag{82}$$

we end up with:

$$\sum_n \hat{\mathbf{C}}_n^{-1}\left(-\mathbf{M}_n\mathbf{D}_n^1 + \mathbf{D}_n^2\right) = \mathbf{R} \tag{83}$$

where $\mathbf{R}$ and $\mathbf{D}_n^2$ are $K \times K(L-K)$ matrices and $\mathbf{D}_n^1$ is $(L-K) \times K(L-K)$. All these matrices are given, and we have to solve for the $K \times (L-K)$ matrix $\mathbf{M}_n$ and the symmetric, negative definite $K \times K$ matrix $\mathbf{C}_{\hat{n}}^{-1}$.

To prove that this procedure is possible, we need a lemma. Let us first introduce some notation. Set

$$\mathbf{M} = (\mathbf{M}_1, \dots, \mathbf{M}_N) \in \mathbf{R}^{NK(L-K)} \tag{84}$$

and define a map Ψ from $\mathbf{R}^{NK(L-K)}$ into $\mathbf{R}^{(L-K)KK}$ by:

$$\Psi(\mathbf{M}) = \sum_n Q_n \mathbf{M}_n \mathbf{D}_n^1 \tag{85}$$

Lemma 3. *Assume $N \geq K$ and the $N(L-K) \times K(L-K)$ matrix* $\mathbf{D}$ *defined by*

$$\mathbf{D} = \left(\mathbf{D}_1^1 | \mathbf{D}_2^1 | \cdots | \mathbf{D}_N^1\right) \tag{86}$$

has rank $K(L-K)$. Assume also that all the Q_n are invertible. Then Ψ is onto.

Proof. Without loss of generality, we may assume that $Q_n = I$ for every n. Indeed, setting $\overline{\mathbf{M}}_n = Q_n^{-1}\mathbf{M}_n$ and $\overline{\Psi}(\mathbf{M}) = \Psi(\overline{\mathbf{M}})$, we have

$$\overline{\Psi}(\mathbf{M}) = \sum_n \mathbf{M}_n \mathbf{D}_n^1 \tag{87}$$

Since $\mathbf{M} \to \tilde{\mathbf{M}}$ is an isomorphism, Ψ is onto if and only if $\overline{\Psi}$ is onto.

The map $\mathbf{D}$ sends $\mathbf{R}^{K(L-K)}$ into $\mathbf{R}^{N(L-K)}$. By assumption, it is one-to-one. Set $\mathbf{D}(\mathbf{R}^{K(L-K)}) = \mathscr{A}$, so that $\mathbf{R}^{K(L-K)}$ can be identified with its image $\mathscr{A}$.

Any map from $\mathbf{R}^{N(L-K)}$ into $\mathbf{R}^{N\,K}$ can be put in the form

$$(X_1, \dots, \mathbf{X}_N) \to (\mathbf{M}_1 X_1, \dots, \mathbf{M}_N X_N) \tag{88}$$

with suitable $K \times (L-K)$ matrices $\mathbf{M}_n$, $1 \leq n \leq N$. It follows that any map from $\mathscr{A}$ into $\mathbf{R}^K$ can be put in that form, and hence that any map from $\mathbf{R}^{K(L-K)}$ into $\mathbf{R}^K$ can be written as $\overline{\Psi}(\mathbf{M})$, for a suitable choice of $\mathbf{M}$. This means that $\overline{\Psi}$ is onto, as desired.

We now solve Eq. (83). Choose any family $Q = (Q_1, \dots, Q_N)$ with the Q_n symmetric, negative definite, and set $\hat{\mathbf{C}}_n^{-1} = Q_n$. So the Q_n are invertible. Going back to the definition of the $\mathbf{D}_n^1$ (Eq. (78)), and remembering that the family $\bar{\mathbf{x}}_n$, $1 \leq n \leq N$ has rank K, we find that $\mathbf{D}$ has rank $K(L-K)$. We then apply lemma 3 to find the corresponding $\mathbf{M}_n$. This gives us the $\hat{\mathbf{C}}_n$ and the $\mathbf{M}_n$. Going back to Eq. (75), we leave it to the reader to show that the $\overline{\mathbf{C}_n}$ can be found to complete $\mathbf{C}_n$ into a symmetric, negative definite matrix.

This concludes the proof of Lemma 2.

2.5. Computing the Cartan characters

The next step in applying the Cartan–Kähler theorem, in accordance with the procedure described in Sec. 4.5 of Chiappori and Ekeland, is to compute the Cartan characters and compare them with the codimension of the manifold of integral elements in the Grassmannian manifold of tangent $K(L-K)$-planes to $\mathscr{M}$.

As noted above, the manifold $\mathscr{M}$ has codimension $L+(N-1)(L-K)+NL$ in $\mathscr{E}$. Its dimension then is:

$$\begin{aligned} \dim(\mathscr{M}) &= \dim(\mathscr{E}) - \operatorname{codim}(\mathscr{M}) \\ &= K(L-K) + NL + NL + N(L-K) - L - (N-1)(L-K) - NL \\ &= K(L-K) + NL - K \end{aligned} \tag{89}$$

At every point $m = (\pi, x, y, \lambda)$ in $\mathscr{M}$, the dimension of the Grassmannian manifold of $K(L-K)-$ planes in $T_m\mathscr{M}$ then is:

$$K(L-K)(K(L-K)+NL-K-K(L-K)) = K(L-K)(NL-K) \tag{90}$$

On the other hand, at every point m of $\mathscr{M}$, the integral elements in $T_m\mathscr{M}$ are defined by the matrices $\mathbf{A}_n$, $\mathbf{B}_n$ and $\mathbf{C}_n$ written in Eqs. (62), (63) and (65).

The $\mathbf{C}_n$ have to be symmetric. Their lower right and upper left parts $\mathbf{C}_{\hat{n}}$ and $\overline{\mathbf{C}_n}$ have to satisfy open conditions (i.e., to be negative definite, with sufficiently large eigenvalues in the case of $\overline{\mathbf{C}_n}$). The lower left (or upper right) parts $\mathbf{M}_n$ have to satisfy Eq. (83), which boils down to $KK(L-K)$ linearly independent equations in $NK(L-K)$ variables. So the number of degrees of freedom allowed in the choice of the $\mathbf{C}_n$ ends up being:

$$\frac{NL(L+1)}{2} - KK(L-K) \tag{91}$$

Once the $\mathbf{C}_n$ are chosen, the $\mathbf{A}_n$ are determined by Eq. (76) (no degree of freedom allowed). The $\mathbf{B}_n$, on the other hand, have to satisfy Eq. (71), which is really a set of $K(L-K)NL$ linear equations for $K(L-K)N(L-K)$ unknowns. A compatibility condition is required, so that the right-hand side of Eq. (71) belongs to the range of the left-hand side, and this is precisely Eq. (73). Once this condition is satisfied, $K(L-K)NK$ of the Eq. (71) become redundant, and we are left with a system of $K(L-K)N(L-K)$ equations for $K(L-K)N(L-K)$ unknowns, which are thus fully determined. So there is no degree of freedom allowed in the choice of the $\mathbf{B}_n$ either.

Finally, the number of degrees of freedom allowed in the choice of an integral element in $T_m\mathscr{M}$ is given by Eq. (91). So the codimension of the manifold of integral elements in the Grassmannian is found by subtracting Eq. (91) from Eq. (90). We get:

$$K(L-K)(NL-K) - \frac{NL(L+1)}{2} + KK(L-K)$$
$$= K(L-K)NL - \frac{NL(L+1)}{2} \tag{92}$$

We now proceed to computing the Cartan characters. Fix a point $\overline{m} = (\overline{\pi}, \bar{\boldsymbol{x}}, \bar{y}, \overline{\lambda})$ in $\mathscr{M}$. Let us first describe a coordinate system for the cotangent space $T^*_{\overline{m}}\mathscr{M}$.

The cotangent space $T^*_{\overline{m}}\mathscr{M}$ is defined as a subspace of $T^*_{\overline{m}}\mathscr{E}$ by Eqs. (59)–(61). As we mentioned earlier, there is no loss of generality in assuming that the $\overline{\boldsymbol{p}_i} = \phi_i(\overline{\pi})$ are the first $(L-K)$ vectors of the chosen basis in $\mathbf{R}^L$ (Eq. (43)). Writing Eq. (43) into Eqs. (59)–(61), we get

$$\mathrm{d}X^l = \sum_n \mathrm{d}x_n^l \ \forall l \tag{93}$$

$$0 = \sum_l \left(x_n^l \sum_k \frac{\partial \phi_{i,l}}{\partial \pi_k} \mathrm{d}\pi_k + \mathrm{d}x_n^i \right) \forall n, \forall i \leq (L-K) \tag{94}$$

$$\mathrm{d}y_{n,l} = \mathrm{d}\lambda_n^l + \sum_{i,k} \lambda_n^i \frac{\partial \phi_{i,l}}{\partial \pi_k} \mathrm{d}\pi_k \ \forall n, \forall l \leq (L-K) \tag{95}$$

$$\mathrm{d}y_{n,l} = \sum_i \lambda_n^i \mathrm{d}\phi_{i,l} \ \forall n, \forall l \geq (L-K+1) \tag{96}$$

It follows from Eq. (94) that the $\mathrm{d}x_n^l$, for $l \leq L-K$, can be computed from the $\mathrm{d}\pi_k$. In addition, Eq. (93) tells us that the $\mathrm{d}x_N^l$ can be computed from the $\mathrm{d}x_n^l$, $1 \leq n \leq N-1$ and the $\mathrm{d}\pi_k$. Similarly, Eqs. (95) and (96) gives the $\mathrm{d}y_{n,l}$ in terms of the $\mathrm{d}\lambda_n^l$ and the $\mathrm{d}\pi_k$. Finally, we are left with $K(N-1) + K(L-K) + (L-K)N$ independent 1-forms

$$\mathrm{d}x_n^l, n \leq N-1, l \geq L-K+1 \tag{97}$$

$$\mathrm{d}\lambda_n^l, \mathrm{d}\pi_k \tag{98}$$

which provide us with a basis for $T^*_{\overline{m}}\mathscr{M}$.

As we mentioned earlier, at every point m of $\mathscr{M}$, the integral elements in $T_{\overline{m}}\mathscr{M}$ are defined by the Eqs. (59)–(61) (we are not claiming that they are linearly independent). Fix such an integral element $\overline{E}$, that is, prescribe values for the matrices $\mathbf{A}_n$, $\mathbf{B}_n$ and $\mathbf{C}_n$. Since the $\mathbf{C}_n$ are symmetric, we may rewrite the exterior differential system (56) as follows:

$$\begin{aligned} 0 &= \Sigma_l \mathrm{d} y_{n,l} \wedge \mathrm{d} x_n^l \\ &= \Sigma_l \left(\mathrm{d} y_{n,l} - \Sigma_{l'} \mathbf{C}_{n,l,l'} \mathrm{d} x_n^{l'} \right) \wedge \mathrm{d} x_n^l \\ &= \Sigma_l \left(\mathrm{d} y_{n,l} - \Sigma_{l'} \mathbf{C}_{n,l,l'} \mathrm{d} x_n^{l'} \right) \wedge \left(\mathrm{d} x_n^l - \Sigma_k \mathbf{A}_n^{l,k} \mathrm{d} \pi_k \right) \\ &\quad + \Sigma_l \left(\mathrm{d} y_{n,l} - \Sigma_{l'} \mathbf{C}_{n,l,l'} \mathrm{d} x_n^{l'} \right) \wedge \left(\Sigma_k \mathbf{A}_n^{l,k} \mathrm{d} \pi_k \right) \end{aligned} \tag{99}$$

Set $\omega_{n,l} = \mathrm{d} y_{n,l} - \Sigma_{l'} \mathbf{C}_{n,l,l'} \mathrm{d} x_n^{l'}$, so that this last equation can be rewritten as:

$$0 = \sum_l \omega_{n,l} \wedge \left(\mathrm{d} x_n^l - \sum_k \mathbf{A}_n^{l,k} \mathrm{d} \pi_k \right) + \sum_l \omega_{n,l} \wedge \left(\sum_k \mathbf{A}_n^{l,k} \mathrm{d} \pi_k \right) \tag{100}$$

Note that, for $l \leq L - K$, $\omega_{n,l}$ contains a term in $\mathrm{d}\lambda_n^l$ (and is the only one to do so), while for $l \geq L - K + 1$, $\omega_{n,l}$ contains terms in $\mathrm{d} x_n^l$, $n \leq N - 1$, $l \geq L - K + 1$ and terms in $\mathrm{d}\pi_k$, $k \leq K(L-K)$. It follows from the above, and from the properties of the matrix $\mathbf{C}_n$, that the $\omega_{n,l}$, $n \leq N$, $l \leq L$ and $K(L-K) - K$ of the $\mathrm{d}\pi_k$ are linearly independent, and can be completed into a basis of $T_{\overline{m}}^* \mathscr{M}$.

The dimension of the space generated by the $\omega_{n,l}$ is NL. In accordance with the procedure described in Sec. 4.5 of Chiappori and Ekeland), we compute the Cartan characters:

$$c_0 = 0 \tag{101}$$

$$c_m = mN \,\forall m \leq L \tag{102}$$

$$c_m = NL \,\forall m \geq L \tag{103}$$

We have:

$$\sum_{m=0}^{K(L-K)-1} c_m = N \frac{L(L+1)}{2} + \left(K(L-K) - 1 - L \right) NL \tag{104}$$

Comparing Eqs. (92) and (104), we see that they are equal. This means that all points $\overline{m} = (\overline{\pi}, \overline{\mathbf{x}}, \overline{y}, \overline{\lambda})$ in $\mathscr{M}$ are ordinary, so that the Cartan–Kähler theorem holds.

2.6. Conclusion

We have proved that, for every integral element at $\overline{m}$, that is, every choice of $\mathbf{A}_N$, $\mathbf{B}_n$ and $\mathbf{C}_n$, there is an integral manifold of the exterior differential system (56). Eq. (57), having $\overline{E}$ as tangent space at $\overline{m}$. By lemma 1, any such integral manifold provides us with functions $x_n^l(\pi)$, $y_{n,l}(\pi)$, $\lambda_n^i(\pi) > 0$, of $(\pi_1, \ldots,$

$\pi_{K,(L-K)}$) satisfying Eqs. (44), (45), (48)–(50), and with functions $U_n(x)$ satisfying Eqs. (46) and (58). Differentiating the latter equation at $\overline{x_n}$, we get:

$$\frac{\partial^2 U_n}{\partial x^l \partial x^{l'}}(\overline{x_n}) = \mathbf{C}_{n,l,l'} \tag{105}$$

Since the matrices $\mathbf{C}_n$ are negative definite, the functions U_n are strictly concave in a neighbourhood of $\overline{x_n}$. This concludes the proof.

References

Bottazi, J.M., Hens, T., On excess demand functions with incomplete markets, to appear, Journal of Economic Theory.

Bryant, R., Chern, S., Gardner, R., Goldschmidt, H., Griffiths, P., 1991. Exterior Differential Systems, Vol. 18. MSRI Publications, Springer-Verlag.

Cartan, E., 1945. Les systèmes Différentiels Extérieurs et Leurs Applications Géométriques. Hermann.

Chiappori, P.A., Ekeland, I., 1995. Exterior Differential Calculus and Aggregation: a Presentation and Some New Results, Mimeo, DELTA.

Chiappori, P.A., Ekeland, I., 1996. Problèmes d'agrégation en théorie du consommateur et calcul différentiel extérieur. C.R.A.S. Paris 323, 565–570.

Chiappori, P.A., Ekeland, I., Aggregation and market demand: an exterior differential calculus viewpoint, Econometrica, forthcoming.

Duffie, D., 1992. The nature of incomplete security markets. In: Laffont, J.J. (Ed.), Advances in Economic Theory (6th World Congress). Econometric Society Monographs. Cambridge Univ. Press, pp. 214–262.

Geanakoplos, J., Polemarchakis, H., 1980. On the disaggregation of excess demand functions. Econometrica, 315–331.

Gottardi, P., Hens, T., 1995. On the disaggregation of excess demand functions when markets are incomplete: the case of nominal returns, discussion paper, Bonn University.

Mas-Colell, A., Four lectures in the differentiable approach to general equilibrium theory. In: Ambrosetti, Gori, Lucchetti (Eds.), Mathematical Economics. Springer Lecture Notes in Mathematics 1330, pp. 19–43.

MacGill, M., Quinzii, M., 1997. Incomplete Markets, Vol. 1. MIT Press.

Mantel, Debreu, 1974. On the characterization of Aggregate Excess Demand. Journal of Economic Theory 7, 348–353.

Shafer, W., Sonnenschein, H., 1982. Market demand and excess demand functions. In: Arrow, K., Intriligator, M. (Eds.), Handbook of Mathematical Economics, Chap. 14, Vol. 2. Amsterdam, North Holland, pp. 670–693.