On bifurcation of eigenvalues along convex symplectic paths

Yinshan Chang[1]*, Yiming Long[2]†, Jian Wang[3]‡

[1]College of Mathematics, Sichuan University, Chengdu 610065, China

[2]Chern Institute of Mathematics and LPMC, Nankai University, Tianjin 300071, China

[3]Chern Institute of Mathematics, Nankai University, Tianjin 300071, China

Abstract

We consider a continuously differentiable curve $t \mapsto \gamma(t)$ in the space of $2n\times 2n$ real symplectic matrices, which is the solution of the following ODE:

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) = J_{2n}A(t)\gamma(t), \gamma(0) \in \mathrm{Sp}(2n,\mathbb{R}),$$

where $J = J_{2n} \overset{\text{def}}{=} \begin{bmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{bmatrix}$ and $A : t \mapsto A(t)$ is a continuous path in the space of $2n \times 2n$ real matrices which are symmetric. Under a certain convexity assumption (which includes the particular case that $A(t)$ is strictly positive definite for all $t \in \mathbb{R}$), we investigate the dynamics of the eigenvalues of $\gamma(t)$ when t varies, which are closely related to the stability of such Hamiltonian dynamical systems. We rigorously prove the qualitative behavior of the branching of eigenvalues and explicitly give the first order asymptotics of the eigenvalues. This generalizes classical Krein-Lyubarskii theorem on the analytic bifurcation of the Floquet multipliers under a linear perturbation of the Hamiltonian. As a corollary, we give a rigorous proof of the following statement of Ekeland: $\{t \in \mathbb{R} : \gamma(t)$ has a Krein indefinite eigenvalue of modulus $1\}$ is a discrete set.

1 Introduction

1.1 The introduction of the model and the main assumption

We consider linear Hamiltonian equations in $\mathbb{R}^{2n}$ of the following type

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) = J_{2n}A(t)\gamma(t), \gamma(0) \in \mathrm{Sp}(2n,\mathbb{R}), \tag{1}$$

*Supported by the Fundamental Research Funds for the Central Universities (No. YJ201661). E-mail: ychang@scu.edu.cn

†Partially supported by NSFC Grants 11131004 and 11671215, LPMC of Ministry of Education of China, Nankai University, and the Beijing Center for Mathematics and Information Interdisciplinary Sciences at Capital Normal University. E-mail: longym@nankai.edu.cn

‡Supported by China Postdoctoral Science Foundation (No. 2013T60251) and Natural Science Foundation of China (No. 11401320 and No. 11571188). E-mail: wangjian@nankai.edu.cn

where $J = J_{2n} \overset{\text{def}}{=} \begin{bmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{bmatrix}$ and $A : t \mapsto A(t)$ is a continuous periodic curve in the space of $2n \times 2n$ real matrices which are symmetric with the periodicity T. The unique solution is a curve in the space of real symplectic matrices such that

$$\gamma(t+T)\gamma(T)^{-1} = \gamma(t)\gamma(0)^{-1}. \tag{2}$$

The system (1) arises naturally from perturbations of linearized Hamiltonian equations. Indeed, let $\varepsilon \in \mathbb{R}$ be a *real* perturbation parameter. Consider

$$\frac{\partial \gamma}{\partial t}(t,\varepsilon) = J_{2n} A(t,\varepsilon)\gamma(t,\varepsilon), \quad \gamma(0,\varepsilon) = \mathrm{Id}_{2n}, \tag{3}$$

where $t \mapsto A(t,\varepsilon)$ is a locally integrable periodic curve in the space of $2n \times 2n$ real matrices which are symmetric and periodic with the periodicity T. Moreover, we assume that $\varepsilon \mapsto (A(t,\varepsilon), t \in [0,T])$ is a *continuously Fréchet-differentiable* curve in $L^1[0,T]$. Then, for fixed T, as ε varies, the endpoint matrix $\gamma(T,\varepsilon)$ is a C^1-curve satisfying (1). More precisely,

$$\frac{\partial}{\partial \varepsilon}\gamma(T,\varepsilon) = \gamma(T,\varepsilon) J_{2n} C(T,\varepsilon) = J_{2n} B(T,\varepsilon)\gamma(T,\varepsilon), \tag{4}$$

where

$$C(T,\varepsilon) = -\gamma(T,\varepsilon)^T J_{2n} \frac{\partial}{\partial \varepsilon}\gamma(T,\varepsilon) = \int_0^T \gamma(t,\varepsilon)^T \frac{\partial}{\partial \varepsilon} A(t,\varepsilon)\gamma(t,\varepsilon)\,\mathrm{d}t \tag{5}$$

and $B(T,\varepsilon) = (\gamma(T,\varepsilon)^{-1})^T C(T,\varepsilon)\gamma(T,\varepsilon)^{-1}$, where the superscript "T" denotes the transpose of matrices. Note that both C and B are symmetric real matrices and they are continuous in ε. We refer to Subsection A.1 for the second inequality in (5).

Let us go back to the system (1) and recall that a matrix γ is called *stable* if $\sup_{n\in\mathbb{Z}} ||\gamma^n|| < \infty$. We say that the system (1) is *stable* if the matrix $\gamma(T)$ is stable. By (2), we have that $\sup_{t\in\mathbb{R}} ||\gamma(t)|| < \infty$ if $\gamma(T)$ is stable. A symplectic matrix γ is called *strongly stable* if there exists a neighborhood of γ in the space of symplectic matrices containing only stable symplectic matrices. We say that the system (1) is strongly stable if $\gamma(T)$ is strongly stable as a symplectic matrix. In this case, when the system (1) is slightly perturbed, it is still a stable system. The picture is not clear in general if we perturb a stable but not strongly stable system.

The stability is closely related to the eigenvalues of a symplectic matrix. We give a brief explanation in the following. For more details, please refer to [Eke90, Sections 1.1 and 1.2]. The eigenvalues of a sympletic matrix come in 4-tuple like $\{\lambda, \lambda^{-1}, \bar{\lambda}, \bar{\lambda}^{-1}\}$ and hence it is stable iff it is diagonalizable and all its eigenvalues stay on the unit circle $U \subset \mathbb{C}$. The characterization of strong stability was firstly formulated by Krein [Kre50, Kre51], and later independently by Moser [Mos58], as stated in the following: a symplectic matrix γ is strongly stable iff it is stable and all its eigenvalues are *Krein definite*. To be more precise, let $G = -\sqrt{-1}J$ be the Krein form which gives an inner product on $\mathbb{C}^{2n}$ via

$$(x,y)_G = \sqrt{-1}\left\{\sum_{k=1}^{n}(x_k\bar{y}_{n+k} - x_{n+k}\bar{y}_k)\right\}. \tag{6}$$

Then, an eigenvalue $\lambda \in U$ is said to be *Krein positive (resp. negative) definite* if the bilinear form $(x, y) \mapsto (x, y)_G$ is positive (resp. negative) definite on the invariant space E_λ associated with the eigenvalue λ, see Subsection 2.1 for the definition of E_λ. It is called *Krein indefinite* if the bilinear form $(x, y) \mapsto (x, y)_G$ is indefinite on E_λ.

Under the convexity assumption that $A(t)$ *is strictly positive definite for all* $t \in \mathbb{R}$, Ekeland [Eke90, Section 1.3] has investigated the system (1) when $\gamma(0) = \mathrm{Id}$. Among various results, Ekeland has claimed that the following set is isolated:

$$D \stackrel{\text{def}}{=} \{t : \gamma(t) \text{ has a Krein indefinite eigenvalue on } U\}, \tag{7}$$

see [Eke90, Proposition 4, Section 1.3]. However, later, in [Eke90, Erratum], Ekeland wrote that "The proof of Proposition 4 (and probably the proposition itself) is wrong", and he proved a weaker statement for continuous $t \mapsto A(t)$: D is a finite union of isolated sets D_m, where

$$D_m \stackrel{\text{def}}{=} \left\{ t \in D : \begin{array}{l} \text{all Krein indefinite eigenvalues of } \gamma(T) \text{ have algebraic multiplicity} \\ \text{at most } m \text{ and one of them having exactly multiplicity } m \end{array} \right\},$$

see Pages 1 and 2 in [Eke90, Erratum].

We prove that the original statement of Ekeland is still correct under the following weaker assumption on A:

$$A(t) \text{ is strictly positive definite on } \ker(\omega \cdot \mathrm{Id} - \gamma(t)) \text{ for all } t \in \mathbb{R} \text{ and } \omega \in U. \tag{8}$$

Theorem 1.1. *For the system* (1) *with continuous (but not necessarily periodic)* $t \mapsto A(t)$*, if* (8) *holds, then the set* D *defined in* (7) *is discrete.*

To understand the system (1) and prove Theorem 1.1, we need to study the dynamics of the eigenvalues and the associated Krein forms as t varies. There is a rather complete answer for linear perturbations of Hamiltonians of Krein positive type. To be more precise, consider the endpoint matrix $\gamma(T, \varepsilon)$ of the system (3) with $\varepsilon \in \mathbb{C}$ and $A(t, \varepsilon) = H(t) + \varepsilon Q(t)$, where $H(t)$ and $Q(t)$ are both $2n \times 2n$ Hermitian matrices. The perturbation is said to be of Krein positive type if Q is non-negative definite and for all $\omega \in U$, there is no solution of the following equations in $\mathbb{C}^{2n}$:

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = JH(t)x(t) \text{ and } Q(t)x(t) = 0 \text{ a.e., } x(T) = \omega x(0).$$

Although ε is complex, by similar arguments, we see that (4) and (5) also hold. And the condition of Krein positive type perturbation is precisely the condition (8) by replacing t by ε, $A(t)$ by $B(T, \varepsilon)$ and $\gamma(t)$ by $\gamma(T, \varepsilon)$. In this special case, Krein-Lyubarskii theorem [KL62] asserts the analytic properties of the eigenvalues and the eigenvectors.

Theorem 1.2 (Krein-Lyubarski)**.** *Consider the system* (3) *with* $A(t, \varepsilon) = H(t) + \varepsilon Q(t)$ *and assume the perturbation is of Krein positive type. Suppose that* $\varepsilon_0 \in \mathbb{R}$ *and that* $\lambda_0 \in U$ *is an eigenvalue of* $\gamma(T, \varepsilon_0)$*. Then, as* ε *varies from* ε_0*,* λ_0 *continuously branches into* κ*-many eigenvalues, where*

$\kappa = \dim\ker(\lambda_0 \cdot \mathrm{Id} - \gamma(T, \varepsilon_0))^{2n}$ is the algebraic multiplicity of λ_0. These eigenvalues are grouped into m-groups, where $m = \dim\ker(\lambda_0 \cdot \mathrm{Id} - \gamma(T, \varepsilon_0))$ is the geometric multiplicity of λ_0. Each group of eigenvalues forms a multi-valued analytic function with Puiseux expansions: for $i = 1, \ldots, m$,

$$\lambda_i(\varepsilon) - \lambda_0 = \sum_{k=1}^{\infty} c_{i,k}(\varepsilon - \varepsilon_0)^{\frac{k}{j_i}},$$

where the numbers $j_1, \ldots, j_m$ are the sizes of Jordan blocks associated with the eigenvalue λ_0. In each of the expansions, the first coefficient $c_{i,1}$ ($i = 1, \ldots, m$) is non-zero. For each group of eigenvalues $\lambda_i(\varepsilon)$, the eigenvalues branch from λ_0 with tangents as $\varepsilon \in \mathbb{R}$ increases from ε_0. These tangents form a j_i-star with the same angle between consecutive tangents. As ε decreases from ε_0, the trajectories of eigenvalues also form another j_i-star. These two stars differ from each other by a rotation of $\frac{\pi}{j_i}$ radians. Among these $2j_i$ many tangents, exactly two are tangential to the circle at λ_0. If the trajectory of an eigenvalue branching from λ_0 is tangential to the circle U at λ_0 as ε varies, then that eigenvalue is Krein definite and moves on the circle U in a definite direction for ε sufficiently close to ε_0.

See Figure 1 for illustrations of a 2-star and a 3-star. The arrows indicate moving directions of the eigenvalues as ε increases.

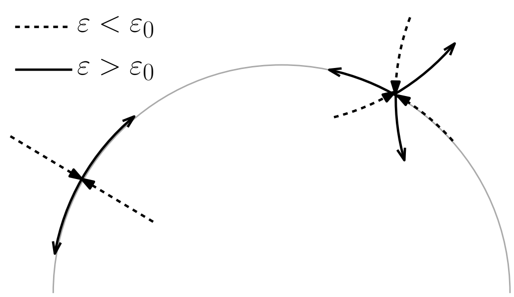

Figure 1: Bifurcation of eigenvalues

Remark 1.1. The eigenvectors also admit expansions in Puiseux series as the eigenvalues, see [YS75].

In the proof of the above theorem, they also gave a recursive way to calculate $c_{i,1}$ via the matrix Q and the generalized eigenvectors of $\gamma(T, \varepsilon)$ associated with λ_0. In the special case that $m = 1$ or $j_1 = \cdots = j_m = 1$, such an expression were obtained earlier by Gelfand and Lidskii [GfL58]. It also implies that Krein positive (resp. negative) definite eigenvalues move counter-clockwise (resp. clockwise) on the circle as the perturbation parameter ε increases along the real axis. If several eigenvalues collide on the circle from U^c, then, necessarily, a Krein indefinite eigenvalue with non-trivial Jordan blocks (Jordan blocks of size ≥ 2) is created. When several eigenvalues of different Krein types meet at λ_0 on the circle, they will continue their movement along the circle iff the geometric multiplicity of λ_0 equals to its algebraic multiplicity.

Particularly, Krein-Lyubarskii theorem implies Theorem 1.1 for the curve $\varepsilon \mapsto \gamma(T, \varepsilon)$ given by (3) when $A(t, \varepsilon) = H(t) + \varepsilon Q(t)$. Indeed, by Krein-Lyubarskii theorem, for all $\varepsilon_0 \in \mathbb{R}$, there exists

$\delta = \delta(\varepsilon_0) > 0$ such that for $\varepsilon \in (\varepsilon_0 - \delta, \varepsilon_0) \cup (\varepsilon_0, \varepsilon_0 + \delta)$, the eigenvalues on the circle are Krein definite.

We would like to obtain a C^1-version of Krein-Lyubarskii theorem for the system (1) and prove that D is isolated. For general C^1-perturbations, the eigenvalues and eigenvectors are no longer multi-valued analytic functions. Instead, we aim to give the first order asymptotic of the deviation of eigenvalues and to verify similar qualitative behavior of the dynamics of eigenvalues.

The argument of Krein and Lyubarskii doesn't directly apply. Their proof relies on a key lemma, which interprets the perturbation parameter ε as an eigenvalue of a certain self-adjoint integral operator depending on $\omega \in U$, see the lemma in [KL62, Section 1]. In this step, the linearity of the perturbation $\varepsilon \mapsto H(t) + \varepsilon Q(t)$ is crucially used. Beyond the scope of linear perturbations of Hamiltonians, if we assume the analyticity of $\varepsilon \mapsto A(t, \varepsilon)$ and follow their idea, we may encounter self-adjoint integral operators $G(\varepsilon, \omega)$ depending on two parameters $\varepsilon \in \mathbb{R}$ and $\omega \in U$. We have to show that $\{(\omega, \varepsilon) : 0 \text{ is an eigenvalue of } G(\varepsilon, \omega)\}$ is actually the graph of an analytic function in ω, which we regard as a difficult question in general. Besides, more seriously, their argument depends heavily on the analyticity of the system. This rules out the possibility of studying C^1-perturbations of the system by following their argument.

Ekeland has investigated the system (1) when $\gamma(0) = \mathrm{Id}$, $t \mapsto A(t)$ is continuous and $A(t)$ is strictly positive definite symmetric matrices for all t, see [Eke90]. It was proved that the moving direction of a Krein definite eigenvalue is determined by its Krein type: as t increases a bit, the Krein positive (resp. negative) definite eigenvalues of $\gamma(t)$ move counter-clockwise (resp. clockwise). Krein indefinite eigenvalues appear when Krein positive definite eigenvalues meet Krein negative definite eigenvalues. He has also described the branching of a Krein indefinite eigenvalue of $\gamma(t)$ when t varies from t_0 if $\gamma(t_0) = \mathrm{Id}$: if $\gamma(t_0) = \mathrm{Id}$, then there exists $\varepsilon_0 > 0$ such that for $t \in (t_0, t_0 + \varepsilon_0]$ (resp. $t \in [t_0 - \varepsilon_0, t_0)$), the eigenvalues of $\gamma(t)$ are all located on the unit circle, the eigenvalues on the upper semi circle are all Krein positive (resp. negative) definite and move counter-clockwise (resp. clockwise), while the eigenvalues on the lower part are all Krein negative (resp. positive) definite and move clockwise (resp. counter-clockwise). We remark that the condition $\gamma(0) = \mathrm{Id}$ is not essential in the above results of Ekeland. It suffices to have $\gamma(0) \in \mathrm{Sp}(2n, \mathbb{R})$.

In the same book, Ekeland has commented that the spirit of the branching mechanism of a Krein indefinite eigenvalue should be the same as in the special case of linear perturbations of Hamiltonians studied by Krein and Lyubarskii. However, to the best of our knowledge, there is no rigorous proof in general. Recently, when the Krein indefinite eigenvalue has algebraic multiplicity 2 and geometric multiplicity 1, Kuwamura and Yanagida [KY06, Theorem 3.2] give a simple and elegant formula on the derivative of the mean of bifurcated eigenvalues, which holds *without* the assumption (8). In our opinion, under the assumption (8), the first order terms of the pair of bifurcated eigenvalues cancel with each other and the second order terms of the pair is the same. And their formula is actually an expression for the second order term.

In the present paper, we focus on the first order term under the assumption (8) (but without any restriction on the multiplicities of the eigenvalues). Naturally, to study the branching of Krein indefinite eigenvalues $e^{\sqrt{-1}\theta_0} \in U$ of $\gamma(0)$, we need information on the Jordan blocks associated with $e^{\sqrt{-1}\theta_0}$. We need to introduce several notations for a precise statement of our C^1-version of Krein-Lyubarskii theorem. Note that there is a basis $\{\xi_{i,j}\}_{i=1,\dots,m;j=1,\dots,j_i}$ of the invariant space $E_{e^{\sqrt{-1}\theta_0}}(\gamma(0)) = \ker(e^{\sqrt{-1}\theta_0}\cdot \mathrm{Id} - \gamma(0))^{2n}$ associated with the eigenvalue $e^{\sqrt{-1}\theta_0}$ of the matrix $\gamma(0)$ such that m is the number of the Jordan blocks associated with the eigenvalue $e^{\sqrt{-1}\theta_0}$ of the matrix $\gamma(0)$, $j_1 \ge j_2 \ge \cdots \ge j_m \ge 1$ are the sizes of the Jordan blocks and $\{\xi_{i,j}\}_{i,j}$ are the corresponding eigenvectors, i.e., for $i = 1,\dots,m$ and $j = 1,\dots,j_i$, we have that

$$\gamma(0)\xi_{i,j} = e^{\sqrt{-1}\theta_0}\xi_{i,j} - \xi_{i,j-1} \text{ for } j = 1,\dots,j_i, \tag{9}$$

with $\xi_{i,0} = 0$ and that

$$\{\xi_{i,j}\}_{i=1,\dots,m;j=1,\dots,j_i} \text{ is a linear basis of } E_{e^{\sqrt{-1}\theta_0}}(\gamma(0)). \tag{10}$$

Note that $j_1 \ge \cdots \ge j_m \ge 1$ is not necessarily strictly decreasing. We break the sequence $\{j_i\}_i$ at the position where a strict decrease occurs. So, there are integers $s \ge 1$, $m_1,\dots,m_s \ge 1$, $n_1 > n_2 > \cdots n_s \ge 1$ such that for $\ell = 1,\dots,s$, the integer number n_ℓ is the ℓ-th largest size of Jordan blocks (in the strict sense) and there are exactly m_ℓ many blocks with the same size n_ℓ. Hence, the total number of blocks $m = \sum_{\ell=1}^{s} m_\ell$ and for $\ell = 1,\dots,s$, we have that

$$j_i = n_\ell, \text{ for } \sum_{1\le k<\ell} m_k + 1 \le i \le \sum_{1\le k\le\ell} m_k. \tag{11}$$

Sometimes, it is convenient[1] to use the following sequence of vectors $\{\eta_{i,j}\}_{i=1,\dots,m;j=1,\dots,j_i}$ instead of $\{\xi_{i,j}\}_{i=1,\dots,m;j=1,\dots,j_i}$, where

$$\eta_{i,j} \stackrel{\text{def}}{=} \left(-\sqrt{-1}e^{\sqrt{-1}\theta_0}\right)^j \xi_{i,j}, \tag{12}$$

for $i = 1,\dots,m$ and $j = 1,\dots,j_i$. We need to introduce more notations to present our results. Define an $m\times m$ square matrix S, which represents the metric $\langle A(0)\cdot,\cdot\rangle$ on the space of eigenvectors associated with $e^{\sqrt{-1}\theta_0}$:

$$S_{i,i'} = \langle A(0)\eta_{i,1}, \eta_{i',1}\rangle = \langle A(0)\xi_{i,1}, \xi_{i',1}\rangle, \quad i,i' = 1,\dots,m. \tag{13}$$

We define an $m \times m$ square matrix X by

$$X_{i,i'} = (\eta_{i,j_i}, \eta_{i',1})_G = (-1)^{j_i-1}\sqrt{-1}^{j_i} e^{(j_i-1)\sqrt{-1}\theta_0}\langle \xi_{i,j_i}, J_{2n}\xi_{i',1}\rangle, \quad i,i' = 1,\dots,m. \tag{14}$$

We write S and X in blocks as follows:

$$S = \begin{bmatrix} S^{(1,1)} & \cdots & S^{(1,s)} \\ \vdots & \ddots & \vdots \\ S^{(s,1)} & \cdots & S^{(s,s)} \end{bmatrix} \text{ and } X = \begin{bmatrix} X^{(1,1)} & \cdots & X^{(1,s)} \\ \vdots & \ddots & \vdots \\ X^{(s,1)} & \cdots & X^{(s,s)} \end{bmatrix}, \tag{15}$$

[1] As we shall see in (14), it helps to simplify the definition of X. Besides, the equation (27) is simpler in terms of $\{\eta_{i,j}\}$: $(\eta_{i,j}, \eta_{i',j'})_G = \sqrt{-1}(\eta_{i,j+1}, \eta_{i',j'})_G - \sqrt{-1}(\eta_{i,j}, \eta_{i',j'+1})_G$.

where $S^{(\ell,\ell')}$ and $X^{(\ell,\ell')}$ are $m_\ell \times m_{\ell'}$ matrices for $\ell, \ell' = 1, \dots, s$. A nice feature of X is that X is upper triangular in block sense and the diagonal blocks are Hermitian, see Corollary 2.4.

Theorem 1.3. *Consider the system* (1) *and assume* (8). *Suppose that* $e^{\sqrt{-1}\theta_0}$ *(*$\theta_0 \in \mathbb{R}$*) is a Krein indefinite eigenvalue of* $\gamma(0)$. *Recall the notations introduced in* (9), (10), (11), (13), (14) *and* (15).

a) As t *varies from* 0, *the eigenvalue* $e^{\sqrt{-1}\theta_0}$ *branches continuously into* $\sum_{\ell=1}^{s} m_\ell n_\ell$ *many eigenvalues with multiplicities, namely* $\{\lambda_{\ell,p,q}(t)\}_{\ell=1,\dots,s;p=1,\dots,m_\ell;q=1,\dots,n_\ell}$.

For $\ell = 1, \dots, s$, *reordering* $\{\lambda_{\ell,p,q}(t)\}_{q=1,\dots,n_\ell}$ *if necessary, we have that*

$$\frac{\lambda_{\ell,p,q}(t) - e^{\sqrt{-1}\theta_0}}{\sqrt{-1}e^{\sqrt{-1}\theta_0}} \overset{t\to 0}{\sim} \begin{cases} \operatorname{sgn}(ta_{\ell,p})|a_{\ell,p}t|^{\frac{1}{n_\ell}} e^{\frac{2\pi}{n_\ell}\sqrt{-1}(q-1)} & \text{if } n_\ell \text{ is odd,} \\ |a_{\ell,p}t|^{\frac{1}{n_\ell}} e^{\frac{2\pi}{n_\ell}\sqrt{-1}(q-1)} e^{\frac{\pi}{2n_\ell}\sqrt{-1}(1-\operatorname{sgn}(ta_{\ell,p}))} & \text{if } n_\ell \text{ is even,} \end{cases} \tag{16}$$

where sgn *denoted the sign function,* $(a_{\ell,p})_{p=1,\dots,m_\ell}$ *are non-zero real numbers and they are exactly the roots with multiplicities of the following polynomial in* z

$$\det \begin{bmatrix} S^{(1,1)} & \cdots & S^{(1,\ell-1)} & S^{(1,\ell)} \\ \vdots & \ddots & \vdots & \vdots \\ S^{(\ell-1,1)} & \cdots & S^{(\ell-1,\ell-1)} & S^{(\ell-1,\ell)} \\ S^{(\ell,1)} & \cdots & S^{(\ell,\ell-1)} & S^{(\ell,\ell)} - zX^{(\ell,\ell)} \end{bmatrix}. \tag{17}$$

b) There exists $\delta_0 > 0$ *such that for* $t \in (-\delta_0, 0) \cup (0, \delta_0)$, $\ell = 1, \dots, s$ *and* $p = 1, \dots, m_\ell$, $(\lambda_{\ell,p,q}(t))_{q=1,\dots,n_\ell}$ *have different behaviors depending on the parity of* n_ℓ *and the sign of* $ta_{\ell,p}$: *if* n_ℓ *is odd, then* $(\lambda_{\ell,p,q}(t))_{q=2,\dots,n_\ell}$ *stay outside of the unit circle* U, *and* $\lambda_{\ell,p,1}$ *is Krein positive definite on* U *(resp. Krein negative definite) if* $ta_{\ell,p} > 0$ *(resp.* $ta_{\ell,p} < 0$*). If* n_ℓ *is even and* $ta_{\ell,p} < 0$, *then* $(\lambda_{\ell,p,q}(t))_{q=1,\dots,n_\ell}$ *stay outside of the unit circle* U*; if* n_ℓ *is even and* $ta_{\ell,p} > 0$, *then* $\lambda_{\ell,p,1}(t) \in U$ *is Krein positive definite,* $\lambda_{\ell,p,n_\ell/2+1}(t) \in U$ *is Krein negative definite, and the other* $\lambda_{\ell,p,q}(t)$ *stay outside of* U.

Remark 1.2. Note that $X^{(\ell,\ell)}$ is Hermitian and non-degenerate, see Corollary 2.4. By Sylvester's law of inertia, the number $\sharp\{p = 1, \dots, m_\ell : a_{\ell,p} > 0\}$ equals the positive index of inertia of $X^{(\ell,\ell)}$. Hence, the instant moving directions of the eigenvalues (when t increases (or decreases) from 0), is purely determined by $\gamma(0)$ under the assumption (8). When t is sufficiently close to 0, the number of the Krein positive (or negative) definite eigenvalues depends only on $\gamma(0)$.

Remark 1.3. If we replace "positive definiteness" by "negative definiteness" in (8), i.e.,

$$A(t) \text{ is strictly negative definite on } \ker(\omega \cdot \mathrm{Id} - \gamma(t)) \text{ for all } t \in \mathbb{R} \text{ and } \omega \in U, \tag{18}$$

then, all the results still hold under a time reversal $t \mapsto A(t)$. But if we remove "positive" from (8), i.e., if we assume

$$A(t) \text{ is strictly definite on } \ker(\omega \cdot \mathrm{Id} - \gamma(t)) \text{ for all } t \in \mathbb{R} \text{ and } \omega \in U, \tag{19}$$

then the system is a mixture of positive and negative systems, which is locally decomposable. To be more precise, we denote by $\Lambda_+(t)$ (resp. $\Lambda_-(t)$) the eigenvalues ω on the unit circle U such that $A(t)$ is strictly positive (resp. negative) definite on $\ker(\omega \cdot \mathrm{Id} - \gamma(t))$. Under the condition (19), the Hausdorff distance between the two sets $\Lambda_+(t)$ and $\Lambda_-(t)$ is strictly positive and lower semi-continuous in t. By Lemma A.2, locally as t varies, the eigenvalues are separated into two groups. The first group corresponds to a possibly smaller system satisfying (8) and the second group corresponds to a system satisfying (18).

The proof of Theorem 1.3 a) is different from previous argument by Krein, Lyubarskii and Ekeland. Besides, our argument is direct and elementary. We analyze the asymptotics of coefficients of the characteristic polynomial of $\gamma(t)$. This is linked to the Jordan structure of the symplectic matrix via exterior products of linear maps. By continuity of roots depending on the coefficients of a certain properly normalized polynomial, we deduce the asymptotics of eigenvalues. This part is some sort of blowup analysis. For the part b) of Theorem 1.3, we use Theorem 1.3 a) together with a local C^1-approximation of $t \mapsto \gamma(t)$ by analytic symplectic paths. Indeed, Theorem 1.3 a) provides an upper bound for the number of Krein definite eigenvalues on the circle by first order asymptotics of the eigenvalues. On the other hand, the approximation argument provides matching lower bounds. However, such an approximation argument alone is not sufficient to predict the movement of eigenvalues. We have to combine it with the monotonicity of a certain index function, see Claim 1. As an intermediate step, in the appendix, we sketch the argument of Theorem 1.3 when $t \mapsto A(t)$ is real analytic.

1.2 Organization of the paper

We collect definitions and notations, prepare some useful properties in Section 2. We prove Theorem 1.3 a) in Section 3 and Theorem 1.3 b) in Section 4. We sketch the argument of Theorem 1.3 for the analytic case in Subsection A.3.

2 Preliminaries

2.1 Notations and definitions

- For two positive integers m and n, we denote by $M_{m\times n}(\mathbb{C})$ (resp. $M_{m\times n}(\mathbb{R})$) the set of $m \times n$ complex (resp. real) matrices. When $m = n$, we use the notations $M_n(\mathbb{C})$ and $M_n(\mathbb{R})$ for simplicity. For a square matrix, we define its size as the number of rows in the matrix.

- For a matrix M, we denote by M^T the transpose of M. For a complex matrix M, we denote by M^* the conjugate transpose of M.

- For $n \geq 1$, we denote by Id_n the $n \times n$ identity matrix and define $J_{2n} \stackrel{\text{def}}{=} \begin{bmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{bmatrix}$. Then,

$J_{2n}^* = J_{2n}^T = -J_{2n}$ and $J_{2n}^2 = -\operatorname{Id}$.

- For a vector space V and a finite number of subspaces $\{V_i\}_{i\in I}$, we denote by $\sum_{i\in I} V_i$ the sum of the vector spaces $\sum_{i\in I} V_i$.

- For vectors $v_1, \ldots, v_n$ in a vector space V, we denote by $\wedge_{j=1}^n v_j$ the exterior product $v_1 \wedge v_2 \wedge \cdots \wedge v_n$. (Note that $\wedge$ is associative.) We denote by $\Lambda^n(V)$ the linear span of all such $\wedge_{j=1}^n v_j$ and denote by $\Lambda(V)$ the direct sum $\bigoplus_{n\geq 0} \Lambda^n(V)$ with the convention that $\Lambda^0(V) = \{0\}$. For a totally ordered set $P = \{p_1, \ldots, p_n\}$ with $p_1 \prec p_2 \prec \cdots \prec p_n$ and vectors $(v_p)_{p\in P}$ indexed by P, we denote by $\wedge_{p\in P} v_p$ the exterior product $v_{p_1} \wedge v_{p_2} \wedge \cdots \wedge v_{p_n}$. (Note that $\Lambda(V)$ is a vector space. Hence, if we take $(v_p)_{p\in P}$ from the vector space $\Lambda(V)$, then we define the exterior products of exterior products in a consistent manner.)

- For $m \geq 1$, the inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{C}^m$ is defined by

$$\langle x, y \rangle = \sum_{j=1}^m x_j \bar{y}_j.$$

  Then, for $x, y \in \mathbb{C}^{2n}$,

$$(x, y)_G = \sqrt{-1}\langle x, J_{2n} y\rangle = -\sqrt{-1}\langle J_{2n} x, y\rangle = \sqrt{-1}\left\{\sum_{k=1}^n (x_n \bar{y}_{n+k} - x_{n+k}\bar{y}_k)\right\}.$$

- For $n \geq 1$ and a linear subspace V of $\mathbb{C}^{2n}$, we denote by $V^{\perp_G}$ the *symplectic orthogonal complement* of V, i.e.,

$$V^{\perp_G} = \{x \in \mathbb{C}^{2n} : (x, y)_G = 0, \forall y \in V\}.$$

  The linear subspace V is *symplectic* if $V \cap V^{\perp_G} = \{0\}$. When V is a linear subspace of $\mathbb{R}^{2n}$, we replace $\mathbb{C}^{2n}$ by $\mathbb{R}^{2n}$ in the above definition.

- For a $k\times k$ complex valued matrix M and an eigenvalue λ of M, the geometric multiplicity of λ is defined as $\dim\ker(\lambda\cdot \operatorname{Id} - M)$ and the algebraic multiplicity is defined as $\dim\ker(\lambda\cdot\operatorname{Id} - M)^k$. We denote by $E_\lambda = E_\lambda(M)$ the invariant subspace of $\mathbb{C}^k$, i.e.,

$$E_\lambda = \{x \in \mathbb{C}^k : (\lambda \cdot \operatorname{Id} - M)^k x = 0\}.$$

- Denote by $p(\lambda, t)$ the characteristic polynomial of the matrix $\gamma(t)$, i.e.,

$$p(\lambda, t) = \det(\lambda \cdot \operatorname{Id} - \gamma(t)).$$

2.2 Exterior powers of linear maps

We recall exterior powers of a linear map A and its relation with its determinant $\det(A)$.

Starting from several linear maps on a vector space V, there are many ways to combine them to define multi-linear skew symmetric maps (or equivalently, linear maps on the exterior products

$\Lambda^m(V)$ of V). We follow the construction in [Win10, Section 3.7]. For natural numbers $k \le m$, the author defines a linear map on $\Lambda^m(V)$ by taking certain "skew symmetrization" of tensors of k many linear maps A with $m-k$ many identity maps. For our purpose, it suffices to take m to be the dimension of V. But we need a slightly generalization to allow the combination of three linear maps A_1, A_2 and the identity map. We introduce these notations in the following definition.

Definition 2.1. Let $A : V \to V$ be a linear map on an n-dimensional vector space V. For $k = 0, \dots, n$, we define the exterior powers $\bigwedge(n,k,A) : \Lambda^n(V) \to \Lambda^n(V)$ as a linear map as follows:

$$\bigwedge(n,k,A)(v_1 \wedge \cdots \wedge v_n) \stackrel{\text{def}}{=} \sum_{\sigma \in \{0,1\}^n : \sum_i \sigma_i = k} \wedge_{i=1}^n (\sigma_i \cdot A v_i + (1-\sigma_i) \cdot v_i). \tag{20}$$

Similarly, for linear maps $A_1, A_2 : V \to V$, for $k_1, k_2 = 0, \dots, n$, we define the linear map $\bigwedge(n, k_1, k_2, A_1, A_2) : \Lambda^n(V) \to \Lambda^n(V)$ as follows:

$$\begin{aligned}&\bigwedge(n,k_1,k_2,A_1,A_2)(v_1 \wedge \cdots \wedge v_n) \\ &\quad\stackrel{\text{def}}{=} \sum_{\sigma \in \{0,1,2\}^n : \sum_i 1_{\sigma_i=1} = k_1, \sum_i 1_{\sigma_i=2} = k_2} \wedge_{i=1}^n (1_{\sigma_i=1} \cdot A_1 v_i + 1_{\sigma_i=2} \cdot A_2 v_i + 1_{\sigma_i=0} \cdot v_i), \end{aligned} \tag{21}$$

Since $\Lambda^n(V)$ is 1-dimensional, we identify the $\bigwedge(n,k,A)$ (or $\bigwedge(n,k_1,k_2,A_1,A_2)$) with the unique scaling factor, which is also denoted by $\bigwedge(n,k,A)$ (or $\bigwedge(n,k_1,k_2,A_1,A_2)$).

In the above definition, for each vector v_i, we choose one from the three linear maps Id, A_1 and A_2 and apply it to v_i. For the assignment of linear maps to the linear basis, the only constraint is that the map A_1 occurs k_1 many times and the map A_2 occurs exactly k_2 many times. All these assignments have equal weight.

Note that $\det(A)$ is identified with the linear map $\bigwedge(n,n,A)$ on the 1-dimensional vector space $\Lambda^n(V)$. In particular, for an eigenvalue λ_0 of the matrix $\gamma(0)$, we have that

$$\begin{aligned} p(\lambda,t) &= \det(\lambda \cdot \mathrm{Id} - \gamma(t)) = \det((\lambda-\lambda_0)\cdot \mathrm{Id} + (\lambda_0 \cdot \mathrm{Id} - \gamma(0)) - (\gamma(t)-\gamma(0))) \\ &= \sum_{k=0}^{2n} (\lambda-\lambda_0)^k \sum_{k_1+k_2=2n-k, k_1\ge 0, k_2 \ge 0} (-1)^{k_2} \cdot \bigwedge(2n, k_1, k_2, \lambda_0 \cdot \mathrm{Id} - \gamma(0), \gamma(t)-\gamma(0)). \end{aligned} \tag{22}$$

In the above calculation, we express the determinant by wedge powers of the sum of linear maps $(\lambda-\lambda_0)\cdot \mathrm{Id}$, $\lambda_0 \cdot \mathrm{Id} - \gamma(0)$ and $\gamma(0) - \gamma(t)$, expand it according to distributive law and collect the terms with the same times of occurrence, where k_1 is the time of occurrence of $\lambda_0 \cdot \mathrm{Id} - \gamma(0)$ and k_2 counts the occurrence of $\gamma(0) - \gamma(t)$.

2.3 Continuity of roots of polynomials

Consider a polynomial with complex coefficients of degree at most n. We will need the following lemma on the continuity of the roots as the coefficients vary.

Lemma 2.1. *Let W be a neighborhood of 0. Let $P_t(z) = \sum_{j=0}^n c_j(t) z^j$, where $c_j(t) \in \mathbb{C}$ and $t \in W$. Suppose that $t \mapsto c_j(t)$ is continuous for $j = 0, \dots, n$ and $t \in W$. Denote by $d(t)$ the degree of the polynomial P_t. Suppose that $d(t) = n$ for $t \in W \setminus \{0\}$ and $d(0) = m \leq n$. Then, there exist m continuous complex valued functions $z_1, \dots, z_m$ on W and $n-m$ continuous complex valued functions $z_{m+1}, \dots, z_n$ on $W \setminus \{0\}$ such that*

- *for $t \neq 0$, $z_1(t), \dots, z_n(t)$ are roots of P_t,*
- *for $t = 0$, $z_1(0), \dots, z_m(0)$ are roots of P_0,*
- *for $i = m+1, \dots, n$, we have that $\lim_{t \to 0} z_i(t) = \infty$.*

Proof. By assumptions, for $t_0 \in W$, $P_t(z) \overset{t \to t_0}{\to} P_{t_0}(z)$ uniformly for z on compacts. Hence, for any continuous loop Γ avoiding the roots of P_{t_0}, for t sufficiently close to t_0, P_t does not vanish on Γ and

$$\lim_{t \to t_0} \frac{1}{2\pi\sqrt{-1}} \int_\Gamma \frac{1}{P_t(z)} \, \mathrm{d}z = \frac{1}{2\pi\sqrt{-1}} \int_\Gamma \frac{1}{P_{t_0}(z)} \, \mathrm{d}z. \tag{23}$$

Also, note that for a simple loop avoiding the roots of P_t, $\frac{1}{2\pi\sqrt{-1}} \int_\Gamma \frac{1}{P_t(z)} \, \mathrm{d}z$ is precisely the number of roots inside the loop. (The interior and exterior region are determined by the orientation of the loop.) Eventually, Lemma 2.1 holds since (23) holds for all continuous loops avoiding the roots of P_{t_0}.

$\square$

2.4 Properties of symplectic matrices

We collect some well-known properties of symplectic matrices in this subsection. The following observations, although elementary, are frequently used in some calculation. For a complex number λ, the adjoint of $\lambda \cdot \mathrm{Id}$ under $(\cdot, \cdot)_G$ is $\bar{\lambda} \cdot \mathrm{Id}$, i.e., $\forall x, y \in \mathbb{C}^{2n}$, we have

$$(\lambda x, y)_G = (x, \bar{\lambda} y)_G. \tag{24}$$

For a symplectic matrix γ, the adjoint of γ under $(\cdot, \cdot)_G$ is γ^{-1}, i.e., $\forall x, y \in \mathbb{C}^{2n}$, we have

$$(\gamma x, y)_G = (x, \gamma^{-1} y)_G. \tag{25}$$

For all symplectic subspaces V, the restriction of the bilinear form $(\cdot, \cdot)_G$ on V is non-degenerate. For a symplectic subspace V, if it is invariant under the linear symplectic transform γ, then so is its symplectic orthogonal complement $V^{\perp_G}$.

The following criteria on the G-orthogonality of invariant spaces is basically [Eke90, Proposition 5, Section 2, Chapter 1].

Lemma 2.2. *Let λ and μ be two eigenvalues of the symplectic matrix $\gamma \in \mathrm{Sp}(2n, \mathbb{R})$. If $\lambda\bar{\mu} \neq 1$, then the invariant spaces E_λ and E_μ are G-othorgonal. Consider a partition $\{P_1, \dots, P_k\}$ of the set of eigenvalues of γ such that each P_i is stable under the circular reflection $z \mapsto \bar{z}^{-1}$. For each i,* *let* $E_i = \cup_{\lambda \in P_i} E_\lambda$. *Then,* $E_1, \ldots, E_k$ *is a* G*-orthogonal decomposition of* $\mathbb{C}^{2n}$. *In particular, when* $\lambda \in U$, *we have the following* G*-othorgonal decomposition of* $\mathbb{C}^{2n}$:

$$\mathbb{C}^{2n} = E_\lambda \oplus F_\lambda, \tag{26}$$

where F_λ *is the direct sum of* $\{E_\mu\}_{\mu \neq \lambda}$. *Hence, if* λ *is a simple eigenvalue on* U, *it is Krein definite.*

The "inner product" under $(\cdot, \cdot)_G$ of the generalized eigenvectors in (9) and (10) must satisfy certain algebraic relations:

Lemma 2.3. *Suppose that* λ *is an eigenvalue of the symplectic matrix* γ. *We use the same notations* $\{\xi_{i,j}\}_{i=1,\ldots,m; j=1,\ldots,j_i}$ *and* $\{\eta_{i,j}\}_{i=1,\ldots,m; j=1,\ldots,j_i}$ *as* (9), (10), (11) *and* (12) *for the eigenvalue* λ *of* γ *instead of the eigenvalue* $e^{\sqrt{-1}\theta_0}$ *of* $\gamma(0)$. *For* $i, i' = 1, \ldots, m$, $j = 0, \ldots, j_i - 1$ *and* $j' = 0, \ldots, j_{i'} - 1$, *we have that*

$$(\lambda^j \xi_{i,j}, \lambda^{j'} \xi_{i',j'})_G = (\lambda^{j+1} \xi_{i,j+1}, \lambda^{j'} \xi_{i',j'})_G + (\lambda^j \xi_{i,j}, \lambda^{j'+1} \xi_{i',j'+1})_G, \tag{27}$$

with the convention that $\xi_{i,0} = 0$ *for all* $i = 1, \ldots, m$. *In particular, when* $j + j' \leq \max(j_i, j_{i'})$, *we have that*

$$(\eta_{i,j}, \eta_{i',j'})_G = (\xi_{i,j}, \xi_{i',j'})_G = 0.$$

For fixed $i, i' = 1, \ldots, s$, *we have the same value* $(\eta_{i,j}, \eta_{i',j'})_G$ *for all* $j = 1, \ldots, j_i$ *and* $j' = 1, \ldots, j_{i'}$ *such that* $j + j' = \max(j_i, j_{i'}) + 1$.

Proof. Since γ is sympletic, we have that $(\gamma v_1, \gamma v_2)_G = (v_1, v_2)_G$ for all $v_1, v_2 \in \mathbb{C}^{2n}$. By taking $v_1 = \xi_{i,j+1}$ and $v_2 = \xi_{i',j'+1}$, we obtain (27). The rest directly follows from (27). $\square$

Note that $(x, y)_G = \overline{(y, x)_G}$. From non-degeneracy of $(\cdot, \cdot)_G$ and Lemmas 2.2 and 2.3, we get

Corollary 2.4. *Recall the notations* (14) *and* (15). *For* $\ell = 1, \ldots, s$, *the matrix* $X^{(\ell,\ell)}$ *is Hermitian and non-degenerate. For* $1 \leq \ell_1 < \ell_2 \leq s$, *we have that* $X^{(\ell_2,\ell_1)} = 0$.

3 Proof of Theorem 1.3 a)

As the proof of Theorem 1.3 a) is long and technical, we decide to give the sketch of the proof and provide some intuitive ideas in advance. Suppose $\lambda_0 = e^{\sqrt{-1}\theta_0} \in U$ is an eigenvalue of $\gamma(0)$. We expand the characteristic polynomial $p(\lambda, t) = \det(\lambda \cdot \mathrm{Id} - \gamma(t))$ at $e^{\sqrt{-1}\theta_0}$:

$$p(\lambda, t) = \sum_{k=0}^{2n} c_k(t)(\lambda - e^{\sqrt{-1}\theta_0})^k. \tag{28}$$

In order to study the asymptotics of the eigenvalues as t varies from 0, we study the asymptotics of the coefficients $\{c_k(t)\}_{k=0,\ldots,2n}$ in Lemma 3.1 in Subsection 3.1. We will illustrate the results of Lemma 3.1 and explain the way to prove Theorem 1.3 a) from Lemma 3.1 by a concrete example. But we will not sketch the technical proof of Lemma 3.1.

In Lemma 3.1, we will show that $c_k(t) = O(t^{\varphi(k)})$ as $t \to 0$, where $\varphi(k)$ is a certain integer valued function in k. Let us precisely give the value of $\varphi(k)$. Denote by N the algebraic multiplicity of λ_0. Then, $\varphi(k)$ is simply 0 for $k \geq N$. For $k = 0, \ldots, N-1$, the value of $\varphi(k)$ can be obtained graphically via Young diagrams as follows: we list the sizes of Jordan blocks associated with λ_0 in non-increasing order $j_1 \geq j_2 \geq \cdots \geq j_m$. The sequence $\{j_i\}_{i=1,\ldots,m}$ forms a partition of N and is represented by a Young diagram. The Young diagram consists of unit squares placed side by side. For $i = 1, \ldots, m$, the i-th row has exactly j_{m+1-i} many squares. All these rows are aligned to the left. Please see Figure 2 for the Young diagram associated with the partition $4+2+2+1 = 9$. To

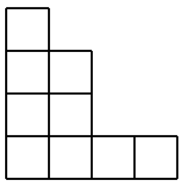

Figure 2: Young diagram

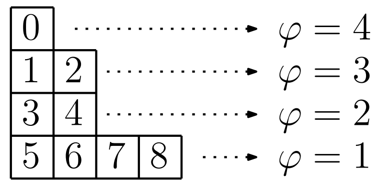

Figure 3: $k \mapsto \varphi(k)$

get the value of φ, we fill the diagram with integers $\{0, \ldots, N-1\}$ from the top row to the bottom row. In each row, we fill the diagram from the left to the right. Then, each integer k is filled in the $\varphi(k)$-th row from the bottom, see Figure 3. Alternatively, from a finite non-increasing sequence of integers $\{j_i\}_{i=1,\ldots,m}$, their partial sums $\{\sum_{i=k}^{m} j_i\}_{k=1,\ldots,m}$ form a strictly decreasing sequence, the upper boundary of the corresponding new Young diagram represents the graph of the function $\varphi(k)$, see Figure 4 for the same sizes of Jordan blocks as Figures 2 and 3. The black and grey points give

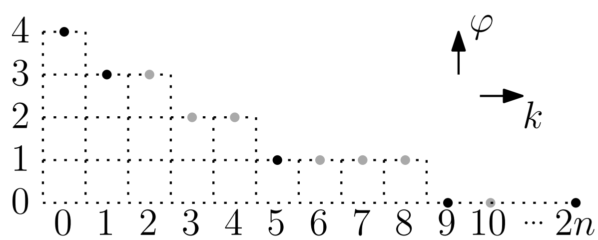

Figure 4: $k \to \varphi(k)$

the graph of φ. (Recall that φ is set to 0 for $k \geq N$ and $N = 9$ in the above figures.)

Let us explain the difference between black and grey points in the following. Roughly speaking, the black points separate the Jordan blocks with different sizes. Alternatively, the black points are exactly the extremal points of the convex hull of the discrete domain $\{(\tilde{k}, \tilde{\varphi}) : \tilde{k} = 0, \ldots, 2n, \tilde{\varphi} \in \mathbb{Z}, \tilde{\varphi} \geq \varphi(\tilde{k})\}$ above the graph of φ, see Figure 5. We will prove in Lemma 3.1 that $c_k(t) = O(t^{\varphi(k)})$ as $t \to 0$. For general k (corresponding to the grey dots), $\varphi(k)$ is not necessarily the exact order of $c_k(t)$. However, for those k corresponding to the black dots, the order $\varphi(k)$ is exact and we calculate $\lim_{t\to 0} c_k(t) t^{-\varphi(k)}$ in (36) of Lemma 3.1.

Next, we sketch the proof of Theorem 1.3 a) from Lemma 3.1. We will carry out certain blow

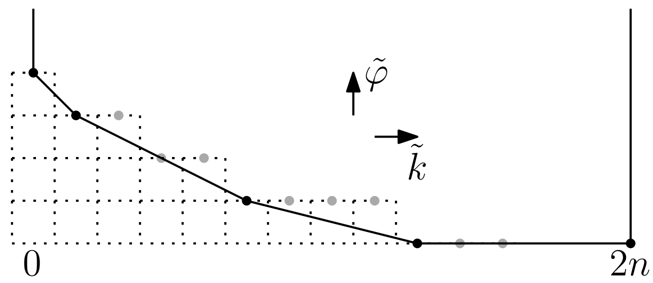

Figure 5: Boundary of the convex hull of $\{(\tilde{k}, \tilde{\varphi}) : \tilde{k} = 0, \dots, 2n, \tilde{\varphi} \in \mathbb{Z}, \tilde{\varphi} \geq \varphi(\tilde{k})\}$

up analysis at $t = 0$ and $\lambda = \lambda_0$. Take $w = \frac{\lambda - \lambda_0}{t^\alpha}$ for $\alpha > 0$[2]. After a change of variable, we obtain another polynomial $q_\alpha(w, t)$ from $p(\lambda, t)$, where $q_\alpha(w, t) = \sum_{k=0}^{2n} c_k(t) t^{\alpha k} w^k$. Note that $\lim_{t \to 0} q_\alpha(w, t) = 0$. To obtain a non-trivial limit, we need to divide $q_\alpha(w, t)$ by $t^{\beta(\alpha)}$, where $\beta(\alpha) = \min\{\varphi(k) + \alpha k : k = 0, \dots, 2n\}$. We are interested in the limiting polynomial

$$r_\alpha(w) = \sum_{k=0}^{2n} \lim_{t \to 0} c_k(t) t^{\alpha k - \beta(\alpha)} \cdot w^k.$$

In order to obtain $\lim_{t \to 0} \frac{\lambda(t) - \lambda_0}{t^\alpha}$ by using Lemma 2.1, we need to answer the following questions: does $r_\alpha(w)$ vanish? If not, how to describe the roots of $r_\alpha(w)$?

Note that the possible minimizers of $\varphi(k) + \alpha k$ are important to us since

$$\lim_{t \to 0} c_k(t) t^{\alpha k - \beta(\alpha)} = \begin{cases} 0 & \text{if } k \text{ is not a minimizer,} \\ \lim_{t \to 0} c_k(t) t^{-\varphi(k)} & \text{if } k \text{ is a minimizer.} \end{cases}$$

Denote by L_α the line through the origin with the slope $-\alpha$. To find the minimizers, we translate L_α upwards until L_α has non-empty intersection with the graph of $\varphi(k)$ for the first time. The k-coordinates of the intersection points are precisely the minimizers. The intersection must contain black points since the black points are extremal points of the convex hull of the discrete domain above the graph of φ, see Figure 5. For the k-coordinates of the black intersection points, the limit $\lim_{t \to 0} c_k(t) t^{\alpha k - \beta(\alpha)} = \lim_{t \to 0} c_k(t) t^{-\varphi(k)} \neq 0$. In particular, $r_\alpha(w) \not\equiv 0$.

When $\frac{1}{\alpha}$ is different from the sizes of Jordan blocks associated with λ_0, the minimizer $k_{\min}$ is the single black intersection point, see Figure 6 for $\alpha = \frac{3}{4}$ and the same sizes of Jordan blocks as in Figure 2. In this case, the limiting polynomial $r_\alpha(w)$ consists of a single term and its roots must be zero. When $\frac{1}{\alpha}$ equals the size of a Jordan block associated with λ_0, there are exactly $m(\alpha) + 1$ many minimizers, where $m(\alpha)$ equals the number of Jordan blocks (associated with λ_0) of the size $\frac{1}{\alpha}$, see Figure 7. The minimizers have equal distance $\frac{1}{\alpha}$ between each other (since the intermediate grey points separate Jordan blocks of the same size). By Lemma 3.1, the coefficients of the limiting polynomial r correspond to the sum of certain principle minors. Finally, we write r as a certain determinant and the asymptotics of eigenvalues are determined by the sizes of Jordan blocks and

[2] For $\alpha = 0$, the following argument simply yields the continuity of the eigenvalues of $\gamma(t)$ as t varies from 0.

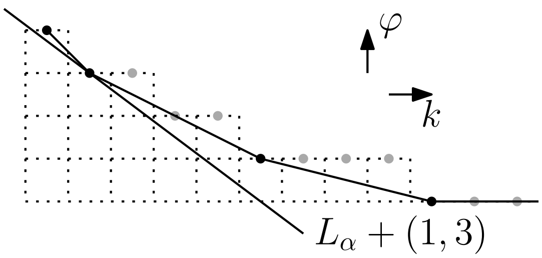

Figure 6: Generic intersection

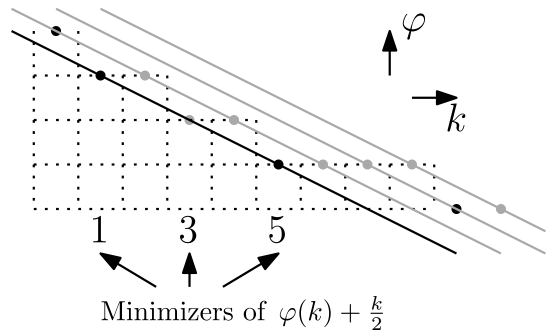

Figure 7: $\alpha = \frac{1}{2}$

the roots of the limiting polynomial r. For instance, in Figure 7, $N = 9$, the sizes of Jordan blocks are 4, 2, 2 and 1, and $\alpha = \frac{1}{2}$ corresponds to the Jordan blocks of size 2. The minimizers are 1, 3 and 5 and $\varphi(1) = 3$, $\varphi(3) = 2$ and $\varphi(5) = 1$. By Lemma 3.1, the limiting polynomial

$$
\begin{aligned}
r_{\frac{1}{2}}(w) &= \sum_{k=1,3,5} \lim_{t \to 0} c_k(t) t^{-\varphi(k)} \cdot w^k \\
&= c_9(0) w \left(\begin{vmatrix} d_{1,1} & d_{1,2} & d_{1,3} \\ d_{2,1} & d_{2,2} & d_{2,3} \\ d_{3,1} & d_{3,2} & d_{3,3} \end{vmatrix} + w^2 \left(\begin{vmatrix} d_{1,1} & d_{1,2} \\ d_{2,1} & d_{2,2} \end{vmatrix} + \begin{vmatrix} d_{1,1} & d_{1,3} \\ d_{3,1} & d_{3,3} \end{vmatrix} \right) + w^4 d_{1,1} \right) \\
&= c_9(0) w \begin{vmatrix} d_{1,1} & d_{1,2} & d_{1,3} \\ d_{2,1} & d_{2,2} + w^2 & d_{2,3} \\ d_{3,1} & d_{3,2} & d_{3,3} + w^2 \end{vmatrix},
\end{aligned}
$$

where the 4×4 matrix d equals $SX^{-1}\Lambda$, where Λ is a diagonal matrix with diagonal elements $-\lambda_0^4$, λ_0^2, λ_0^2 and $\sqrt{-1}\lambda_0$ from the top to the bottom. (Recall the definition of S and X in (13) and (14).) A non-trivial root of $r_{\frac{1}{2}}(w)$ corresponds to a non-zero finite limit of $\lim_{t \to 0} \frac{\lambda(t) - \lambda_0}{t^{\frac{1}{2}}}$ where $\lambda(t)$ is an eigenvalue of $\gamma(t)$. It is Hölder-$\frac{1}{2}$ continuous at $t = 0$. The trivial root 0 of $r_{\frac{1}{2}}(w)$ corresponds to the zero limit of $\lim_{t \to 0} \frac{\lambda(t) - \lambda_0}{t^{\frac{1}{2}}}$, where $\lambda(t)$ corresponds to a certain Jordan block of strictly smaller size and has better regularity at $t = 0$. The non-trivial roots of $r_\alpha(w)$ are important and they are also the roots of the polynomial $Q(w)$, where

$$
Q(w) = \begin{vmatrix} d_{1,1} & d_{1,2} & d_{1,3} \\ d_{2,1} & d_{2,2} + w^2 & d_{2,3} \\ d_{3,1} & d_{3,2} & d_{3,3} + w^2 \end{vmatrix}. \tag{29}
$$

Write the matrices d, S, X and Λ as in (15) with $s = 3$:

$$
d = \left(\begin{array}{c|cc|c} d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} \\ \hline d_{2,1} & d_{2,2} & d_{2,3} & d_{2,4} \\ d_{3,1} & d_{3,2} & d_{3,3} & d_{3,4} \\ \hline d_{4,1} & d_{4,2} & d_{4,3} & d_{4,4} \end{array} \right), \qquad \Lambda = \left(\begin{array}{c|cc|c} -\lambda_0^4 & 0 & 0 & 0 \\ \hline 0 & \lambda_0^2 & 0 & 0 \\ 0 & 0 & \lambda_0^2 & 0 \\ \hline 0 & 0 & 0 & \sqrt{-1}\lambda_0 \end{array} \right),
$$

$$S=\left(\begin{array}{c|cc|c} S_{1,1} & S_{1,2} & S_{1,3} & S_{1,4} \\ \hline S_{2,1} & S_{2,2} & S_{2,3} & S_{2,4} \\ S_{3,1} & S_{3,2} & S_{3,3} & S_{3,4} \\ \hline S_{4,1} & S_{4,2} & S_{4,3} & S_{4,4} \end{array}\right), \qquad X=\left(\begin{array}{c|cc|c} X_{1,1} & X_{1,2} & X_{1,3} & X_{1,4} \\ \hline 0 & X_{2,2} & X_{2,3} & X_{2,4} \\ 0 & X_{3,2} & X_{3,3} & X_{3,4} \\ \hline 0 & 0 & 0 & X_{4,4} \end{array}\right).$$

By calculation in blocks, we get that

$$\left(\begin{array}{c|cc} d_{1,1} & d_{1,2} & d_{1,3} \\ \hline d_{2,1} & d_{2,2} & d_{2,3} \\ d_{3,1} & d_{3,2} & d_{3,3} \end{array}\right) = \left(\begin{array}{c|cc} S_{1,1} & S_{1,2} & S_{1,3} \\ \hline S_{2,1} & S_{2,2} & S_{2,3} \\ S_{3,1} & S_{3,2} & S_{3,3} \end{array}\right) \left(\begin{array}{c|cc} X_{1,1} & X_{1,2} & X_{1,3} \\ \hline 0 & X_{2,2} & X_{2,3} \\ 0 & X_{3,2} & X_{3,3} \end{array}\right)^{-1} \left(\begin{array}{c|cc} -\lambda_0^4 & 0 & 0 \\ \hline 0 & \lambda_0^2 & 0 \\ 0 & 0 & \lambda_0^2 \end{array}\right).$$

Write the above equation by $\tilde{d}=\tilde{S}\tilde{X}^{-1}\tilde{\Lambda}$. Denote by $\tilde{I}$ the 3×3 square matrix $\begin{pmatrix} 0 & 0 \\ 0 & \mathrm{Id}_2 \end{pmatrix}$. Then, we have that

$$Q(w)=\det(\tilde{d}+w^2\tilde{I})=\det(\tilde{S}\tilde{X}^{-1}\tilde{\Lambda}+w^2\tilde{I})=\frac{\det\tilde{\Lambda}}{\det\tilde{X}}\cdot\det(S+w^2\tilde{I}\tilde{\Lambda}^{-1}\tilde{X}).$$

Hence, the roots of Q coincide with the root of $\tilde{Q}(w)$, where

$$\tilde{Q}(w)=\det(S+w^2\tilde{I}\tilde{\Lambda}^{-1}\tilde{X})=\det\left\{\left(\begin{array}{c|cc} S_{1,1} & S_{1,2} & S_{1,3} \\ \hline S_{2,1} & S_{2,2} & S_{2,3} \\ S_{3,1} & S_{3,2} & S_{3,3} \end{array}\right)+w^2\lambda_0^{-2}\left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & X_{2,2} & X_{2,3} \\ 0 & X_{3,2} & X_{3,3} \end{array}\right)\right\}. \tag{30}$$

The above method also works in general case as we shall see in Subsection 3.2. In the formal proof, we will replace the geometric arguments by explicit and rigorous analysis.

We state and prove Lemma 3.1 in Subsection 3.1, where we use the exterior powers of linear maps. We deduce Theorem 1.3 a) from Lemma 3.1 in Subsection 3.2. The reader may firstly skip the technical proof of Lemma 3.1 and go directly to the proof of Theorem 1.3 a).

3.1 Proof of Lemma 3.1

Lemma 3.1. *Consider the solution $\gamma(t)\in\mathrm{Sp}(2n,\mathbb{R})$ of (1) without assuming (8). Recall the notations (9), (10), (11), (13), (14) and (28). Denote by $N=N(e^{\sqrt{-1}\theta_0})$ the dimension of the invariant space $E_{e^{\sqrt{-1}\theta_0}}(\gamma(0))$. (Note that $N=\sum_{i=1}^m j_i$.) Then, we have that*

$$c_N(0)=\lim_{\lambda\to e^{\sqrt{-1}\theta_0}}\frac{p(\lambda,0)}{(\lambda-e^{\sqrt{-1}\theta_0})^N}. \tag{31}$$

For $k=0,\ldots,N-1$, as $t\to 0$,

$$c_k(t)=(-t)^{\varphi(k)}\cdot\bigwedge(2n,2n-k-\varphi(k),\varphi(k),e^{\sqrt{-1}\theta_0}\cdot\mathrm{Id}-\gamma(0),\frac{\mathrm{d}\gamma}{\mathrm{d}t}(0))+o(t^{\varphi(k)}), \tag{32}$$

where

$$\varphi(k)=\varphi(k,\gamma(0))\stackrel{\mathrm{def}}{=}\min\left\{i=1,\ldots,m:\sum_{i'=1}^{i}j_{i'}\geq N-k\right\}$$

$$= \min\left\{ i = 1, \dots, m : \sum_{i < i' \le m} j_{i'} \le k \right\}. \tag{33}$$

(Consider the Jordan blocks associated with the eigenvalue $e^{\sqrt{-1}\theta_0}$. Then, $\varphi(k)$ is precisely the minimal number of blocks such that their total size is not less than $N - k$. By definition, we have that $k \ge \sum_{i>\varphi(k)} j_i$.) In particular, when $k = \sum_{i>\varphi(k)} j_i$, as $t \to 0$,

$$c_k(t) = (-1)^{N-k} t^{\varphi(k)} c_N(0) \sum_{I \in \mathcal{I}_k} \det\left[(d_{i,i'})_{i,i' \in I}\right] + o(t^{\varphi(k)}), \tag{34}$$

where

$$\mathcal{I}_k \overset{\text{def}}{=} \left\{ I \subset \{1, \dots, m\} : \begin{array}{l} \sharp I = \varphi(k); \forall i \in I, j_i \ge j_{\varphi(k)} \\ \text{and } \forall i \in \{1, \dots, m\} \setminus I, j_i \le j_{\varphi(k)} \end{array} \right\} \quad [3]$$

and for $i, i' = 1, \dots, m$,

$$d_{i,i'} \overset{\text{def}}{=} (-1)^{j_{i'}-1} (\sqrt{-1} e^{\sqrt{-1}\theta_0})^{j_{i'}} (SX^{-1})_{i,i'}. \quad [4] \tag{35}$$

Particularly, if $k = \sum_{\ell' > \ell} m_{\ell'} n_{\ell'}$ for some $\ell = 1, \dots, s$ (or equivalently, $\varphi(k) = \sum_{\ell' \le \ell} m_{\ell'}$), we have that $\mathcal{I}_k = \left\{ \{1, \dots, \sum_{\ell'=1}^{\ell} m_{\ell'}\} \right\}$ and as $t \to 0$,

$$c_k(t) = (-1)^{N-k} t^{\varphi(k)} c_N(0) \det\left[(d_{i,i'})_{i,i'=1,\dots,\varphi(k)}\right] + o(t^{\varphi(k)}). \tag{36}$$

Remark 3.1. Lemma 3.1 is valid without (8). However, to get the exact order of asymptotics, we need to ensure that the determinant in (36) does not vanish, which follows from (8).

Proof of Lemma 3.1. Note that

$$p(\lambda, 0) = \det(\lambda \cdot \mathrm{Id} - \gamma(0)) = (\lambda - e^{\sqrt{-1}\theta_0})^N \prod_{\mu \ne e^{\sqrt{-1}\theta_0}} ((\lambda - e^{\sqrt{-1}\theta_0}) + (e^{\sqrt{-1}\theta_0} - \mu)),$$

where μ is an eigenvalue of $\gamma(0)$. Comparing this with the expansion of $p(\lambda, 0)$ at $e^{\sqrt{-1}\theta_0}$ in (28), we conclude that $c_k(0) = 0$ for $k = 0, \dots, N-1$ and $c_N(0)$ is given by (31). Next, we will estimate $c_k(t)$ for $k = 0, \dots, N-1$. We will expand $c_k(t)$ by using exterior powers of linear maps, identify and calculate the major terms. For simplicity of notation, we give the proof for $N = n$ and $e^{\sqrt{-1}\theta_0} \ne \pm 1$. The argument for the general case is quite similar. We briefly explain necessary modifications in Remark 3.2 and omit the details.

In this case, we see that

$$c_N(0) = (e^{\sqrt{-1}\theta_0} - e^{-\sqrt{-1}\theta_0})^n.$$

[3] To get the quantity on the right hand side of (34), we select the biggest $\varphi(k)$-many Jordan blocks. However, due to possible presence of Jordan blocks of equal size, such a selection is not unique and $\mathcal{I}_k$ is introduced to represent all such choices.

[4] By Corollary 2.4, X is invertible.

Recall the definitions and (22) in Subsection 2.2. Note that

$$c_k(t) = \sum_{\substack{k_1+k_2=2n-k,\\ k_1,k_2\geq 0}} (-1)^{k_2} t^{k_2} c_{k_1,k_2}(t), \tag{37}$$

where

$$c_{k_1,k_2}(t) = \bigwedge(2n, k_1, k_2, e^{\sqrt{-1}\theta_0}\cdot \mathrm{Id} - \gamma(0), \frac{1}{t}(\gamma(t)-\gamma(0))).$$

Note that $t \mapsto c_{k_1,k_2}(t)$ is continuous and

$$c_{k_1,k_2}(0) = \bigwedge(2n, k_1, k_2, e^{\sqrt{-1}\theta_0}\cdot \mathrm{Id} - \gamma(0), \frac{\mathrm{d}}{\mathrm{d}t}\gamma(0)).$$

To calculate $c_k(t)$ and $c_{k_1,k_2}(t)$, we need to fix a basis of $\mathbb{C}^{2n}$. Recall the notations given by (9) and (10). Then, $\sum_{i=1}^m j_i = N = n$. By taking complex conjugates, we see that $\{\bar{\xi}_{i,j}\}_{i,j}$ is a basis of the invariant space $E_{e^{-\sqrt{-1}\theta_0}}(\gamma(0))$ associated with the eigenvalue $e^{-\sqrt{-1}\theta_0}$ of the matrix $\gamma(0)$ with properties similar to (9). Moreover, by Lemma 2.2 and non-degeneracy of $(\cdot,\cdot)_G$, $\{\xi_{i,j}, \bar{\xi}_{i,j}\}_{i,j}$ is a basis of $\mathbb{C}^{2n}$.

Before we proceed with the expansion of $c_{k_1,k_2}(t)$, let us firstly fix several notations. We define $M_0 = \mathrm{Id}$, $M_1 = e^{\sqrt{-1}\theta_0}\cdot \mathrm{Id} - \gamma(0)$ and $M_2 = \frac{1}{t}(\gamma(t)-\gamma(0))$. Let $P = \{(i,j) : i = 1,\dots,m; j = 1,\dots,j_i\}$. Then, the generalized eigenvectors $\{\xi_{i,j}\}_{i,j}$ are indexed by P. We fix the lexicographic order on P so that P is totally ordered. In the definition of $c_{k_1,k_2}(t)$, for each vector ξ_p ($p \in P$), we apply to it some linear map selected from the three different linear maps M_0, M_1 and M_2, and then multiply the resulting vectors via wedge products. Let $\Omega = \{0,1,2\}^P$. Then, the choice of linear maps is represented by an element in Ω. For instance, for $\sigma = (\sigma_p)_{p\in P} \in \Omega$, for a vector ξ_p, we apply to it the map M_{σ_p}. For the vectors $\{\bar{\xi}_p\}_{p\in P}$, we use the similar notations $\tilde{\sigma}$. In the definition of $c_{k_1,k_2}(t)$, we don't sum over all possible assignment $\sigma, \tilde{\sigma} \in \Omega$. The requirement is that we use k_1 times the map M_1, k_2 times the map M_2 and $2n-k_1-k_2$ times the map M_0. To count the number of occurrence of a particular map M_i ($i = 0,1,2$), we introduce the following notation: for $\sigma \in \Omega$, a subset of indices $Q \subset P$ and $\alpha = 0,1,2$, we define

$$N_\alpha(\sigma, Q) = \sum_{p\in Q} 1_{\sigma_p = \alpha}.$$

For $q_1 + q_2 \leq n$, we define

$$\Omega_{q_1,q_2} \overset{\mathrm{def}}{=} \{\sigma \in \Omega : N_1(\sigma,P) = q_1, N_2(\sigma,P) = q_2\}.$$

Then, we express $c_{k_1,k_2}(t)$ as follows:

$$\sum_{\substack{q_1+\tilde{q}_1=k_1,\\ q_2+\tilde{q}_2=k_2}} \sum_{\substack{\sigma\in\Omega_{q_1,q_2},\\ \tilde{\sigma}\in\Omega_{\tilde{q}_1,\tilde{q}_2}}} \left(\wedge_{p\in P} M_{\sigma_p}\xi_p\right) \wedge \left(\wedge_{p\in P} M_{\tilde{\sigma}_p}\bar{\xi}_p\right) = c_{k_1,k_2}(t)\left(\wedge_{p\in P}\xi_p\right) \wedge \left(\wedge_{p\in P}\bar{\xi}_p\right).$$

At the first sight, the above expression may seem to be impractical as it evolves lots of terms. However, not all the terms in the above summation contribute to $c_{k_1,k_2}(t)$. For instance, if we apply

M_1 to an eigenvector associated with the eigenvalue $e^{\sqrt{-1}\theta_0}$ of $\gamma(0)$, then we immediately get a zero. The other possibility to get a zero contribution is due to the skew-symmetry of the wedge product. For instance, for an eigenvector v_1 and a generalized eigenvector v_2 such that $M_1v_2 = v_1$ and $M_1v_1 = 0$, we see that $M_1v_2 \wedge M_0v_1 = 0$. We will combine these two observations and give a necessary condition for non-trivial contributions. For $i = 1, \ldots, m$, we define $P_i = \{(i, 1), \ldots, (i, j_i)\}$ with the lexicographic order. The index set P_i corresponds to the generalized eigenvectors associated with the i-th Jordan block. Note that for $i = 1, \ldots, m$, we have that $\wedge_{p\in P_i} M_{\sigma_p}\xi_p = 0$ if $N_2(\sigma, P_i) = 0$ and $N_1(\sigma, P_i) \geq 1$. So, roughly speaking, in order that the term $\left(\wedge_{p\in P} M_{\sigma_p}\xi_p\right) \wedge \left(\wedge_{p\in P} M_{\tilde{\sigma}_p}\bar{\xi}_p\right)$ is not vanishing, the following condition is necessary: for the generalized eigenvectors corresponding to some Jordan block, if we don't apply M_2 to them, then we have to apply M_0 to all these vectors. In this sense, we need a certain minimal amount of M_0 available. To be more precise, if the number of M_2 available is strictly less than the total number m of the Jordan blocks associated with $e^{\sqrt{-1}\theta_0}$, then at least $m - N_2(\sigma, P)$ blocks are free of M_2 and we have to apply M_0 to all the corresponding generalized eigenvectors. The minimum of the total size of $m - N_2(\sigma, P)$ many Jordan blocks is $\sum_{i>N_2(\sigma,P)} j_i$. Hence, in order to get non-zero contribution, we need that $N_0(\sigma, P) \geq \sum_{i>N_2(\sigma,P)} j_i$. Noting that $N_2(\sigma, P) \leq k_2$ and $2n - k_1 - k_2 = N_0(\sigma, P) + N_0(\tilde{\sigma}, P) \geq N_0(\sigma, P)$, we need that $2n - k_1 - k_2 \geq \sum_{i>k_2} j_i$, which is equivalent to $k_2 \geq \varphi(2n - k_1 - k_2)$. Hence, for $k = 2n - k_1 - k_2$, we have that

$$c_{k_1,k_2}(t) = 0 \text{ if } k_2 < \varphi(k). \tag{38}$$

By (37) and (38), for $k = 0, \ldots, n-1$, as $t \to 0$,

$$c_k(t) = \sum_{k_2=\varphi(k)}^{2n-k} (-t)^{k_2} c_{2n-k-k_2,k_2}(t) = (-t)^{\varphi(k)} c_{2n-k-\varphi(k),\varphi(k)}(0) + o(t^{\varphi(k)}), \tag{39}$$

which is precisely Eq. (32).

Next, we will calculate $c_{2n-k-\varphi(k),\varphi(k)}(0)$ when $k = \sum_{i>\varphi(k)} j_i$. For simplicity of notation, let $K_0 = e^{\sqrt{-1}\theta_0} \cdot \mathrm{Id} - \gamma(0)$ and $\Delta_0 = \frac{\mathrm{d}\gamma}{\mathrm{d}t}(0)$. (We decide to abandon the use of notations M_0, M_1 and M_2 since we would like to emphasize the difference between K_0 and Δ_0.) We have that

$$c_{2n-k-\varphi(k),\varphi(k)}(0) = \bigwedge(2n, 2n - k - \varphi(k), \varphi(k), K_0, \Delta_0),$$

which can be expanded as before. From previous discussion above (38), to get non-zero contributions, there aren't many choices for the assignments of the maps Id, K_0 and Δ_0: for the vectors $\bar{\xi}_{i,j}$, we apply K_0 to them; for the generalized eigenvectors of the biggest $\varphi(k)$ Jordan blocks associated with $e^{\sqrt{-1}\theta_0}$, we apply Δ_0 to each eigenvector and K_0 to the remainder so that we use only one Δ_0 for each big Jordan blocks; for the generalized eigenvectors of the remainder small Jordan blocks associated with $e^{\sqrt{-1}\theta_0}$, we apply the map Id to them. Accordingly, we have that

$$\sum_{I\in\mathcal{I}_k} \left(\wedge_{i=1}^m \omega_{i,I}\right) \wedge \left(\wedge_{p\in P} K_0\bar{\xi}_p\right) = c_{2n-k-\varphi(k),\varphi(k)}(0) \left(\wedge_{p\in P}\xi_p\right) \wedge \left(\wedge_{p\in P}\bar{\xi}_p\right), \tag{40}$$

where $\mathcal{I}_k$ represents different choices of the biggest $\varphi(k)$ many Jordan blocks and

$$\omega_{i,I} = 1_{i\in I} \cdot \Delta_0 \xi_{i,1} \wedge (\wedge_{j=2}^{j_i} K_0 \xi_{i,j}) + 1_{i\notin I} \cdot \wedge_{j=1}^{j_i} \xi_{i,j}.$$

By (9), for $i = 1, \ldots, m$ and $j = 1, \ldots, j_i$, we have that

$$K_0 \xi_{i,j} = \xi_{i,j-1} \text{ and } K_0 \bar{\xi}_{i,j} = (e^{\sqrt{-1}\theta_0} - e^{-\sqrt{-1}\theta_0}) \bar{\xi}_{i,j} + \bar{\xi}_{i,j-1}$$

where $\xi_{i,0} = 0$. Hence, we have that

$$\wedge_{p\in P} K_0 \bar{\xi}_p = (e^{\sqrt{-1}\theta_0} - e^{-\sqrt{-1}\theta_0})^n \cdot \wedge_{p\in P} \bar{\xi}_p = c_N(0) \cdot \wedge_{p\in P} \bar{\xi}_p \tag{41}$$

and that

$$\omega_{i,I} = 1_{i\in I} \cdot \Delta_0 \xi_{i,1} \wedge (\wedge_{j=1}^{j_i-1} \xi_{i,j}) + 1_{i\notin I} \cdot \wedge_{j=1}^{j_i} \xi_{i,j}.$$

The vector $\Delta_0 \xi_{i,1}$ can be uniquely expressed as a linear combination of the basis $(\xi_{i,j}, \bar{\xi}_{i,j})_{i,j}$. We denote by $\tilde{d}_{i,i'}$ the coefficient of $\Delta_0 \xi_{i,1}$ before $\xi_{i',j_{i'}}$. Denote by $\mathcal{S}_I$ all permutations of the set $I \subset \{1, \ldots, m\}$ and by $\mathrm{Sgn}(g)$ the signature of a permutation g. Then, we have that

$$\begin{aligned}
\wedge_{i=1}^m w_{i,I} =& \sum_{g\in\mathcal{S}_I} \wedge_{i=1}^m (1_{i\in I} \cdot (-1)^{j_i-1} \cdot \tilde{d}_{i,g(i)} \cdot (\wedge_{j=1}^{j_i-1} \xi_{i,j}) \wedge \xi_{g(i),j_{g(i)}} + 1_{i\notin I} \cdot \wedge_{j=1}^{j_i} \xi_{i,j}) \pmod{\wedge_{p\in P} \bar{\xi}_p}\\
=&(-1)^{\sum_{i\in I}(j_i-1)} \sum_{g\in\mathcal{S}_I} (-1)^{\mathrm{Sgn}(g)} \prod_{i\in I} \tilde{d}_{i,g(i)} \cdot \wedge_{i=1}^m (\wedge_{j=1}^{j_i} \xi_{i,j}) \pmod{\wedge_{p\in P} \bar{\xi}_p}\\
=&(-1)^{\sum_{i\in I}(j_i-1)} \cdot \det(\tilde{d}_{i,i'})_{i,i'\in I} \cdot \wedge_{i=1}^m (\wedge_{j=1}^{j_i} \xi_{i,j}) \pmod{\wedge_{p\in P} \bar{\xi}_p}
\end{aligned}$$

By definition of $\mathcal{I}_k$, for $k = \sum_{i>\varphi(k)} j_i$ and $I \in \mathcal{I}_k$, we have that $\sharp I = \varphi(k)$ and $\sum_{i\in I}(j_i - 1) = N - k - \varphi(k)$. Hence, we obtain that

$$\wedge_{i=1}^m w_{i,I} = (-1)^{N-k-\varphi(k)} \cdot \det(\tilde{d}_{i,i'})_{i,i'\in I} \cdot \wedge_{i=1}^m (\wedge_{j=1}^{j_i} \xi_{i,j}) \pmod{\wedge_{p\in P} \bar{\xi}_p}. \tag{42}$$

Next, we will show that $\tilde{d}_{i,i'}$ equals $d_{i,i'}$ defined by (35). On one hand, since $\Delta_0 = J_{2n} A(0) \gamma(0)$, $J_{2n}^* J_{2n} = \mathrm{Id}_{2n}$ and $\gamma(0)\xi_{i,1} = e^{\sqrt{-1}\theta_0} \xi_{i,1}$, we have that

$$\langle \Delta_0 \xi_{i,1}, J_{2n} \xi_{i',1} \rangle = \langle J_{2n} A(0) \gamma(0) \xi_{i,1}, J_{2n} \xi_{i',1} \rangle = e^{\sqrt{-1}\theta_0} \langle A(0) \xi_{i,1}, \xi_{i',1} \rangle = e^{\sqrt{-1}\theta_0} S_{i,i'}. \tag{43}$$

On the other hand, by Lemmas 2.2 and 2.3, we see that $\langle \bar{\xi}_{i,j}, J_{2n} \xi_{i',1} \rangle = 0$ for all $i, i' = 1, \ldots, m$ and $j = 1, \ldots, j_i$, and that $\langle \xi_{i,j}, J_{2n} \xi_{i',1} \rangle = 0$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, j_i - 1$. Hence, together with the definition of the matrix X given by (14), we get that

$$\langle \Delta_0 \xi_{i,1}, J_{2n} \xi_{i',1} \rangle = \sum_{i''=1}^m \tilde{d}_{i,i''} \langle \xi_{i'',j_{i''}}, J_{2n} \xi_{i',1} \rangle = e^{\sqrt{-1}\theta_0} \sum_{i''=1}^m (-1)^{1-j_{i''}} (\sqrt{-1} e^{\sqrt{-1}\theta_0})^{-j_{i''}} \tilde{d}_{i,i''} X_{i'',i'}. \tag{44}$$

Combining (43) and (44), we see that the expression of $\tilde{d}$ is given by (35).

Together with (40), (41) and (42), we get that

$$c_{2n-k-\varphi(k),\varphi(k)}(0) = c_N(0)(-1)^{N-k-\varphi(k)} \sum_{I\in\mathcal{I}_k} \det\left[(d_{i,i'})_{i,i'\in I}\right], \tag{45}$$

where d is given by (35). Then, (34) follows from (39) and (45). Particularly, when $k = \sum_{\ell'>\ell} m_{\ell'} n_{\ell'}$ for some $\ell = 1, \ldots, s$, we have that $\mathcal{I}_k = \{\{1, \ldots, \varphi(k)\}\}$ and (36) follows. $\square$

The above proof is written for the case $N = n$. We briefly explain the modifications for $N \neq n$ in the following remark.

Remark 3.2. Instead of the eigenvectors $\{\bar{\xi}_{i,j}\}_{i,j}$, for each eigenvalue $\mu \neq e^{\sqrt{-1}\theta_0}$ with algebraic multiplicity $N(\mu)$, we take generalized eigenvectors $\{\xi_k^{(\mu)}\}_{k=1,\dots,N(\mu)}$ as $\{\xi_{i,j}\}_{i,j}$ for the eigenvalue $e^{\sqrt{-1}\theta_0}$. Then, instead of (41), we have that

$$K_0\xi_1^{(\mu)} \wedge \cdots \wedge K_0\xi_{N(\mu)}^{(\mu)} = (e^{\sqrt{-1}\theta_0} - \mu)^{N(\mu)}\xi_1^{(\mu)} \wedge \cdots \wedge \xi_{N(\mu)}^{(\mu)}.$$

Instead of $\langle \bar{\xi}_{i,j}, J_{2n}\xi_{i',1}\rangle = 0$, we use the G-orthogonality of the invariant spaces E_μ and $E_{e^{\sqrt{-1}\theta_0}}$ for $\mu \neq e^{\sqrt{-1}\theta_0}$.

3.2 Proof of Theorem 1.3 a) from Lemma 3.1

Recall the notations introduced in (9), (10), (11), (13), (14) and (15). As t varies from 0, the continuous branching of the eigenvalue $e^{\sqrt{-1}\theta_0}$ follows from the continuity of $t \mapsto p(\lambda, t) = \det(\lambda \cdot \mathrm{Id} - \gamma(t))$ and Lemma 2.1.

Next, note that S is Hermitian and strictly positive definite, $X^{(\ell,\ell)}$ is Hermitian (see Corollary 2.4). Hence, the roots of the polynomial (17) are non-zero real numbers.

We prove the asymptotic of eigenvalues when $t > 0$. The proof for $t < 0$ is similar.

By Lemma A.2, without loss of generality, we assume that the eigenvalues of $\gamma(t)$ are $e^{\sqrt{-1}\theta_0}$ and $e^{-\sqrt{-1}\theta_0}$. There are two possibilities: $e^{\sqrt{-1}\theta_0} \in U \setminus \mathbb{R}$ or $e^{\sqrt{-1}\theta_0} = \pm 1$. Again, the proofs in both cases are quite similar and we only present the proof for the first case, which appears to be a bit more complicated. In this case, $p(\lambda, 0) = (\lambda - e^{\sqrt{-1}\theta_0})^n(\lambda - e^{-\sqrt{-1}\theta_0})^n$.

Suppose that $\lambda(t) \in \mathbb{C}$ is a root of the polynomial $p(\lambda, t)$. For $\ell = 1, \dots, s$ and $t > 0$, we consider

$$w_\ell(t) \stackrel{\text{def}}{=} t^{-\frac{1}{n_\ell}}(\lambda(t) - e^{\sqrt{-1}\theta_0}). \tag{46}$$

By (28), it is a root of the polynomial $\sum_{k=0}^{2n} c_k(t)t^{\frac{k}{n_\ell}}w^k$ in w. Since the polynomial p has $2n$ roots, there are $2n$ continuous curves $t \mapsto w(t)$ for $t \neq 0$. We will show that there are exactly $n_\ell m_\ell$ many curves with non-zero limits as t tends to 0, there are exactly $\sum_{\ell<\ell'\leq s} m_{\ell'}n_{\ell'}$ many curves with the limit 0 as t tends to 0, and the remainder tends to ∞ as t tends to 0. So, there are exactly n curves $t \mapsto \lambda(t)$ of eigenvalues of $\gamma(t)$ tending to $e^{-\sqrt{-1}\theta_0}$ and the remainder tends to $e^{\sqrt{-1}\theta_0}$ with possibly different speeds. Roughly speaking, each Jordan block associated with $e^{\sqrt{-1}\theta_0}$ of the size n_ℓ corresponds to n_ℓ many curves of eigenvalues, these curves are exactly Hölder-$\frac{1}{n_\ell}$ continuous at $t = 0$ and they form an n_ℓ-star at $e^{\sqrt{-1}\theta_0}$.

Our task is to find the limit of (46) by applying Lemma 2.1. Although $w(t)$ is a root of the polynomial $\sum_{k=0}^{2n} c_k(t)t^{\frac{k}{n_\ell}}w^k$, we cannot apply Lemma 2.1 directly to that polynomial since it has a trivial limit 0 as $t \to 0$. Instead, we will divide that polynomial by certain fractal powers $t^{\tau(\ell)/n_\ell}$ of t, which is "the biggest common factor" of $\{c_k(t)t^{\frac{k}{n_\ell}}\}_k$, and obtain a new polynomial $q(w, t)$ with the same roots and a non-trivial limit as $t \to 0$. To get the exponent $\tau(\ell)/n_\ell$, we will use the asymptotics of $t \mapsto c_k(t)$ summarized in Lemma 3.1. By Lemma 3.1, for $k = 0, \dots, n$, if $k = \sum_{\ell' > \ell} m_{\ell'} n_{\ell'} + u n_\ell$ for some $u = 0, 1, \dots, m_\ell$, then $\varphi(k)$ defined in (33) equals $\sum_{\ell' \leq \ell} m_{\ell'} - u$ and

$$c_k(t) t^{\frac{k}{n_\ell}} = t^{\frac{\tau(\ell)}{n_\ell}} (-1)^{\sum_{\ell' \leq \ell} m_{\ell'} n_{\ell'} - u n_\ell} (e^{\sqrt{-1}\theta_0} - e^{-\sqrt{-1}\theta_0})^n \sum_{I \in \mathcal{I}_{\ell,u}} \det(d_{i,i'})_{i,i' \in I} + o(t^{\tau(\ell)/n_\ell}),$$

where $\tau(\ell) \overset{\text{def}}{=} \sum_{\ell'=1}^{s} m_{\ell'} \min(n_{\ell'}, n_\ell)$ and

$$\mathcal{I}_{\ell,u} \overset{\text{def}}{=} \left\{ I \subset \{1, 2, \dots, \sum_{\ell' \leq \ell} m_{\ell'}\} : \sharp I = \sum_{\ell' \leq \ell} m_{\ell'} - u, \{1, 2, \dots, \sum_{\ell' < \ell} m_{\ell'}\} \subset I \right\}.$$

Otherwise, for $k \notin \{\sum_{\ell' > \ell} m_{\ell'} n_{\ell'}, \sum_{\ell' > \ell} m_{\ell'} n_{\ell'} + n_\ell, \dots, \sum_{\ell' > \ell} m_{\ell'} n_{\ell'} + m_\ell n_\ell\}$,

$$c_k(t) t^{\frac{k}{n_\ell}} = o(t^{\tau(\ell)/n_\ell}) \text{ as } t \to 0.$$

Hence, we define

$$q(w,t) = \sum_{k=0}^{2n} c_k(t) t^{\frac{k - \tau(\ell)}{n_\ell}} w^k. \tag{47}$$

Note that the limiting polynomial $q(w,0) \overset{\text{def}}{=} \lim_{t \to 0} q(w,t)$ exists and

$$q(w,0) = (-1)^{\sum_{\ell' \leq \ell} m_{\ell'} n_{\ell'}} (e^{\sqrt{-1}\theta_0} - e^{-\sqrt{-1}\theta_0})^n w^{\sum_{\ell' > \ell} m_{\ell'} n_{\ell'}} \sum_{u=0}^{m_\ell} (-w)^{u n_\ell} \sum_{I \in \mathcal{I}_{\ell,u}} \det(d_{i,i'})_{i,i' \in I}. \tag{48}$$

We write d in block matrix as S and X in (15), i.e., $d = \begin{bmatrix} d^{(1,1)} & \cdots & d^{(1,s)} \\ \vdots & \ddots & \vdots \\ d^{(s,1)} & \cdots & d^{(s,s)} \end{bmatrix}$. (For $1 \leq \ell_1, \ell_2 \leq s$, we note that $d^{(\ell_1,\ell_2)}$ is an $m_{\ell_1} \times m_{\ell_2}$-matrix.) For $I \in \mathcal{I}_{\ell,u}$, $(d_{i,i'})_{i,i' \in I}$ is the square matrix obtained by deleting u elements on the diagonal of $d^{(\ell,\ell)}$ together with the rows and columns containing them from the matrix $\begin{bmatrix} d^{(1,1)} & \cdots & d^{(1,\ell)} \\ \vdots & \ddots & \vdots \\ d^{(\ell,1)} & \cdots & d^{(\ell,\ell)} \end{bmatrix}$. When we sum over $\mathcal{I}_{\ell,u}$ in (48), we sum over all such choices of principle minors. Hence, we see that

$$q(w,0) = (-1)^{\sum_{\ell' \leq \ell} m_{\ell'} n_{\ell'}} (e^{\sqrt{-1}\theta_0} - e^{-\sqrt{-1}\theta_0})^n w^{\sum_{\ell' > \ell} m_{\ell'} n_{\ell'}} Q_\ell(w), \tag{49}$$

where

$$Q_\ell(w) = \det \begin{bmatrix} d^{(1,1)} & \cdots & d^{(1,\ell-1)} & d^{(1,\ell)} \\ \vdots & \ddots & \vdots & \vdots \\ d^{(\ell-1,1)} & \cdots & d^{(\ell-1,\ell-1)} & d^{(\ell-1,\ell)} \\ d^{(\ell,1)} & \cdots & d^{(\ell,\ell-1)} & d^{(\ell,\ell)} + (-w)^{n_\ell} \cdot \mathrm{Id}_{m_\ell} \end{bmatrix}. \tag{50}$$

By expanding the determinant $Q_\ell(w)$ in polynomials of w, we find that (48) and (49) coincide. Similarly to the calculation from (29) to (30), by the relation (35) between the matrices d, S and X

and the fact that X is upper triangular in the block sense (Corollary 2.4), we get that $Q_\ell(w) = 0$ iff w is the root of the polynomial

$$\tilde{Q}_\ell(w) \stackrel{\text{def}}{=} \det \begin{bmatrix} S^{(1,1)} & \dots & S^{(1,\ell-1)} & S^{(1,\ell)} \\ \vdots & \ddots & \vdots & \vdots \\ S^{(\ell-1,1)} & \dots & S^{(\ell-1,\ell-1)} & S^{(\ell-1,\ell)} \\ S^{(\ell,1)} & \dots & S^{(\ell,\ell-1)} & S^{(\ell,\ell)} - w^{n_\ell}(\sqrt{-1}e^{\sqrt{-1}\theta_0})^{-n_\ell} X^{(\ell,\ell)} \end{bmatrix}. \tag{51}$$

Hence, there are $m_\ell n_\ell$ many roots $\{\omega_{\ell,p,q}\}_{p=1,\dots,m_\ell;q=1,\dots,n_\ell}$ such that for fixed integers ℓ and p, $\left\{\frac{\omega_{\ell,p,q}}{\sqrt{-1}e^{\sqrt{-1}\theta_0}}\right\}_{q=1,\dots,n_\ell}$ are the n_ℓ-th roots of $a_{\ell,p}$ with multiplicities. (Recall that $a_{\ell,p}$ are the roots of (17).) By Lemma 2.1, there are corresponding $w_{\ell,p,q}(t)$ and $\lambda_{\ell,p,q}(t) = e^{\sqrt{-1}\theta_0} + t^{\frac{1}{n_\ell}} w_{\ell,p,q}(t)$ for $p = 1, \dots, m_\ell$ and $q = 1, \dots, n_\ell$ such that $w_{\ell,p,q}(0) = \lim_{t\to 0} w_{\ell,p,q}(t)$ exists and $(w_{\ell,p,q}(0))_{p=1,\dots,m_\ell;q=1,\dots,n_\ell}$ are roots of $\tilde{Q}_\ell(w)$. Or equivalently, (16) holds.

Remark 3.3. During the proof of Theorem 1.3 a), the only purpose of assuming (8) is to ensure that $\tilde{Q}_\ell(w)$ has non-zero roots. Hence, Theorem 1.3 a) still holds under the following weaker condition:

$$\det \begin{bmatrix} S^{(1,1)} & \dots & S^{(1,\ell)} \\ \vdots & \ddots & \vdots \\ S^{(\ell,1)} & \dots & S^{(\ell,\ell)} \end{bmatrix} \neq 0 \text{ for all } \ell = 1, \dots, s. \tag{52}$$

Or equivalently in the following coordinate-free form: the bilinear form $\langle A(0)\cdot, \cdot\rangle$ is non-degenerate on the spaces V_ℓ for all integer ℓ, where

$$V_\ell = \ker(e^{\sqrt{-1}\theta_0} \cdot \mathrm{Id} - \gamma(t)) \cap (e^{\sqrt{-1}\theta_0} \cdot \mathrm{Id} - \gamma(t))^\ell \left(\ker(e^{\sqrt{-1}\theta_0} \cdot \mathrm{Id} - \gamma(t))^{2n}\right).$$

4 Proof of Theorem 1.3 b)

Our proof strategy is to approximate the continuous curve $t \mapsto A(t)$ by analytic curves. To prove Theorem 1.3 b), we use Theorem 1.3 a) proved in Section 3 and Theorem 1.3 for the analytic case. We present a sketch of Theorem 1.3 when $t \mapsto A(t)$ is real analytic in Subsection A.3.

We choose to present the proof for n_ℓ odd, $t > 0$ and $a_{\ell,p} > 0$. The proofs for other cases are similar and we left them to the reader. By Theorem 1.3 a), we see that $(\lambda_{\ell,p,q}(t))_{q=2,\dots,n_\ell}$ are outside of U for sufficiently small t. It remains to prove that $\lambda_{\ell,p,1}(t)$ is a Krein positive definite eigenvalue on U. By Theorem 1.3 a), we have that

$$\lim_{t\downarrow 0} \frac{\lambda_{\ell,p,1}(t) - e^{\sqrt{-1}\theta_0}}{\sqrt{-1}e^{\sqrt{-1}\theta_0} t^{\frac{1}{n_\ell}}} > 0.$$

Hence, as t increases from 0, tangent to the circle and counter-clockwise, $\lambda_{\ell,p,1}(t)$ continuously branches from $e^{\sqrt{-1}\theta_0}$. We need to show that $\lambda_{\ell,p,1}(t) \in U$ for sufficiently small t.

We define

$$I_+ = \left\{(\ell, p, q) : \lim_{t\downarrow 0} \frac{\lambda_{\ell,p,q}(t) - e^{\sqrt{-1}\theta_0}}{\sqrt{-1}e^{\sqrt{-1}\theta_0} t^{\frac{1}{n_\ell}}} \in (0, +\infty)\right\},$$

$$I_- = \left\{ (\ell, p, q) : \lim_{t \downarrow 0} \frac{\lambda_{\ell,p,q}(t) - e^{\sqrt{-1}\theta_0}}{\sqrt{-1} e^{\sqrt{-1}\theta_0} t^{\frac{1}{n_\ell}}} \in (-\infty, 0) \right\},$$

$$J_+(t) = \{(\ell, p, q) : \lambda_{\ell,p,q}(t) \text{ is a Krein positive definite eigenvalue on } U\},$$

$$J_-(t) = \{(\ell, p, q) : \lambda_{\ell,p,q}(t) \text{ is a Krein negative definite eigenvalue on } U\},$$

$$K_+(t) = \left\{ (\ell, p, q) : \lambda_{\ell,p,q}(t) \in U \setminus \{e^{\sqrt{-1}\theta_0}\} \text{ and it is on the counter-clockwise side of } e^{\sqrt{-1}\theta_0} \right\},$$

$$K_-(t) = \left\{ (\ell, p, q) : \lambda_{\ell,p,q}(t) \in U \setminus \{e^{\sqrt{-1}\theta_0}\} \text{ and it is on the clockwise side of } e^{\sqrt{-1}\theta_0} \right\}.$$ [5]

We will show that

$$\lim_{t \downarrow 0} J_+(t) = \lim_{t \downarrow 0} K_+(t) = I_+ \text{ and } \lim_{t \downarrow 0} J_-(t) = \lim_{t \downarrow 0} K_-(t) = I_-. \tag{53}$$

The continuity of $t \mapsto \det(\lambda \cdot \mathrm{Id} - \gamma(t))$ implies the continuity of the eigenvalues as t varies. Also, by the first order asymptotics in Theorem 1.3 a), we see that $e^{\sqrt{-1}\theta_0}$ is no longer an eigenvalue of $\gamma(t)$ if t varies from 0 a bit. Hence, there exist $r > 0$ and $\delta > 0$ such that for $t \in (0, \delta]$, $(\lambda_{\ell,p,q}(t))_{\ell=1,\dots,s;p=1,\dots,m_\ell;q=1,\dots,n_\ell}$ are located in the punctured open disk $B(e^{\sqrt{-1}\theta_0}, r) \setminus \{e^{\sqrt{-1}\theta_0}\}$ centered at $e^{\sqrt{-1}\theta_0}$ with the radius $r < 0.1$, and the other eigenvalues of $\gamma(t)$ stay outside of $B(e^{\sqrt{-1}\theta_0}, r)$. Shrinking δ if necessary, for $t \in (0, \delta]$, for $(\ell, p, q) \in I_+$ (resp. $(\ell, p, q) \in I_-$), $\lambda_{\ell,p,q}(t)$ stays on the counter-clockwise side (resp. clockwise side) of $e^{\sqrt{-1}\theta_0}$, and for $(\ell, p, q) \notin I_- \cup I_+$, $\lambda_{\ell,p,q}(t) \notin U$. Hence, $K_+(t) \subset I_+$ and $K_-(t) \subset I_-$ for $t \in (0, \delta]$.

Next, we prove that $\lim_{t \downarrow 0} \sharp K_+(t) \geq \sharp I_+$ and $\lim_{t \downarrow 0} \sharp K_-(t) \geq \sharp I_-$. For that purpose, we approximate the continuous curve $t \mapsto A(t)$ by analytic curves for $t \in [-1, 1]$ by using Bernstein polynomials. For positive integers M, we define

$$A^{(M)}(t) = \sum_{k=-M}^{M} A\left(\frac{k}{M}\right) \binom{2M}{M+k} \left(\frac{1-t}{2}\right)^{M-k} \left(\frac{1+t}{2}\right)^{M+k}.$$

As a polynomial in t, the function $t \mapsto A^{(M)}(t)$ is analytic. By classical results on Bernstein polynomials, for continuous $t \mapsto A(t)$, $A^{(M)}(t)$ converges to $A(t)$ as $M \to \infty$ uniformly for $t \in [-1, 1]$. Hence, the corresponding solution $\gamma^{(M)}(t)$ of (1) (with the same initial condition) also converges to $\gamma(t)$, uniformly for $t \in [-1, 1]$.

We wish to use Krein-Lyubarskii theorem for approximated analytic systems, see Subsection A.3 for a proof in analytic case. For that purpose, we need to verify the condition (8) for large enough M. By taking a subsequence, we may assume that (8) holds for each M and $t \in [-1, 1]$. Otherwise, if (8) is violated for infinitely many M, then there exist sequences $\{M_n\}_n$, $\{t_n\}_n$, $\{\xi_n\}_n$ and $\{\lambda_n\}_n$ such that $\lim_{n \to +\infty} M_n = +\infty$, $\{t_n\}_n$ is bounded and for all n, $\lambda_n \in U$, $||\xi_n||_2 = 1$, $\gamma^{(M_n)}(t_n)\xi_n = \lambda_n \xi_n$ and $\langle A^{(M_n)}(t_n)\xi_n, \xi_n \rangle = 0$. By compactness, taking subsequence if necessary, we may further assume that $\lim_{n \to +\infty} t_n = t$, $\lim_{n \to +\infty} \xi_n = \xi$ and $\lim_{n \to +\infty} \lambda_n = \lambda$. Then, by taking the limit, we see that

[5] When t is sufficiently close to 0, $\lambda_{\ell,p,q}(t)$ locates near $e^{\sqrt{-1}\theta_0}$. Thus, it makes sense to use the notions "counter-clockwise side" and "clockwise side".

$||\xi||_2 = 1$, $\lambda \in U$, $\gamma(t)\xi = \lambda\xi$ and $\langle A(t)\xi, \xi\rangle = 0$, which contradicts with the assumption (8) on the continuous curve $t \mapsto \gamma(t)$.

In the following, we assume that (8) holds for each M.

For approximated systems, we analogously define the notations $\{\lambda^{(M)}_{\ell,p,q}(t)\}_{\ell=1,\ldots,s;p=1,\ldots,m_\ell;q=1,\ldots,n_\ell}$, $I^{(M)}_+$, $I^{(M)}_-$, $J^{(M)}_+(t)$, $J^{(M)}_-(t)$, $K^{(M)}_+(t)$, $K^{(M)}_-(t)$ and $D^{(M)}$ (see (7)). In the following, we take M large enough such that $(\lambda^{(M)}_{\ell,p,q}(t))_{\ell=1,\ldots,s;p=1,\ldots,m_\ell;q=1,\ldots,n_\ell}$ are located in $B(e^{\sqrt{-1}\theta_0}, r)$ for $t \in (0, \delta]$. For $t \notin D^{(M)}$, we define an index

$$\nu^{(M)}_+(t) = \sharp(K^{(M)}_+(t) \cap J^{(M)}_+(t)) - \sharp(K^{(M)}_+(t) \cap J^{(M)}_-(t)).$$

Since $(D^{(M)})^c$ is dense, for $t \in D^{(M)}$, we may define $\nu^{(M)}_+(t) = \limsup_{s\uparrow t, s\notin D^{(M)}} \nu^{(M)}_+(s)$. Direct approximation argument relying on the convergence $\lim_{M\to\infty} \gamma^{(M)} = \gamma$ is not sufficient to conclude the desired result. Instead, we will crucially use the following feature of $\nu^{(M)}_+(t)$ in the argument.

Claim 1. *For large enough M, as t increases from 0 to δ, the index $\nu^{(M)}_+(t)$ is non-decreasing and integer-valued.*

We focus on the application of Claim 1 and postpone its proof in the end of this section.

Since $\lim_{M\to\infty} K^{(M)}_+(t) = K_+(t)$, to prove $\lim_{t\downarrow 0} \sharp K_+(t) \geq \sharp I_+$, it suffices to show that for M large enough, for all $t \in (0, \delta)$, $\sharp K^{(M)}_+(t) \geq I_+$. By upper semi-continuity[6] of $t \mapsto \sharp K^{(M)}_+(t)$, it suffices to show the inequality for t in a dense set of $(0, \delta)$, say $(0, \delta) \setminus D^{(M)}$. By definition of $\nu^{(M)}_+(t)$, $\sharp K^{(M)}_+(t) \geq \nu^{(M)}_+(t)$ for $t \in (0, \delta) \setminus D^{(M)}$. Hence, it is enough to show that $\inf\{\nu^{(M)}_+(t) : t \in (0, \delta)\} \geq \sharp I_+$. By Claim 1, we see that $\inf\{\nu^{(M)}_+(t) : t \in (0, \delta)\}$ equals the right limit $\nu^{(M)}_+(0+)$ of $\nu^{(M)}_+$ at 0. Hence, it suffices to show that $\nu^{(M)}_+(0+) \geq \sharp I_+$. Note that $\lim_{t\downarrow 0} \sharp(K^{(M)}_+(t) \cap J^{(M)}_+(t)) = \sharp I^{(M)}_+$ and $\lim_{t\downarrow 0} \sharp(K^{(M)}_+(t) \cap J^{(M)}_-(t)) = 0$ by Theorem 1.3 in the analytic case. Moreover, by Remark 1.2, since $\gamma^{(M)}(0) = \gamma(0)$ by construction, we have that $\lim_{M\to\infty} \sharp I^{(M)}_+ = \sharp I_+$. Hence, $\nu^{(M)}_+(0+)$ precisely equals $\sharp I_+$ for M large enough. Therefore, for M large enough, we have that

$$\sharp K_+(t) \geq \sharp I_+ \text{ for } t \in (0, \delta). \tag{54}$$

and similarly, we see that $\sharp K_-(t) \geq \sharp I_-$.

Hence, together with the inclusion $K_+(t) \subset I_+$ and $K_-(t) \subset I_-$ for small enough $t > 0$, we get that $K_+(t) = I_+$ and $K_-(t) = I_-$. From the argument for (54), for $t \in (0, \delta)$ with δ small enough, $\nu^{(M)}_+(t) = \sharp I_+$ as long as M is large enough such that $\sharp K^{(M)}_+(t) = \sharp K_+(t)$.

To finish the proof of (53), consider the invariant space $W_+(t)$ (resp. $W_-(t)$) spanned by the invariant spaces associated with the eigenvalues indexed by $K_+(t)$ (resp. $K_-(t)$), i.e., $W_+(t) \overset{\text{def}}{=} \sum_{(\ell,p,q)\in K_+(t)} E_{\lambda_{\ell,p,q}(t)}$ (resp. $W_-(t) \overset{\text{def}}{=} \sum_{(\ell,p,q)\in K_-(t)} E_{\lambda_{\ell,p,q}(t)}$). We use similar notations $W^{(M)}_+(t)$ and $W^{(M)}_-(t)$ for the approximated systems. By Lemma 2.2, the Krein form $(\cdot,\cdot)_G$ is non-degenerate on these spaces. It suffices to show that the negative index of $(\cdot,\cdot)_G|_{W_+(t)}$ is zero and the positive

[6]Note that $\sharp K^{(M)}_+(t)$ counts the multiplicity.

index of $(\cdot,\cdot)_G|_{W_-(t)}$ is zero for small enough $t > 0$. Again, we will use the same approximated systems, analyze the analytical systems and pass to the limit in the end. The non-degeneracy of the Krein forms is an important sufficient condition for the continuity of indices.

In the following, we will give the proof for $W_+(t)$. The other part is similar and is left to the reader. Note that there exists small enough $\delta > 0$ such that $K_+^{(M)}(t) = I_+$ for M large enough and $t \in (0,\delta]$, $K_+(t) = I_+$ for $t \in (0,\delta]$ and $t \mapsto W_+(t)$ is continuous[7] for $t \in (0,\delta]$. By non-degeneracy of the Krein form on $W_+(t)$, the positive and negative indices are invariant for $t \in (0,\delta]$. Note that $\cup_{M\in\mathbb{N}} D^{(M)}$ is countable. Hence, by decreasing δ if necessary, we assume that $\delta \notin \cup_{M\in\mathbb{N}} D^{(M)}$. We will show that the Krein form is strictly positive definite on $W_+(\delta)$. Note that $\lim_{M\to\infty} K_+^{(M)}(\delta) = I_+ = K_+(\delta)$ and hence, $\lim_{M\to\infty} W_+^{(M)}(\delta) = W_+(\delta)$ (in certain Grassmannian). Therefore, as $M \to \infty$, the positive and negative indices of the restriction of the Krein form $(\cdot,\cdot)_G$ on $W_+^{(M)}(\delta)$ converge to those of $W_+(\delta)$. As $\delta \notin D^{(M)}$, the positive index of $(\cdot,\cdot)_G|_{W_+^{(M)}(\delta)}$ is precisely $\sharp(K_+^{(M)}(\delta) \cap J_+^{(M)}(\delta))$, which is not less than $\nu_+^{(M)}(\delta)$ by definition. Recall that $\nu_+^{(M)}(t)$ is non-decreasing and $\lim_{t\downarrow 0} \nu_+^{(M)}(t) = \sharp I_+^{(M)}$. Hence, the positive index of $(\cdot,\cdot)_G|_{W_+^{(M)}(\delta)}$ is at least $\sharp I_+^{(M)}$. On the other hand, $\dim W_+^{(M)}(\delta) = \sharp K_+^{(M)}(\delta) \le \sharp I_+^{(M)}$. Hence, the positive and negative index of $W_+^{(M)}(\delta)$ are respectively $\sharp I_+^{(M)}$ and 0. Also, recall that $\lim_{M\to\infty} \sharp I_+^{(M)} = \sharp I_+$. Therefore, for M sufficient large, the positive and negative index of $W_+^{(M)}(\delta)$ are respectively $\sharp I_+$ and 0. Hence, by taking $M \to \infty$, the Krein form $(\cdot,\cdot)_G$ must be strictly positive definite on $W_+(t)$ for $t \in (0,\delta]$.

We finish this section by verifying Claim 1.

Proof of Claim 1. Note that $\nu_+^{(M)}(t)$ is integer-valued by definition. It remains to prove its monotonicity, which follows from Theorem 1.3 for the analytic case.

Firstly, let us recall the definition of the index of an eigenvalue on U (cf. [Eke90, Section 1.3]). For $t_0 \in \mathbb{R}$ and an eigenvalue $\lambda \in U$ of $\gamma^{(M)}(t_0)$, we will define an index $\mathrm{ind}^{(M)}(\lambda, t_0)$ as in [Eke90, Section 1.3]. As t varies from t_0, the eigenvalue λ branches into N eigenvalues. (For instance, when no bifurcation occurs, we have that $N = 1$.) Among these eigenvalues we denote by p_t the number of Krein positive definite eigenvalues and by q_t the number of Krein negative definite eigenvalues. For t close to t_0, $t \notin D^{(M)}$. Thus, (p_t, q_t) is defined in a punctured neighborhood of t_0. By Corollary 5 in [Eke90, Section 1.3], the difference $p_t - q_t$ is locally constant near t_0. (Alternatively, we can deduce that from Theorem 1.3 in the analytic case. For instance, one can check this for each group of eigenvalues $\{\lambda_{\ell,p,q}(t)\}_{q=1,\dots,n_\ell}$ forming an n_ℓ-star, see (16).) The index $\mathrm{ind}^{(M)}(\lambda, t_0)$ is defined to be the integer $p_t - q_t$ for t close to t_0. For a Krein positive definite eigenvalue, its index is simply its algebraic (and geometric) multiplicity. For a Krein negative definite eigenvalue, the index is the opposite of its algebraic (and geometric) multiplicity. Hence, if an eigenvalue λ branches into several ones, the sum of the indices of the eigenvalues branched from λ must equal to the index of λ (in the analytic case).

[7] See e.g. [Kat95, Section 5.1, Chapter 2].

Note that $\nu_+^{(M)}(t) = \sum_{(\ell,p,q)\in K_+^{(M)}(t)} \mathrm{ind}^{(M)}(\lambda_{\ell,p,q}, t)$, i.e., it is the sum of the indices of eigenvalues indexed by $K_+(t)$. Recall that the eigenvalues $\lambda(t)$ branched from $e^{\sqrt{-1}\theta_0}$ are located in a small disk $B(e^{\sqrt{-1}\theta_0}, r)$ for $t \in (0, \delta]$. In the following, we assume that M is sufficient large such that $\gamma^{(M)}(t)$ has no eigenvalue on the boundary of $B(e^{\sqrt{-1}\theta_0}, r)$ for $t \in (0, \delta]$. The part of U inside $B(e^{\sqrt{-1}\theta_0}, r)$ is an arc with a mid-point at $e^{\sqrt{-1}\theta_0}$. The point $e^{\sqrt{-1}\theta_0}$ separates the arc into two smaller arcs. We denote by arc_+ the open half arc on the counter-clockwise side of $e^{\sqrt{-1}\theta_0}$, see Figure 8. Then, for

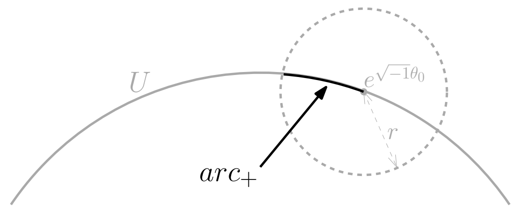

Figure 8: arc_+

$t \in (0, \delta)$, $\nu_+^{(M)}(t)$ is the sum of indices of eigenvalues in the interior of arc_+. By the local constancy on the sum of the indices of branched eigenvalues, we see that $\nu_+^{(M)}(t)$ doesn't vary around $t_0 \in (0, \delta)$ except that $\gamma^{(M)}(t_0)$ has an eigenvalue on the boundary of arc_+. In the exceptional case, $\gamma^{(M)}(t_0)$ has no eigenvalue on the boundary of the disk $B(e^{\sqrt{-1}\theta_0}, r)$ and $e^{\sqrt{-1}\theta_0}$ is an eigenvalue of $\gamma^{(M)}(t_0)$. By Theorem 1.3 for the analytic case, when t increases through t_0, the eigenvalues entered in arc_+ from $e^{\sqrt{-1}\theta_0}$ must move counter-clockwise and be Krein positive definite, the eigenvalues left arc_+ from $e^{\sqrt{-1}\theta_0}$ must move clockwise and be Krein negative definite. Hence, $\nu_+^{(M)}(t)$ strictly increases in this case. Thus, we see that $t \mapsto \nu_+^{(M)}(t)$ is non-deceasing for $t \in (0, \delta)$ for M sufficiently large. $\square$

A Appendix

A.1 Alternative expression for $C(t, \varepsilon)$

We verify the second equality in (5).

Lemma A.1. *Let* $C(t, \varepsilon) \overset{\text{def}}{=} -\gamma(t, \varepsilon)^T J_{2n} \frac{\partial}{\partial \varepsilon} \gamma(t, \varepsilon)$. *Then,*

$$C(t, \varepsilon) = \int_0^t \gamma(u, \varepsilon)^T \frac{\partial}{\partial \varepsilon} A(u, \varepsilon) \gamma(u, \varepsilon) \, \mathrm{d}t.$$

Proof. Note that for all ε, $\gamma(0, \varepsilon) = \mathrm{Id}$. Hence, $\frac{\partial}{\partial \varepsilon} \gamma(0, \varepsilon) = 0$ and $C(0, \varepsilon) = 0$. Thus, it remains to show that

$$\frac{\partial}{\partial t} C(t, \varepsilon) = \gamma(t, \varepsilon)^T \frac{\partial}{\partial \varepsilon} A(t, \varepsilon) \gamma(t, \varepsilon).$$

By a standard contraction argument with Grönwell's inequality, we have that

$$\frac{\partial}{\partial t} \frac{\partial}{\partial \varepsilon} \gamma(t, \varepsilon) = J_{2n} \frac{\partial}{\partial \varepsilon} A(t, \varepsilon) \gamma(t, \varepsilon) + J_{2n} A(t, \varepsilon) \frac{\partial}{\partial \varepsilon} \gamma(t, \varepsilon). \tag{55}$$

By (3) and the symmetry of A, we have that

$$\frac{\partial}{\partial t}\gamma(t,\varepsilon)^T = (J_{2n}A(t,\varepsilon)\gamma(t,\varepsilon))^T = \gamma(t,\varepsilon)^T A(t,\varepsilon)J_{2n}^T. \tag{56}$$

Hence, combining the definition of C, (55) and (56), we obtain that

$$\begin{aligned}\frac{\partial}{\partial t}C(t,\varepsilon) =& -\frac{\partial}{\partial t}\gamma(t,\varepsilon)^T J_{2n}\frac{\partial}{\partial \varepsilon}\gamma(t,\varepsilon) - \gamma(t,\varepsilon)^T J_{2n}\frac{\partial}{\partial t}\frac{\partial}{\partial \varepsilon}\gamma(t,\varepsilon)\\ =&\gamma(t,\varepsilon)^T\frac{\partial}{\partial \varepsilon}A(t,\varepsilon)\gamma(t,\varepsilon),\end{aligned}$$

which completes the proof. □

A.2 Dimension reduction

The following lemma helps to simplify certain notations and proofs (since it allows us to focus on one eigenvalue and to reduce the dimension in many cases). Besides, it is of independent interest. Therefore, we choose to present it here.

Lemma A.2. *For all $n \geq 2$, let $\Lambda(t_0)$ and $\tilde{\Lambda}(t_0)$ be a division of the eigenvalues of $\gamma(t_0)$ for $t_0 \in \mathbb{R}$, where $t \mapsto \gamma(t)$ is the solution of (1). Assume that $\Lambda(t_0)$ is closed under the conjugation $\lambda \to \bar{\lambda}$ and the circular reflection $\lambda \mapsto \bar{\lambda}^{-1}$ with respect to U. There exists $\varepsilon > 0$ such that for $t \in [t_0-\varepsilon, t_0+\varepsilon]$, there exists a division of the eigenvalues of $\gamma(t)$ into $\Lambda(t)$ and $\tilde{\Lambda}(t)$ such that $\Lambda(t)$ is closed under the conjugation $\lambda \mapsto \bar{\lambda}$ and the circular reflection $\lambda \mapsto \bar{\lambda}^{-1}$, and $\Lambda(t)$ (resp. $\tilde{\Lambda}(t)$) converges to $\Lambda(t_0)$ (resp. $\tilde{\Lambda}(t_0)$) as t tends to t_0. Denote by E_t (resp. $\tilde{E}_t$) the sum of invariant spaces $(E_\lambda)_{\lambda\in\Lambda(t)}$ (resp. $(E_\lambda)_{\lambda\in\tilde{\Lambda}(t)}$). Then, by decreasing ε if necessary, we also require that $\dim(E_t) = \dim(E_{t_0})$, $\dim(\tilde{E}_t) = \dim(\tilde{E}_{t_0})$ for $t \in [t_0-\varepsilon, t_0+\varepsilon]$ and $\lim_{t\to t_0} E_t = E_{t_0}$. Moreover, there exists a C^1 curve $t \mapsto Q(t) \in M_{2n\times 2k}(\mathbb{R})$ where $2k = \dim(E_{t_0})$ such that*

- *the column vectors of $Q(t)$ form a basis of E_t and $Q^*(t)J_{2n}Q(t) = J_{2k}$, i.e., the column vectors of Q form a symplectic basis of E_t,*
- *$\gamma(t)Q(t) = Q(t)M_Q(t)$ uniquely determines a C^1 curve $t \mapsto M_Q(t) \in \mathrm{Sp}(2k,\mathbb{R})$,*
- *$\mathrm{d}M_Q/\mathrm{dt} = J_{2k}Q^*(t)A(t)Q(t)M_Q(t)$.*

Remark A.1. Note that the eigenvalues of $M_Q(t)$ are precisely those in $\Lambda(t)$.

Remark A.2. Under the assumption of Lemma A.2, similar to $Q(t)$ and $M_Q(t)$, we may take $\tilde{Q}(t)$ and $M_{\tilde{Q}}(t)$ for $\tilde{\Lambda}(t)$ and $\tilde{E}_t$. Write $Q(t)$ into two $2n \times k$ blocks: $Q(t) = \begin{pmatrix} Q_1(t) & Q_2(t)\end{pmatrix}$. Similarly, we write $\tilde{Q}(t) = \begin{pmatrix}\tilde{Q}_1(t) & \tilde{Q}_2(t)\end{pmatrix}$. Define $Y(t) = \begin{pmatrix} Q_1(t) & \tilde{Q}_1(t) & Q_2(t) & \tilde{Q}_2(t)\end{pmatrix}$. Then, $Y(t) \in \mathrm{Sp}(2n,\mathbb{R})$ and $\gamma(t)Y(t) = Y(t)(M_Q(t)\diamond M_{\tilde{Q}}(t))$, where "$\diamond$" denotes the symplectic summation (cf. [Lon99, Lon02]). To be more precise, we write $M_Q(t) = \begin{pmatrix} M_Q^{11}(t) & M_Q^{12}(t)\\ M_Q^{21}(t) & M_Q^{22}(t)\end{pmatrix}$, where the four sub-matrices are of equal size. We divide $M_{\tilde{Q}}(t)$ in a similar way. The symplectic sum of $M_Q(t)$ and

$M_{\tilde{Q}}(t)$ is defined to be the square matrix

$$\begin{pmatrix} M_Q^{11}(t) & 0 & M_Q^{12}(t) & 0 \\ 0 & M_{\tilde{Q}}^{11}(t) & 0 & M_{\tilde{Q}}^{12}(t) \\ M_Q^{21}(t) & 0 & M_Q^{22}(t) & 0 \\ 0 & M_{\tilde{Q}}^{21}(t) & 0 & M_{\tilde{Q}}^{22}(t) \end{pmatrix}.$$

Then, the original system is decomposed into two sub-systems. Moreover, these two sub-systems satisfy (8) if the original system satisfies such condition.

Proof of Lemma A.2. Since $\Lambda(t)$ is closed under conjugation, we have that $E_t = \mathbb{C} \otimes (\mathbb{R}^{2n} \cap E_t)$. In this sense, $E_t \subset \mathbb{R}^{2n}$ and we replace E_t by $E_t \cap \mathbb{R}^{2n}$ in the following context. By continuity, there exists $\varepsilon > 0$ such that for $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$, there exists a simple smooth curve Γ surrounding all $\Lambda(t)$ and separating $\Lambda(t)$ from $\tilde{\Lambda}(t)$. Then, we may take $P(t) = \frac{1}{2\pi\sqrt{-1}} \int_\Gamma (z \cdot \mathrm{Id} - \gamma(t))^{-1} \, \mathrm{d}z$, which projects $\mathbb{R}^{2n}$ onto E_t, see e.g. [Kat95, Section 1.4, Chapter 2]. Note that E_{t_0} is a symplectic subspace. We choose a symplectic basis $(\xi_1, \ldots, \xi_k, \eta_1, \ldots, \eta_k)$ of E_{t_0} such that $\langle \xi_i, J_{2n}\eta_j \rangle = 1_{i=j}$ and $\langle \xi_i, J_{2n}\xi_j \rangle = \langle \eta_i, \eta_j \rangle = 0$ for $i, j = 1, \ldots, k$. Decreasing ε if necessary, $(P(t)\xi_1, \ldots, P(t)\xi_k, P(t)\eta_1, \ldots, P(t)\eta_k)$ is a linear basis for E_t for $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$. However, it is in general no longer a symplectic basis. Nevertheless, by shrinking ε if necessary, after Gram-Schmidt operation, we obtain a time dependent symplectic basis of E_t, which forms a $2n \times 2k$ matrix $T(t)$. Note that $t \mapsto T(t)$ is continuously differentiable and that

$$T^* J_{2n} T = J_{2k}. \tag{57}$$

In general, we should not take $Q = T$. We consider the following ODE where the solution corresponds to a dynamic change of sympletic basis:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = J_{2k} T^* J_{2n} \frac{\mathrm{d}T}{\mathrm{d}t} V, \quad V(t_0) = \mathrm{Id}\,. \tag{58}$$

By differentiating both sides of (57), we get that $T^* J_{2n} \frac{\mathrm{d}T}{\mathrm{d}t}$ is self-adjoint, which implies that $t \mapsto V(t)$ is a sympletic path, see e.g. [Eke90, Prop. 3, Section 1, Chapter 1]. We define $Q \overset{\text{def}}{=} TV$. By sympleticity of V and (57), we see that $Q(t)^* J_{2n} Q(t) = J_{2k}$. Also, the equation

$$\gamma(t) Q(t) = Q(t) M_Q(t) \tag{59}$$

uniquely determines a C^1 curve $t \mapsto M_Q(t) \in \mathrm{Sp}(2k, \mathbb{R})$. Indeed, by multiplying $Q(t)^* J_{2n}$ on both sides of (59), we obtain that $M_Q(t) = -J_{2k} Q(t)^* J_{2n} \gamma(t) Q(t)$. By taking the derivatives and using (59), we obtain that

$$\frac{\mathrm{d}M_Q}{\mathrm{d}t} = J_{2k} B_Q M_Q, \tag{60}$$

where

$$B_Q = Q^* A Q - \left(\frac{\mathrm{d}Q}{\mathrm{d}t}\right)^* J_{2n} Q - Q^* J_{2n} \gamma \frac{\mathrm{d}Q}{\mathrm{d}t} M_Q^{-1}. \tag{61}$$

By $Q = TV$ and (58), we get that

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{\mathrm{d}T}{\mathrm{d}t}V + TJ_{2k}T^*J_{2n}\frac{\mathrm{d}T}{\mathrm{d}t}V. \tag{62}$$

Hence, together with (57), $Q = TV$ and $J_{2n} + J_{2n}^* = 0$, we get that

$$\left(\frac{\mathrm{d}Q}{\mathrm{d}t}\right)^* J_{2n}Q = V^*\left(\frac{\mathrm{d}T}{\mathrm{d}t}\right)^* J_{2n}TV + V^*\left(\frac{\mathrm{d}T}{\mathrm{d}t}\right)^* J_{2n}^*TJ_{2k}^*T^*J_{2n}TV = 0. \tag{63}$$

It remains to prove that

$$Q^*J_{2n}\gamma\frac{\mathrm{d}Q}{\mathrm{d}t}M_Q^{-1} = 0. \tag{64}$$

By multiplying $(V^*)^{-1}$ on the left and M_QV^{-1} on the right, using $Q = TV$ and (62), we find that (64) is equivalent to

$$(T^*J_{2n}\gamma + T^*J_{2n}\gamma TJ_{2k}T^*J_{2n})\frac{\mathrm{d}T}{\mathrm{d}t} = 0.$$

It would be sufficient to prove that

$$T^*J_{2n}\gamma TJ_{2k}T^*J_{2n} = -T^*J_{2n}\gamma.$$

Note that $\gamma T = TM_T$ uniquely determines a sympletic $2k \times 2k$ matrix M_T since $T^*J_{2n}T = J_{2k}$ and γ is sympletic. Indeed, we have that $M_T^*J_{2k}M_T = M_T^*T^*J_{2n}TM_T = T^*\gamma^*J_{2n}\gamma T = T^*J_{2n}T = J_{2k}$. By symplecity of M_T, we have that

$$\gamma TJ_{2k}T^*\gamma^* = TM_TJ_{2k}M_T^*T^* = TJ_{2k}T^*. \tag{65}$$

By writing J_{2n} as $\gamma^*J_{2n}\gamma$, using (65) and (57), we see that

$$T^*J_{2n}\gamma TJ_{2k}T^*J_{2n} = T^*J_{2n}\gamma TJ_{2k}T^*\gamma^*J_{2n}\gamma = T^*J_{2n}TJ_{2k}T^*J_{2n}\gamma = -T^*J_{2n}\gamma,$$

which implies (64). By (60), (61), (63) and (64), we get that $\frac{\mathrm{d}M_Q}{\mathrm{d}t} = J_{2k}Q^*AQM_Q$. □

A.3 Analytic Krein-Lyubarskii theorem

In this subsection, we provide a proof of Theorem 1.3 when $t \mapsto A(t)$ is real analytic. We partially follow the argument in [YS75] for (3) when $\varepsilon \mapsto A(t, \varepsilon)$ is affine in ε. The connection between the first order asymptotics of the eigenvalues and the Jordan structure has already been established in Section 3. We will only prove the analyticity of the eigenvalues as t varies and the part b) of Theorem 1.3.

By analytic continuation, the real parameter t of (1) is extended to the complex parameter $z \in \mathbb{C}$ around 0:

$$\frac{\mathrm{d}}{\mathrm{d}z}\gamma(z) = J_{2n}A(z)\gamma(z), \gamma(0) \in \mathrm{Sp}(2n, \mathbb{R}). \tag{66}$$

By analyticity of $z \mapsto A(z)$, $z \mapsto \gamma(z)$ is analytic. Since the zero set of an analytic function is isolated, the following two equations are extended to complex z: $A^T(z) = A(z)$ and $\gamma(z)^TJ_{2n}\gamma(z) = J_{2n}$.

In [YS75], they crucially used the key feature of the system that when $\gamma(z)$ has eigenvalue ω on U, the parameter z has to be real. (Roughly speaking, the reason is that z happens to be the eigenvalue of a self-adjoint operator when $\omega \in U$.) Such a phenomenon also appears for our general system (66), as stated in the following lemma.

Lemma A.3. *Consider the ODE (66). We assume that $z \mapsto \gamma(z)$ is analytic (or equivalently, $z \mapsto A(z)$ is analytic), $A(t)$ is real symmetric for $t \in \mathbb{R}$ and for any eigenvector ξ of $\gamma(0)$ associated with an eigenvalue on U, $\langle A(0)\xi, \xi\rangle > 0$. Then, there exists $\delta > 0$, for all $z \in \mathbb{C} \setminus \mathbb{R}$ and $|z| < \delta$, $\gamma(z)$ has no eigenvalue on U.*

To prove Lemma A.3, we need to modify the argument in [YS75, Section 4.1].

Proof of Lemma A.3. It suffices to prove the following cannot happen: there exist non-real complex numbers z_n tending to 0 such that for each z_n, $\gamma(z_n)$ has an eigenvector ξ_n with $||\xi_n||_2 = 1$ associated with some eigenvalue $\lambda_n \in U$. We write z_n in polar coordinate as $r_n e^{\sqrt{-1}\theta_n}$ with $r_n > 0$ and $\theta_n \in (-\pi, 0) \cup (0, \pi)$. By taking subsequence if necessary, we assume that $\lim_{n\to+\infty} \lambda_n = \lambda$, $\lim_{n\to+\infty} \xi_n = \xi$ and $\lim_{n\to+\infty} \theta_n = \theta$.

We expand $A(z)$ in Taylor series as $\sum_{j\geq 0} z^j A^{(j)}$ around 0. Since $A(t)$ is real symmetric for $t \in \mathbb{R}$, $A^{(j)}$ are real symmetric for all $j \geq 0$. For $r \geq 0$ and $\theta \in \mathbb{R}$, we define $A_1(re^{\sqrt{-1}\theta}) = \sum_{j\geq 0} \cos(j\theta) r^j A^{(j)}$ and $A_2(re^{\sqrt{-1}\theta}) = \sum_{j\geq 1} \sin(j\theta) r^j A^{(j)}$. Then, $A_1(z)$ and $A_2(z)$ are real symmetric matrices and $A(z) = A_1(z) + \sqrt{-1}A_2(z)$. Moreover, there exists $C = C(A) < \infty$ such that for all $r \in [0, C^{-1})$ and $\theta \in [-\pi, \pi]$, for all $\xi \in \mathbb{C}^{2n}$ with $||\xi||_2 = 1$,

$$\left|\langle A_2(re^{\sqrt{-1}\theta})\xi, \xi\rangle\right| \leq C \cdot r \cdot |\sin(\theta)|, \tag{67}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{C}^{2n}$, which is linear in the first vector.

For $\rho \in U$, we denote by $\mathfrak{X}(\gamma(0), \rho)$ the space of analytic paths $y : [0, 1] \to \mathbb{C}^{2n}$ with the boundary condition $\gamma(0)y(1) = \rho y(0)$. Define three functions L_0, L_1 and L_2 on $\cup_{\rho\in U}\mathfrak{X}(\gamma(0), \rho) \times \mathfrak{X}(\gamma(0), \rho)$ as follows: for $y_1, y_2 \in \mathfrak{X}(\gamma(0), \rho)$,

$$\begin{aligned}
L_0(y_1, y_2) &= \int_0^1 \left\langle J_{2n} \frac{\mathrm{d}}{\mathrm{d}s} y_1(s), y_2(s) \right\rangle \mathrm{d}s, \\
L_{1,z}(y_1, y_2) &= \int_0^1 \langle A_1(sz)y_1(s), y_2(s)\rangle \,\mathrm{d}s, \\
L_{2,z}(y_1, y_2) &= \int_0^1 \langle A_2(sz)y_1(s), y_2(s)\rangle \,\mathrm{d}s.
\end{aligned}$$

Note that $L_0(y_1, y_2) = \overline{L_0(y_2, y_1)}$, $L_{1,z}(y_1, y_2) = \overline{L_{1,z}(y_2, y_1)}$ and $L_{2,z}(y_1, y_2) = \overline{L_{2,z}(y_2, y_1)}$. Hence, $L_0(y, y), L_{1,z}(y, y), L_{2,z}(y, y) \in \mathbb{R}$ for $y \in \mathfrak{X}(\gamma(0), \rho)$.

Define $x_n(s) = \gamma(sz_n)\xi_n$ for $s \in [0, 1]$. Then, $x_n \in \cup_{\rho\in U}\mathfrak{X}(\gamma(0), \rho)$. By (66), we have that

$$L_0(x_n, x_n) + z_n(L_{1,z_n}(x_n, x_n) + \sqrt{-1}L_{2,z_n}(x_n, x_n)) = 0. \tag{68}$$

Necessarily, the argument θ_n of z_n and the argument ψ_n of the complex number $L_{1,z_n}(x_n, x_n) + \sqrt{-1}L_{2,z_n}(x_n, x_n)$ differ by a multiple of π, or equivalently,

$$|\sin(\theta_n)| = |\sin(\psi_n)|. \tag{69}$$

By (67), there exists $C = C(A) < \infty$ such that for large enough n,

$$L_{2,z_n}(x_n, x_n) \leq C \cdot r_n |\sin(\theta_n)|.$$

By continuity, $\lim_{n\to+\infty} L_{1,z_n}(x_n, x_n) = \langle A(0)\xi, \xi\rangle > 0$. Hence, as $n \to +\infty$, ψ_n and $|\sin(\psi_n)|$ are of the order $r_n|\sin(\theta_n)|$, which contradicts with (69) since $\lim_{n\to+\infty} r_n = 0$. $\square$

Consider the characteristic polynomial $p(\lambda, z) = \det(\lambda\cdot\mathrm{Id} - \gamma(z))$. Assume that $\lambda_0 = e^{\sqrt{-1}\theta_0} \in U$ is an eigenvalue of $\gamma(0)$. By Weierstrass's preparation theorem of the local form of analytic functions in multi-variables, there exist integers ℓ and M such that for (λ, z) close to $(\lambda_0, 0)$, we have that

$$p(\lambda, z) = (\lambda - \lambda_0)^\ell (z^M + a_{M-1}(\lambda)z^{M-1} + \cdots + a_0(\lambda))b(\lambda, z),$$

where $b(\lambda, z)$ is non-zero and analytic, $\{a_i\}_{i=0,\dots,M-1}$ are analytic in λ and vanish at λ_0. Note that $\ell = 0$ and hence,

$$p(\lambda, z) = (z^M + a_{M-1}(\lambda)z^{M-1} + \cdots + a_0(\lambda))b(\lambda, z). \tag{70}$$

(Otherwise, λ_0 is an eigenvalue of $\gamma(z)$ as long as z is sufficient close to 0, which contradicts with Lemma A.3. Alternatively, we could see that from the first order asymptotics in Theorem 1.3 a) proved in Section 3. Or simply follow the argument of [Eke90, Proposition 2, Section 3, Chapter 1].) The solution of $p(\lambda, z) = 0$ coincides with the solution of $z^M + a_{M-1}(\lambda)z^{M-1} + \cdots + a_0(\lambda) = 0$, which is the union of the graphs of several multi-valued analytic functions $\{z_i(\lambda)\}_{i=1,\dots,\tilde{M}}$ ($\tilde{M} \leq M$) in λ. By Lemma A.3, when λ is on U, $z_i(\lambda)$ must lie on $\mathbb{R}$. This forces that each z_i is actually single-valued analytic functions and $\tilde{M} = M$, see the lemma in [YS75, Section 1.5, Chapter 3]. Hence,

$$z^M + a_{M-1}(\lambda)z^{M-1} + \cdots + a_0(\lambda) = \prod_{i=1}^{M}(z - z_i(\lambda)). \tag{71}$$

Let

$$z_i(\lambda) = \sum_{k\geq j_i} e_{i,k}(\lambda - \lambda_0)^k, \quad e_{i,j_i} \neq 0 \tag{72}$$

be the Taylor expansion of $z_i(\lambda)$. Inverting that expansion, we see that $\lambda = \lambda_0 + h_i(z^{\frac{1}{j_i}})$, where h_i is analytic, $h_i(0) = 0$ and $h_i'(0) \neq 0$. Note that $\lambda - \lambda_0 \overset{z\to 0}{\sim} h_i'(0)z^{\frac{1}{j_i}}$. Compared with (16), we need to show that $\{j_i\}_{i=1,\dots,M}$ are exactly the sizes of Jordan blocks of $\gamma(0)$ associated with λ_0. Then, $\{h_i'(0)\}_{i=1,\dots,M}$ will be given by (16).

Firstly, let us show that M is precisely the number of Jordan blocks associated with λ_0. By multiplying the first order asymptotics of the eigenvalues in (16), we see that $p(\lambda_0, t)$ is of the order

t^m as $t \to 0$, where m is the geometric multiplicity of λ_0, or equivalently, m is the number of Jordan blocks. On the other hand, by (70), $p(\lambda_0, z)$ is of the order z^M as $z \to 0$. Hence, M equals m.

Next, we show that $\{j_i\}_{i=1,\dots,M}$ are the sizes of the Jordan blocks. Again, by Weierstrass preparation theorem, the analytic function $g_i(\lambda, z) = z - z_i(\lambda)$ in variables λ and z has the following local form near $(\lambda_0, 0)$:

$$z - z_i(\lambda) = z^{\ell_i} \left((\lambda - \lambda_0)^{\tilde{j}_i} + c_{i,\tilde{j}_i-1}(z)(\lambda - \lambda_0)^{\tilde{j}_i-1} + \dots + c_{i,0}(z) \right) b_i(\lambda, z), \tag{73}$$

where ℓ_i and $\tilde{j}_i$ are integers, the analytic function $b_i(\lambda, z)$ doesn't vanish near $(\lambda_0, 0)$ and the analytic functions $\{c_{i,k}(z)\}_{k=0,\dots,\tilde{j}_i-1}$ vanish at 0. Clearly, ℓ_i is zero. Otherwise, the set of eigenvalues of $\gamma(0)$ would contain an open neighborhood of λ_0. Taking $z = 0$ and compare with the expansion (72) of $z_i(\lambda)$, we find that $\tilde{j}_i = j_i$. Combining (70), (71) and (73), we get that

$$p(\lambda, z) = \prod_{i=1}^{m} p_{\lambda_0,i}(\lambda, z) \cdot f(\lambda, z), \tag{74}$$

where $f(\lambda, z) = b(\lambda, z) \prod_{i=1}^m b_i(\lambda, z)$ and

$$p_{\lambda_0,i}(\lambda, z) = (\lambda - \lambda_0)^{j_i} + c_{i,j_i-1}(z)(\lambda - \lambda_0)^{j_i-1} + \dots + c_{i,0}(z).$$

Taking $z = 0$, we see that $\sum_{i=1}^m j_i$ equals the algebraic multiplicity of λ_0. Moreover, for z close to 0, the roots of $p(\lambda, z)$ near λ_0 coincide with those of $\prod_{i=1}^m p_{\lambda_0,i}(\lambda, z)$ with multiplicities. Comparing (16) with the asymptotics $\lambda - \lambda_0 \sim h_i'(0) z^{\frac{1}{j_i}}$ of the roots of $\{p_{\lambda_0,i}(\lambda, z)\}_{i=1,\dots,m}$, we conclude that $\{j_i\}_{i=1}^m$ are precisely the sizes of Jordan blocks. This completes the argument for the analyticity of eigenvalues and their first order asymptotics when t varies from 0.

Next, we prove the part b) of Theorem 1.3. We only present the proof for the case that t increases from 0. The other case is essentially the same and is left to the reader. Together with the first order asymptotics in (16), it suffices to show that for t close to 0,

i) the eigenvalues moving tangential to the circle at $t = 0$ actually move along the circle for a period of time,

ii) they are Krein definite.

By Theorem 1.3 a), i) implies the semi-simplicity of these eigenvalues on the circle for non-zero real t close to 0.

If i) fails, then there exist an integer j, a real number v, an analytic function h and a sequence (t_n, λ_n) such that t_n decreases to 0 as n increases to infinity, $\lambda_n \notin U$, λ_n is an eigenvalue of $\gamma(t_n)$, $\lambda_n - \lambda_0 = h(t_n^{\frac{1}{j}})$ and $\lambda_n - \lambda_0 \overset{n\to\infty}{\sim} \sqrt{-1}\lambda_0 \cdot v \cdot t_n^{\frac{1}{j}}$. For each n, let us consider the eigenvalues $\lambda_0 + h(t_n^{\frac{1}{j}} e^{\sqrt{-1}\varphi_n/j})$ of $\gamma(t_n e^{\sqrt{-1}\varphi_n})$. As φ_n increases from $-\pi$ to π, they rotate around λ_0 for roughly $\frac{2\pi}{j}$ radians. By first order estimates of the eigenvalues, for n sufficient large, there exists $\phi_n \notin \pi\mathbb{Z}$ such that $\gamma(t_n e^{\sqrt{-1}\phi_n})$ has an eigenvalue on U. (Indeed, $\phi_n \to 0$ as $n \to \infty$.) This contradicts with Lemma A.3 since $t_n e^{\sqrt{-1}\phi_n} \notin \mathbb{R}$.

Next, we show that the eigenvalues moving on the circle are Krein definite when t is sufficiently close to 0 with their Krein types determined by their moving directions.

Let us verify the statement as t increases from 0. The other case is similar and we left the proof to the reader. We have seen that the eigenvalue $\lambda(t) = \lambda_0 + h(t^{\frac{1}{j}})$ for a certain integer j and a certain analytic function h. Note that $\lambda'(t) = \frac{1}{j}h'(t^{\frac{1}{j}})t^{\frac{1}{j}-1}$. By continuity of h' and $h'(0) \neq 0$, we see that the eigenvalue on U has a deterministic moving direction along U as t increases from 0 a bit. If the eigenvalue on U situates on the counter-clockwise direction of λ_0 in the local sense, then the eigenvalue moves counter-clockwise along the circle as t slightly increases from 0. Hence, by Theorem 1.3 a), there is no Krein indefinite eigenvalue on U situating on the counter-clockwise side of λ_0. Together with their moving direction, by Theorem 1.3 a), we see that those eigenvalues must be semi-simple and Krein positive definite. (Otherwise, if there exists $t_1 > 0$ such that one of those eigenvalues on the circle is Krein indefinite, then according to the branching mechanism described in Theorem 1.3 a) together with the fact that no eigenvalue entrances or escapes U during this period of time, there must exist eigenvalues with different moving directions on the counter-clockwise side of λ_0 on U, which is a contradiction.) Similarly, if an eigenvalue on U situates on the clockwise side of λ_0, then it is Krein negative definite and moves clockwise along the circle.

Finally, we will see that the eigenvalues of $\gamma(z)$ branching from λ_0 are semi-simple for z in a small enough punctured disk of 0. Moreover, the corresponding eigenvectors are also multi-valued analytic functions and admit Puiseux expansion. To see this, it suffices to prove that there exist m $\mathbb{C}^{2n}$-valued analytic functions $\{v_i\}_{i=1,\ldots,m}$ such that $\{v_i(z^{\frac{1}{j_i}})\}_{i=1,\ldots,m}$ is a set of $\sum_{i=1}^m j_i$ many linearly independent vectors for sufficiently small z and for $i = 1, \ldots, m$, we have that

$$\gamma(z^{j_i})v_i(z) = (h_i(z) + \lambda_0)v_i(z). \tag{75}$$

Define a family of operators analytic in z:

$$T_i(z) \stackrel{\text{def}}{=} (\gamma(z^{j_i})^T - (\bar{\lambda_0} - \overline{h_i(\bar{z})}) \cdot \mathrm{Id})(\gamma(z^{j_i}) - (\lambda_0 - h_i(z)) \cdot \mathrm{Id}).$$

Note that $T_i(z)^* = T_i(z)$ for real valued z. Such a family of operator is said to be *symmetric*. By perturbation theories of symmetric operators [Kat95, Sections 6.1 and 6.2, Chapter 2], eigenvalues and corresponding eigenvectors of T_i are analytical for real z. More precisely, there exist m analytic complex-valued functions $\mu_{i,1}, \mu_{i,2}, \ldots, \mu_{i,m}$ and m analytic $\mathbb{C}^{2n}$-valued functions $\zeta_{i,1}, \ldots, \zeta_{i,m}$ such that $\zeta_{i,1}, \ldots, \zeta_{i,m}$ are orthonormal and $T_i(t)\zeta_{i,k}(t) = \mu_{i,k}(t)\zeta_{i,k}(t)$ for $k = 1, \ldots, m$ and real t close to 0. Since non-zero analytic functions have isolated zeros, there exist $\delta > 0$ and an integer $g := g(i) = g(h_i)$ such that $\mu_{i,1}, \ldots, \mu_{i,g}$ are identically zero and $\mu_{i,g+1}, \ldots, \mu_{i,m}$ are non-zero on $[-\delta, 0) \cup (0, \delta]$. Note that $T_i(t)\zeta_{i,k}(t) = 0$ iff

$$\gamma(t^{j_i})\zeta_{i,k}(t) = (h_i(t) + \lambda_0)\zeta_{i,k}(t). \tag{76}$$

Hence, for real and sufficiently small t, $g(i)$ equals to the geometric multiplicity of the eigenvalue $h_i(t) + \lambda_0$ of the matrix $\gamma(t^{j_i})$.

We define an equivalence relation on the set $\{1,\ldots,m\}$: $i \sim i'$ if either $i = i'$ or $j_i = j_{i'}$ and $h_i(z) = h_{i'}(ze^{2k(i,i')\pi\sqrt{-1}/j_i})$ for some integer $k(i,i')$. Then, $i \sim i'$ iff $h_i(z^{\frac{1}{j_i}})$ and $h_{i'}(z^{\frac{1}{j_{i'}}})$ are the same multi-valued analytic functions. In particular, $i \sim i'$ implies that $j_i = j_{i'}$.

Let us firstly consider a special (yet generic) case that the equivalence relation "$\sim$" coincides with the standard one "$=$". Since non-zero analytic functions have isolated zeros, for different i and i', $h_i(z^{\frac{1}{j_i}})$ and $h_i'(z^{\frac{1}{j_i}})$ are disjoint in a punctured neighbourhood of 0. In this case, together with the first order asymptotics in (16), we see that the eigenvalues of $\gamma(z)$ branching from λ_0 have algebraic multiplicity 1 as z varies from 0. For $i = 1,\ldots,m$, we take v_i to be the direct analytic continuation of $\zeta_{i,1}$. Then, they satisfy (75). Moreover, for z in a small enough punctured neighborhood of 0, the set $\{v_i(z^{\frac{1}{j_i}})\}_{i=1,\ldots,m}$ is linearly independent since they are the eigenvectors of different eigenvalues of $\gamma(z)$.

The general case is more complicated. For the set of eigenvectors $\{v_i(z^{\frac{1}{j_i}})\}_{i=1,\ldots,m}$, we wish to take all $\zeta_{i,k}(z^{\frac{1}{j_i}})$ for $i = 1,\ldots,m$ and $k \le g(i)$. However, there exist duplications: if $i \sim \tilde{i}$, then $h_i(z^{\frac{1}{j_i}})$ and $h_{i'}(z^{\frac{1}{j_{i'}}})$ are the same multi-valued analytic functions and hence, the linear spaces $\mathrm{Span}\{\zeta_{i,k}(z^{\frac{1}{j_i}}) : k \le g(i)\}$ and $\mathrm{Span}\{\zeta_{i',k}(z^{\frac{1}{j_{i'}}}) : k \le g(i')\}$ are identical. Instead of collecting vectors $\zeta_{i,k}(z^{\frac{1}{j_i}})$ for each $i = 1,\ldots,m$, we collect vectors for each equivalence class $[i]$, where we denote by $[i]$ the equivalence class of i with respect to the equivalence relation $\sim$. We will show that for $i = 1,\ldots,m$, $g(i)$ equals the cardinality $\sharp[i]$ of $[i]$ so that we have the correct number of eigenvectors, i.e., $\sum_{[i]} g(i)j_i = \sum_{i=1}^m j_i$. Clearly, $g(i) \le \sharp[i]$ since the algebraic multiplicity dominates the geometric multiplicity. To get the converse inequality, recall that the eigenvalues branching from λ_0 are semi-simple for real t close to 0, which implies that $g(i) = \sharp[i]$ if $h_i(t) \in U$ for real t close to 0. To obtain the inequality in the general case, we may perform a rotation $t \mapsto te^{2\ell\pi\sqrt{-1}/j_i}$ in (76) for some properly chosen integer ℓ. Eventually, for $i = 1,\ldots,m$, we define v_i in the following way.

1) Take the equivalence class $[i]$ of i and list the integers in $[i]$ in increasing order.

2) Find the smallest element $\ell(i)$ in $[i]$ and define $k(i) = \sharp\{i' \in [i] : i' \le i\}$.

3) Define v_i to be the direct analytic continuation of $\zeta_{\ell(i),k(i)}$.

The linear independence of the set of vectors $\{v_i(z^{\frac{1}{j_i}})\}_{i=1,\ldots,m}$ is left to the reader.

References

[Eke90] Ivar Ekeland, *Convexity methods in Hamiltonian mechanics*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 19, Springer-Verlag, Berlin, 1990. MR 1051888

[GfL58] I. M. Gel′fand and V. B. Lidskiĭ, *On the structure of the regions of stability of linear canonical systems of differential equations with periodic coefficients*, Amer. Math. Soc. Transl. (2) **8** (1958), 143–181. MR 0091390

[Kat95] Tosio Kato, *Perturbation theory for linear operators*, Classics in Mathematics, Springer-Verlag, Berlin, 1995, Reprint of the 1980 edition. MR 1335452

[KL62] M. G. Kreĭn and G. Ja. Ljubarskiĭ, *Analytic properties of the multipliers of periodic canonical differential systems of positive type*, Izv. Akad. Nauk SSSR Ser. Mat. **26** (1962), 549–572. MR 0142832

[Kre50] M. G. Kreĭn, *A generalization of some investigations of A. M. Lyapunov on linear differential equations with periodic coefficients*, Doklady Akad. Nauk SSSR (N.S.) **73** (1950), 445–448. MR 0036379

[Kre51] ______, *On certain problems on the maximum and minimum of characteristic values and on the Lyapunov zones of stability*, Akad. Nauk SSSR. Prikl. Mat. Meh. **15** (1951), 323–348. MR 0043980

[KY06] Masataka Kuwamura and Eiji Yanagida, *Krein's formula for indefinite multipliers in linear periodic Hamiltonian systems*, J. Differential Equations **230** (2006), no. 2, 446–464. MR 2271499

[Lon99] Yiming Long, *Bott formula of the Maslov-type index theory*, Pacific J. Math. **187** (1999), no. 1, 113–149. MR 1674313

[Lon02] ______, *Index theory for symplectic paths with applications*, Progress in Mathematics, vol. 207, Birkhäuser Verlag, Basel, 2002. MR 1898560

[Mos58] Jürgen Moser, *New aspects in the theory of stability of Hamiltonian systems*, Comm. Pure Appl. Math. **11** (1958), 81–114. MR 0096872

[Win10] Sergei Winitzki, *Linear algebra via exterior products*, lulu.com, 2010.

[YS75] V. A. Yakubovich and V. M. Starzhinskii, *Linear differential equations with periodic coefficients. 1, 2*, Halsted Press [John Wiley & Sons] New York-Toronto, Ont.,; Israel Program for Scientific Translations, Jerusalem-London, 1975, Translated from Russian by D. Louvish. MR 0364740