Regularity in an unusual variational problem

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Abstract: We prove regularity up to the boundary of the solution of an unusual variational problem arising in mathematical finance.

This paper is dedicated to Victor Yudovich on his 70th birthday.

1 The variational problem.

In this paper we consider a variational problem arising in mathematical finance (see [1]). It concerns functions defined in the open positive cone R_+^n in R^n. The problem is to minimize a positive definite quadratic form in derivatives up to order n. The unusual feature here is that not all derivatives up to order n are involved, only special ones. The principal result of the paper, Theorem 1, is a proof of smoothness of the solution in the closure of R_+^n, including the origin.

We begin with notation For any multi-index $\alpha = (\alpha_1, ..., \alpha_n)$, we denote:

$$D^\alpha u = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} ... \partial_{x_n}^{\alpha_n} u$$

where $\partial_{x_i}^{\alpha_i} = \partial^{\alpha_i} / \partial x_i$.

Set:

$$A_n = \{\alpha = (\alpha_1, ..., \alpha_n) \mid \forall i, \quad \alpha_i = 0 \text{ or } 1\}$$

We will consider only multi-indices α in A_n, and the corresponding derivatives $D^\alpha u$. We shall set:

$$\tilde{D}^n u = \{D^\alpha u \mid \alpha \in A_n\}$$

For instance, we have:

- for $n = 1$, $\widetilde{D}^1 u = \{u, \partial u\}$.
- for $n = 2$, $\widetilde{D}^2 u = \{u, \partial_x u, \partial_y u, \partial_x \partial_y u\}$.

- for $n=3$, $\widetilde{D}^3 u = \{u, \partial_x u, \partial_y u, \partial_z u, \partial_x \partial_y u, \partial_x \partial_z u, \partial_y \partial_z u, \partial_x \partial_y \partial_z u\}$.

The highest-order derivative in $\tilde{D}^n u$ is $\partial^1 \partial^2 ... \partial^n u$. Denote by $\tilde{D}^n_{n-1}$ the set of all other derivatives:

$$\tilde{D}^n_{n-1} u = \{D^\alpha u \mid \alpha \in A_n,\ \alpha \neq (1, ..., 1)\}$$

Here is the problem coming from [1]. Given a symmetric, positive definite $(2^n - 1) \times (2^n - 1)$ matrix Q, find a function $u : R^n_+ \to R$ which minimizes:

$$J(u) := \int \int ... \int_{R^n_+} \left[(\partial_{x_1} \partial_{x_2} ... \partial_{x_n} u)^2 + \left(Q \tilde{D}^n_{n-1} u, \tilde{D}^n_{n-1} u \right) \right] dx_1 dx_2 ... dx_n \tag{1}$$

subject to:

$$u(0) = 1$$

In the paper [1] it is proved that there is a unique solution. The proof consists of introducing the quadratic form:

$$H_n(u) := \sum_{\alpha \in A} \|D^\alpha u\|^2_{L^2}$$

and the Hilbert space E_n consisting of all functions u such that $H_n(u) < \infty$, with the norm $H_n(u)^{1/2}$. The problem then becomes:

$$(\mathrm{P}) \ \left\{ \begin{array}{c} \inf\ J(u) \\ u \in E_n, u(0) = 1 \end{array} \right.$$

It is easily seen that all functions in E_n are continuous and go to zero at infinity. In fact, the injection $E_n \to C^0\left(R^n_+\right)$ is continuous, so that the boundary condition $u(0) = 1$ defines an affine linear subset of E_n with codimension 1. Since J is a continuous, positive definite quadratic form on E_n it attains its minimum at a unique point, which is the only solution of the problem.

If u is that minimum, it must satisfy the usual optimality conditions, namely:

$$\int_{R^n_+} \left[(\partial_{x_1} \partial_{x_2} ... \partial_{x_n} u)(\partial_{x_1} \partial_{x_2} ... \partial_{x_n} \varphi) + \left(Q \tilde{D}^n_{n-1} u, \tilde{D}^n_{n-1} \varphi \right) \right] dx = 0 \tag{2}$$

for all $\varphi \in E_n$ such that $\varphi(0) = 0$.

As an example, let us work out these conditions in the special case when $n = 2$ and:

$$J(u) = \int \int_{R^2_+} \left[u^2_{xy} + a u^2_x + 2b u_x u_y + c u^2_y + d u^2 \right] dxdy \tag{3}$$

Note that, since Q is positive definite, we must have $d > 0$.

We then get:

$$\int \int_{R^2_+} \left[u_{xy} \varphi_{xy} + a u_x \varphi_x + b(u_x \varphi_y + u_y \varphi_x) + c u_y \varphi_y + d u \varphi \right] dxdy = 0 \tag{4}$$

for all φ such that $\varphi(0,0) = 0$. Taking a function φ with compact support contained in the interior of R^2_+, we get the Euler-Lagrange equation:

$$u_{xxyy} - au_{xx} - 2bu_{xy} - cu_{yy} + du = 0 \text{ in } \quad R^2_+ \tag{5}$$

and any smooth solution u in the closure of R^n_+ is subject to the boundary conditions:

$$\begin{aligned} u_{xyy} - au_x - bu_y = 0 &\quad \text{on } x = 0,\ y \geq 0 \\ u_{xxy} - bu_x - cu_y = 0 &\quad \text{on } \ y = 0,\ x \geq 0 \\ u(0,0) = 1 & \end{aligned}$$

In the case $n = 1$, the solution is an exponential, $u(x) = e^{-\lambda x}$, for some $\lambda > 0$. In higher dimension, one no longer gets explicit solutions, except in very particular cases. In the case $n = 2$, for instance, taking for $J(u)$ the particular form (4) with $b = 0$ and $ac = d$, the solution will be $e^{-\lambda x - \mu y}$, for $\lambda = \sqrt{c}$ and $\mu = \sqrt{a}$.

2 Regularity.

In this paper, we prove that the solution u of problem (P) is C^∞ up to and including the boundary. This is a somewhat surprising result, for the corresponding Euler-Lagrange equation, such as (5), is not elliptic.

Theorem 1 *Set $R^n_+ = \{x \mid x_i > 0 \ \ \forall i\}$. For every multi-index $\alpha \in N^n$, there is a constant $C(\alpha)$ such that:*

$$\int_{R^n_+} |D^\alpha u|^2 \leq C(\alpha) \tag{6}$$

Corollary 2 *The solution u of problem (P) is C^∞ up to and including the boundary.*

The result does not extend to other domains, such as the half-plane, as the following counterexample shows. In the case $n = 2$, take for $J(u)$ the particular form (3) with $b = 0$ and $ac = d$. The solution of problem (P) then is $u_0(x, y) = \exp(-\lambda x - \mu y)$, for some $\lambda > 0$ and $\mu > 0$, as we pointed out earlier. Consider the problem of minimizing the integral (1) on the half-plane $\Omega = \{y > 0\}$, subject to $u(0,0) = 1$. Call the functional $\bar{J}$ and denote its solution by $\bar{u}$. Define v by $v(x,y) = \bar{u}(-x,y)$. It is clear that $\bar{J}(v) = \bar{J}(\bar{u}) = \min \bar{J}$, and since $\bar{J}$ is strictly convex, the minimizer is unique, and $v = \bar{u}$. So $\bar{u}$ is even. Now set $\bar{w}(x,y) = \exp(-\lambda|x| - \mu y)$ for $y \geq 0$. Denote by u_1 the restriction of $\bar{u}$ to the positive cone. Since u_0 minimizes J on the positive cone, subject to $u(0,0) = 1$, we have:

$$\bar{J}(\bar{u}) = 2J(u_1) \geq 2J(u_0) = \bar{J}(\bar{w})$$

So w must be the minimizer, and by uniqueness again, $\bar{u} = w$. But w is not smooth, in fact it is not even C^1. If, instead of the upper half-plane, we work in an angle $0 < \arg(x + iy) < \alpha$, with $\alpha > \pi/2$, we suspect that the minimizer will not be smooth near the y-axis, but we have no proof of this.

The usual way to get estimates like (6) is to use the optimality condition (2), taking $\varphi = \xi D^{\beta} u$, where $D^{\beta} u$ is some suitable derivative of u and $\xi(x)$ some suitable cutoff function, and then integrating by parts. For this approach to succeed, however, we need the problem to be elliptic, which is not the case. We thereferore introduce an elliptic penalization.

For $\varepsilon > 0$, consider the problem of minimizing the functional:

$$J_{\varepsilon}(u) = J(u) + \varepsilon \int_{R_+^n} \sum_{i=1}^{n} \left| \partial_i^{n+1} u \right|^2$$

under the boundary condition $u(0) = 1$. The solution exists and is unique; denote by u_{ε} its solution. By standard elliptic regularity results, we find that u_{ε} is smooth in the closed orthant except possibly at the edges, where $x_i = x_j = 0$ for $i \neq j$. Note that u_{ε} has to satisfy additional boundary conditions, namely $\partial_i^{2n+1} u_{\varepsilon} = 0$ on $x_i = 0$, which are independent of ε.

The key to the proof is to take advantage of the special shape of the domain first by using cutoff functions which depend only on one of the variables $x_1, ..., x_n$.

3 The proof in the case $n = 2$.

Notations quickly become overwhelming. For this reason, we will begin by giving the proof in the case when $n = 2$ and $J(u)$ has the special form (3), so that there are no terms in $u_x u$ or $u_y u$. The penalized functional then is:

$$J_{\varepsilon}(u) = J(u) + \varepsilon \int \int_{R_+^2} \left(u_{xxx}^2 + u_{yyy}^2 \right) dxdy$$

We shall denote by u_{ε} its solution. Since the variational problem is uniformly elliptic, u_{ε} is C^{∞} up to the boundary, except possibly at the origin. Note that $J \leq J_{\varepsilon} \leq J_{\eta}$ for $0 \leq \varepsilon \leq \eta$, so that:

$$J(u_0) \leq J_{\varepsilon}(u_{\varepsilon}) \leq J_{\eta}(u_{\eta}) \tag{7}$$

where u_0 denotes the solution of problem (P). The optimality conditions are:

$$\int \int_{R_+^2} \left[B(u_{\varepsilon}, \varphi) + (\partial_x \partial_y u_{\varepsilon})(\partial_x \partial_y \varphi) \right] dxdy + \varepsilon \int \int_{R_+^2} \left[\left(\partial_x^3 u\right)\left(\partial_x^3 \varphi\right) + \left(\partial_y^3 u\right)\left(\partial_y^3 \varphi\right) \right] dxdy = 0 \tag{8}$$

$$\forall \varphi \in E_2 : \quad \varphi(0) = 0$$

where B stands for:

$$B(u, \varphi) = a(\partial_x u)(\partial_x \varphi) + b\left[(\partial_x u)(\partial_y \varphi) + (\partial_y u)(\partial_x \varphi) \right] + c(\partial_y u)(\partial_y \varphi)$$

We now introduce a special cutoff function. Let $\sigma : [0, \infty) \to R$ be a C^∞ function. Assume σ is non-decreasing, vanishes in some neighbourhood of the origin,and $\sigma(t) = 1$ for $t \geq 1$. For $s > 1$, set:

$$\xi_s(t) = \sigma(t/s)$$

Our proof involves (i) deriving preliminary estimates for u_ε, (ii) showing that these hold for u, and (iii) using these preliminary estimates to repeat the argument, applied to u, in order to establish (6).

Proposition 3 *For any integer $m \geq 0$ and any $s > 0$, there are constants $C_1(m,s)$ and $C_2(m,s)$ such that the inequalities:*

$$\left\|\xi_s^{4+m}\partial_x^{m+2}\partial_y u_\varepsilon\right\|_{L^2}^2 + \left\|\xi_s^{4+m}\partial_x^{m+2}u_\varepsilon\right\|_{L^2}^2 \leq C_1(m,s) + \varepsilon C_2(m,s) \tag{9}$$

hold for every $\varepsilon > 0$.

The proof will use the following result (Lemma 8.2, page 856, of [2]), with ξ_s as above.

Lemma 4 *Let $w : [0,\ \infty] \to \infty$ be such that:*

$$\int_0^\infty |\partial_x^p w|^2 \xi_s^{2p} dt < \infty \quad \forall p \in [0,\ m]$$

Then for any $\delta > 0$, there is some $A(s,\delta)$ such that:

$$\sum_{p<m} \int_0^\infty |\partial_x^p w|^2 \xi_s^{2p} \leq \delta \int_0^\infty |\partial_x^m w|^2 \xi_s^{2m} + A(s,\delta) \int_{\xi\neq 0} w^2$$

The lemma is proved using integration by parts.

3.1 Proof of proposition 3

We take:

$$\varphi(x,y) = (-1)^{m-1}\partial_x^m\left[\xi_s^{8+2m}(x)\,\partial_x^{m+2}u_\varepsilon(x,y)\right]$$

in condition (8), which is meaningful since u_ε is C^∞ for $x > 0$. Integrating by parts, we find:

$$M(u_\varepsilon) + H_1(u_\varepsilon,\varepsilon) + H_2(u_\varepsilon,\varepsilon) = 0 \tag{10}$$

where:

$$\begin{aligned} M(u) = &\int\int \left|\partial_x^{m+2}u_y\right|^2 \xi_s^{8+2m} + a\int\int \left|\partial_x^{m+2}u\right|^2 \xi_s^{8+2m} \\ &- b\int\int \left(\partial_x^{m+1}u\right)\ \left(\partial_x^{m+2}u_y\right)\ \xi_s^{8+2m} + b\int\int \left(\partial_x^{m+1}u_y\right)\ \left(\partial_x^{m+2}u\right)\ \xi_s^{8+2m} \\ &- c\int\int \left(\partial_x^m u\right)\ \left(\partial_x^{m+2}u_y\right)\xi_s^{8+2m} - d\int\int \left(\partial_x^m u\right)\left(\partial_x^{m+2}u\right)\xi_s^{8+2m} \end{aligned}$$

$$H_1(u,\varepsilon) = \varepsilon \int\int \left(\partial_x^{m+4}u\right)\ \partial_x^2\left(\xi_s^{8+2m}\partial_x^{m+2}u\right)$$

$$H_2(u,\varepsilon) = -\varepsilon \int\int \left(\partial_x^m\partial_y^3 u\right)\ \left(\partial_x^{m+2}\partial_y^3 u\right)\ \xi_s^{8+2m}$$

Note that H_1 is the only term which involves derivatives of ξ. From our assumptions on ξ, it follows that there is a constant K_s such that

$$\left|\partial_x^k\left(\xi_s^{8+2m}\right)\right| \le K_s\left|\xi_s^{8+2m-k}\right| \quad \forall x>0,\quad 1\le k\le 3 \tag{11}$$

Let us first estimate H_1. Using (11), we have:

$$\begin{aligned}H_1(u,\varepsilon) &= \varepsilon\int\int\left(\partial_x^{m+4}u\right)\left[\xi_s^{8+2m}\partial_x^{m+4}u + 2\left(\partial_x^{m+3}u\right)\partial_x\left(\xi_s^{8+2m}\right) + \left(\partial_x^{m+2}u\right)\partial_x^2\left(\xi_s^{8+2m}\right)\right]\\
&\ge \varepsilon\left\|\left(\partial_x^{m+4}u\right)\xi_s^{4+m}\right\|_{L^2}^2\\
&- \varepsilon K_s^{'}\left\|\left(\partial_x^{m+4}u\right)\xi_s^{m+4}\right\|_{L^2}\left(\left\|\left(\partial_x^{m+3}u\right)\xi_s^{m+3}\right\|_{L^2} + \left\|\left(\partial_x^{m+2}u\right)\xi_s^{m+2}\right\|_{L^2}\right)\\
&\ge \frac{\varepsilon}{2}\left\|\left(\partial_x^{m+4}u\right)\xi_s^{4+m}\right\|_{L^2}^2\\
&- \varepsilon K_s''\left(\left\|\left(\partial_x^{m+3}u\right)\xi_s^{m+3}\right\|_{L^2}^2 + \left\|\left(\partial_x^{m+2}u\right)\xi_s^{m+2}\right\|_{L^2}^2\right)\end{aligned}$$

where K_s' and K_s'' are constants depending only on s. Applying Lemma 4 to the second term on the right-hand side, with suitable δ, we find that there is a constant $K_1(s)$ such that:

$$H_1(u,\varepsilon) \ge \frac{\varepsilon}{4}\left\|\left(\partial_x^{m+4}u\right)\xi_s^{4+m}\right\|_{L^2}^2 - \varepsilon K_1(s)\left\|u\right\|_{L^2}^2 \tag{12}$$

Next, look at H_2 and integrate by parts in x. We find, using (11) again:

$$\begin{aligned}H_2(u,\varepsilon) &= \varepsilon\int\int\left|\partial_x^{m+1}\partial_y^3 u\right|^2\xi^{8+2m} + \varepsilon\int\int\left(\partial_x^m\partial_y^3 u\right)\left(\partial_x^{m+1}\partial_y^3 u\right)\partial_x\left(\xi^{8+2m}\right)\\
&\ge \varepsilon\int\int\left|\partial_x^{m+1}\partial_y^3 u\right|^2\xi^{8+2m} - \varepsilon K_s'''\left\|\left(\partial_x^{m+1}\partial_y^3 u\right)\xi^{4+m}\right\|_{L^2}\left\|\left(\partial_x^m\partial_y^3 u\right)\xi^{3+m}\right\|_{L^2}\\
&\ge \frac{\varepsilon}{2}\left\|\left(\partial_x^{m+1}\partial_y^3 u\right)\xi^{4+m}\right\|_{L^2}^2 - \varepsilon K_s''''\left\|\left(\partial_x^m\partial_y^3 u\right)\xi^{3+m}\right\|_{L^2}^2\end{aligned}$$

where K_s'' and K_s''' are constants depending only on s. Applying Lemma 4 to the second term on the right-hand side, with suitable δ, we find that there is a constant $K_2(s)$ such that:

$$H_2(u,\varepsilon) \ge \frac{\varepsilon}{4}\left\|\left(\partial_x^{m+1}\partial_y^3 u\right)\xi^{4+m}\right\|_{L^2}^2 - \varepsilon K_2(s)\left\|\partial_y^3 u\right\|_{L^2}^2 \tag{13}$$

We now use equation (10). We have:

$$\int\int \left|\partial_x^{m+2}\partial_y u_\varepsilon\right|^2 \xi_s^{8+2m} + a\int\int \left|\partial_x^{m+2}u_\varepsilon\right|^2 \xi_s^{8+2m} + H_1\left(u_\varepsilon,\varepsilon\right) + H_2\left(u_\varepsilon,\varepsilon\right) =$$
$$\int\int \left[b\left(\partial_x^{m+1}u_\varepsilon\right)\ \left(\partial_x^{m+2}\partial_y u_\varepsilon\right) - b\left(\partial_x^{m+1}\partial_y u_\varepsilon\right)\ \left(\partial_x^{m+2}u_\varepsilon\right)\right]\ \xi_s^{8+2m}+$$
$$\int\int \left[c\left(\partial_x^{m}u_\varepsilon\right)\ \left(\partial_x^{m+2}\partial_y u_\varepsilon\right) + d\left(\partial_x^{m}u_\varepsilon\right)\left(\partial_x^{m+2}u_\varepsilon\right)\right]\ \xi_s^{8+2m}$$
$$\leq\ C\left\|\xi_s^{4+m}\partial_x^{m+2}\partial_y u_\varepsilon\right\|_{L^2}\left(\left\|\xi_s^{4+m}\partial_x^{m+1}u_\varepsilon\right\|_{L^2} + \left\|\xi_s^{4+m}\partial_x^{m}u_\varepsilon\right\|_{L^2}\right)$$
$$+\,C\left\|\xi_s^{4+m}\partial_x^{m+2}u_\varepsilon\right\|\left(\left\|\xi_s^{4+m}\partial_x^{m+1}\partial_y u_\varepsilon\right\|_{L^2} + \left\|\xi_s^{4+m}\partial_x^{m}u_\varepsilon\right\|_{L^2}\right)$$

where $C > 0$ is a suitable constant. It follows that:

$$\frac{1}{2}\left\|\xi_s^{4+m}\partial_x^{m+2}\partial_y u_\varepsilon\right\|_{L^2}^2 + \frac{a}{2}\left\|\xi_s^{4+m}\partial_x^{m+2}u_\varepsilon\right\|_{L^2}^2 + H_1\left(u_\varepsilon,\varepsilon\right) + H_2\left(u_\varepsilon,\varepsilon\right) \leq$$
$$C'\left(\left\|\xi_s^{4+m}\partial_x^{m+1}u_\varepsilon\right\|_{L^2}^2 + \left\|\xi_s^{4+m}\partial_x^{m}u_\varepsilon\right\|_{L^2}^2 + \left\|\xi_s^{4+m}\partial_x^{m+1}\partial_y u_\varepsilon\right\|_{L^2}^2\right)$$

where C' is another constant. Using the estimates (12) and (13), this becomes:

$$\left\|\xi_s^{4+m}\partial_x^{m+2}\partial_y u_\varepsilon\right\|_{L^2}^2 + \left\|\xi_s^{4+m}\partial_x^{m+2}u_\varepsilon\right\|_{L^2}^2$$
$$+\,\frac{\varepsilon}{4}\left(\left\|\left(\partial_x^{m+4}u_\varepsilon\right)\xi_s^{4+m}\right\|_{L^2}^2 + \left\|\left(\partial_x^{m+1}\partial_y^3 u_\varepsilon\right)\xi_s^{4+m}\right\|_{L^2}^2\right) \leq$$
$$C''\left(\left\|\xi_s^{4+m}\partial_x^{m+1}u_\varepsilon\right\|_{L^2}^2 + \left\|\xi_s^{4+m}\partial_x^{m}u_\varepsilon\right\|_{L^2}^2 + \left\|\xi_s^{4+m}\partial_x^{m+1}\partial_y u_\varepsilon\right\|_{L^2}^2\right)$$
$$+\,\varepsilon\bar{K}\left(s\right)\left(\left\|u_\varepsilon\right\|_{L^2}^2 + \left\|\partial_y^3 u_\varepsilon\right\|_{L^2}^2\right)$$

where $\bar{K}\left(s\right)$ depends on ε and s.

Because of (7), and the fact that Q is positive definite, $\left\|u_\varepsilon\right\|_{L^2}^2$ and $\varepsilon\left\|\partial_y^3 u_\varepsilon\right\|_{L^2}^2$ are bounded from above when $\varepsilon \to 0$. We therefore end up with the following estimate:

$$\left\|\xi_s^{4+m}\partial_x^{m+2}\partial_y u_\varepsilon\right\|_{L^2}^2 + \left\|\xi_s^{4+m}\partial_x^{m+2}u_\varepsilon\right\|_{L^2}^2 \leq$$
$$C_m\left(\left\|\xi_s^{4+m}\partial_x^{m+1}u_\varepsilon\right\|_{L^2}^2 + \left\|\xi_s^{4+m}\partial_x^{m}u_\varepsilon\right\|_{L^2}^2 + \left\|\xi_s^{4+m}\partial_x^{m+1}\partial_y u_\varepsilon\right\|_{L^2}^2\right) + \varepsilon K_m\left(s\right) + R_m\left(s\right) \tag{14}$$

where $K_m\left(s\right)$ and $R_m\left(s\right)$ depend on s, and of course m.

We now proceed to the proof of inequality (9). It will be done by induction on m.

For $m = 0$, inequality (14) becomes:

$$\left\|\xi_s^4\partial_x^2\partial_y u_\varepsilon\right\|_{L^2}^2 + \left\|\xi_s^4\partial_x^2 u_\varepsilon\right\|_{L^2}^2$$
$$\leq C_0\left(\left\|\xi_s^4\partial_x u_\varepsilon\right\|_{L^2}^2 + \left\|\xi_s^4 u_\varepsilon\right\|_{L^2}^2 + \left\|\xi_s^4\partial_x\partial_y u_\varepsilon\right\|_{L^2}^2\right) + \varepsilon K_0\left(s\right) + R_0\left(s\right)$$
$$\leq C_0\left(\left\|\partial_x u_\varepsilon\right\|_{L^2}^2 + \left\|u_\varepsilon\right\|_{L^2}^2 + \left\|\partial_x\partial_y u_\varepsilon\right\|_{L^2}^2\right) + \varepsilon K_0\left(s\right) + R_0\left(s\right)$$

because $0 \leq \xi \leq 1$. Using (7) again, we find that the first term stays bounded when $\varepsilon \to 0$. Hence:

$$\left\|\xi_s^4 \partial_x^2 \partial_y u_\varepsilon\right\|_{L^2}^2 + \left\|\xi_s^4 \partial_x^2 u_\varepsilon\right\|_{L^2}^2 \leq \; C_0 + R_0(s) + \varepsilon K_0(s)$$

which is of the form (9) for $m = 0$.

Suppose now (9) has been proved for $m \leq M-1$. This means that we have:

$$\left\|\xi_s^{4+m} \partial_x^{m+2} \partial_y u_\varepsilon\right\|_{L^2}^2 + \left\|\xi_s^{4+m} \partial_x^{m+2} u_\varepsilon\right\|_{L^2}^2 \leq C_1(m,s) + \varepsilon C_2(m,s)$$

for $m \leq M-1$. Then, using (14), we get, since $0 \leq \xi \leq 1$:

$$\begin{aligned}
&\left\|\xi_s^{4+M} \partial_x^{M+2} \partial_y u_\varepsilon\right\|_{L2}^2 + \left\|\xi_s^{4+M} \partial_x^{M+2} u_\varepsilon\right\|_{L^2}^2 \\
&\leq C_M \left(\left\|\xi_s^{4+M} \partial_x^{M+1} u_\varepsilon\right\|_{L^2}^2 + \left\|\xi_s^{4+M} \partial_x^{M} u_\varepsilon\right\|_{L^2}^2 + \left\|\xi_s^{4+M} \partial_x^{M+1} \partial_y u_\varepsilon\right\|_{L^2}^2\right) \\
&+ \varepsilon K_M(s) + R_M(s) \\
&\leq C_M \left(\left\|\xi_s^{4+M-1} \partial_x^{M+1} u_\varepsilon\right\|_{L^2}^2 + \left\|\xi_s^{4+M-1} \partial_x^{M+1} \partial_y u_\varepsilon\right\|_{L^2}^2\right) \\
&+ C_M \left\|\xi_s^{4+M-2} \partial_x^{M} u_\varepsilon\right\|_{L^2}^2 + \varepsilon K_M(s) + R_M(s) \\
&\leq C_M \left[C_1(M-1,s) + \varepsilon C_2(M-1,s)\right] + \\
&C_M \left[C_1(M-2,s) + \varepsilon C_2(M-2,s)\right] + \varepsilon K_M(s) + R_M(s)
\end{aligned}$$

and the inequality has the desired form (9) for $m = M$. Proposition 3 is proved.

3.2 Interior regularity

We will now prove the following:

Proposition 5 *Set $R_s^2 = \{x \mid x_i > s \;\; \forall i\}$. For every multi-index $\alpha \in N^2$, and every $s > 0$, there is a constant $C(\alpha, s)$ such that:*

$$\int_{R_s^2} |D^\alpha u|^2 \leq C(\alpha, s)$$

Corollary 6 *The solution u of problem (P) is C^∞ in the interior.*

From Proposition 3, we have for any $m \geq 0$ and $\varepsilon > 0$,

$$\int\int_{x>s} \left[\left(\partial_x^{m+2} \partial_y u_\varepsilon\right)^2 + \left(\partial_x^{m+2} u_\varepsilon\right)^2\right] \leq C(m,s)$$

and in a similar way, by interchanging x and y,. we find:

$$\int\int_{y>s} \left[\left(\partial_y^{m+2} \partial_x u_\varepsilon\right)^2 + \left(\partial_y^{m+2} u_\varepsilon\right)^2\right] \leq C(m,s)$$

We will first extend these inequalities to the function u. There is a sequence $\varepsilon_n \to 0$ such that $u_n := u_{\varepsilon_n}$ converges weakly in $L^2\left(R_+^2\right)$ to some v, while

$\partial_x u_n, \partial_y u_n$ and $\partial_x \partial_y u_n$ converge weakly in $L^2\left(R_+^2\right)$ to $\partial_x v, \partial_y v$ and $\partial_x \partial_y v$. Similarly, from (9), we find that in $x \geq s > 0$, the derivatives $\partial_x^{m+2} \partial_y u_\varepsilon$ and $\partial_x^{m+2} u_\varepsilon$ converge weakly to the corresponding derivatives of u. Similarly for $\partial_y^{m+2} \partial_x u_\varepsilon$ and $\partial_y^{m+2} u_\varepsilon$ in $y \geq s$ for any $s > 0$.

If φ is some C^∞ function with compact support such that $\varphi(0,0) = 1$, we have:

$$
\begin{aligned}
1 = u_n(0,0) &= \int\int \partial_x \partial_y (\varphi u_n) \\
&= \int\int \left[(\partial_x \partial_y \varphi) u_n + (\partial_x \varphi)(\partial_y u_n) + (\partial_x u_n)(\partial_y \varphi) + (\partial_x \partial_y u_n)\varphi\right] \\
&\to \int\int \left[(\partial_x \partial_y \varphi) v + (\partial_x \varphi)(\partial_y v) + (\partial_x v)(\partial_y \varphi) + (\partial_x \partial_y v)\varphi\right] \\
&= \int\int \partial_x \partial_y (\varphi v) = v(0,0)
\end{aligned}
$$

so $v(0,0) = 1$.

Consider now $J(v)$. We have, since J is convex and continuous, hence weakly lower semi-continuous and $J \leq J_\varepsilon$:

$$
J(v) \leq \liminf_n J(u_n) \leq \liminf_n J_{\varepsilon_n}(u_n) \tag{15}
$$

Let u be the minimizer of J. For any $\eta > 0$, there is a C^∞ function with compact support u_η such that $J(u_\eta) \leq J(u) + \eta$. We clearly have $\lim_{\varepsilon \to 0} J_\varepsilon(u_\eta) = J(u_\eta)$, so that:

$$
\liminf_n J_{\varepsilon_n}(u_n) \leq \lim_n J_{\varepsilon_n}(u_\eta) \leq J(u) + \eta
$$

and since this holds for any $\eta > 0$, we have $\liminf_n J_{\varepsilon_n}(u_n) = J(u) = \inf J$. Comparing with (15), we find that $J(v) = \inf J$, so that $v = u$, the unique minimizer of J.

We now set $\varepsilon = \varepsilon_n$ in formula (15), and let $n \to \infty$. We get:

$$
\int\int_{x \geq s} \left[\left(\partial_x^{m+2} \partial_y u\right)^2 + \left(\partial_x^{m+2} u\right)^2\right] dxdy \leq C_1(m,s) \tag{16}
$$

Similarly:

$$
\int\int_{y \geq s} \left[\left(\partial_y^{m+2} \partial_x u\right)^2 + \left(\partial_y^{m+2} u\right)^2\right] dxdy \leq C_3(m,s) \tag{17}
$$

So we have L^2 estimates in the domain $x > s, y > s$ for all derivatives of the form $\partial_x^k u, \partial_y^k u, \partial_x^k \partial_y u$ and $\partial_y^k \partial_x u$. There are still many derivatives missing. We find them by using the Euler equation (5) satisfied by u in the interior:

$$
\partial_x^2 \partial_y^2 u - a \partial_x^2 u - 2b \partial_x \partial_y u - c \partial_y^2 u + du = 0
$$

By themselves, (16) and (17) give us all the derivatives of order 3. Let us look at the derivatives of order 4. We already have $\partial_x^4 u, \partial_y^4 u, \partial_x^3 \partial_y u$ and $\partial_y^3 \partial_x u$ from (16) and (17). The only missing one is $\partial_y^2 \partial_x^2 u$. But it is provided by the Euler equation. Now let us look at the derivatives of order 5. The only missing ones are $\partial_x^2 \partial_y^3 u$ and $\partial_y^2 \partial_x^3 u$. Differentiating the Euler equation, we get:

$$\partial_x^2 \partial_y^3 u = a\partial_x^2 \partial_y u + 2b\partial_x \partial_y^2 u + c\partial_y^3 u + d\partial_y u$$
$$\partial_y^2 \partial_x^3 u = a\partial_x^3 u + 2b\partial_x^2 \partial_y u + c\partial_x \partial_y^2 u + d\partial_x u$$

and so on, so interior regularity is proved. In fact, we have estimates (16) and (17) up to parts of the boundary and these estimates will now be used. We call these partial boundary estimates.

3.3 Regularity.

We now look at (1). In view of (16), we may now use as first test function:

$$\varphi(x,y) = (-1)^{m-1} \partial_x^m \left[\xi_s^{8+2m}(x)\, \partial_x^{m+2} u(x,y)\right] \tag{18}$$

We write the optimality condition for the (unpenalized) problem (P):

$$\int\int_{R_+^2} \left[B(u,\varphi) + (\partial_x \partial_y u)(\partial_x \partial_y \varphi)\right] dxdy = 0 \tag{19}$$

for all $\varphi \in E_2$ such that $\varphi(0) = 0$. Here B stands for:

$$B(u,\varphi) = a(\partial_x u)(\partial_x \varphi) + b\left[(\partial_x u)(\partial_y \varphi) + (\partial_y u)(\partial_x \varphi)\right] + c(\partial_y u)(\partial_y \varphi)$$

Plugging formula (18) into (19), and performing the same calculations as before, we find:

$$\begin{aligned} &\frac{1}{2}\left\|\xi_s^{4+m}\partial_x^{m+2}\partial_y u\right\|_{L^2}^2 + \frac{a}{2}\left\|\xi_s^{4+m}\partial_x^{m+2} u\right\|_{L^2}^2 \leq \\ &C_m\left(\left\|\xi_s^{4+m}\partial_x^{m+1} u\right\|_{L^2}^2 + \left\|\xi_s^{4+m}\partial_x^{m} u\right\|_{L^2}^2 + \left\|\xi_s^{4+m}\partial_x^{m+1}\partial_y u\right\|_{L^2}^2\right) \end{aligned} \tag{20}$$

For $m = 0$, we know that:

$$\left\|\xi_s^4 \partial_x u\right\|_{L^2}^2 + \left\|\xi_s^4 u\right\|_{L^2}^2 + \left\|\xi_s^4 \partial_x \partial_y u\right\|_{L^2}^2 \leq \|\partial_x u\|_{L^2}^2 + \|u\|_{L^2}^2 + \|\partial_x \partial_y u\|_{L^2}^2 \leq C(0)$$

since u minimizes J which is positive definite. Plugging into (20), we find that:

$$\frac{1}{2}\left\|\xi_s^{4+m}\partial_x^{m+2}\partial_y u\right\|_{L^2}^2 + \frac{a}{2}\left\|\xi_s^{4+m}\partial_x^{m+2} u\right\|_{L^2}^2 \leq C_1 C(0)$$

where the right-hand side does not depend on s. Letting $s \to 0$, we get:

$$\left\|\partial_x^2 \partial_y u\right\|_{L^2}^2 + \left\|\partial_x^2 u\right\|_{L^2}^2 \leq C_1 C(0) := C(1)$$

Arguing by induction, we find that there for every m there is a constant C_m such that:

$$\left\|\partial_x^{m+2}\partial_y u\right\|_{L^2}^2 + \left\|\partial_x^{m+2}u\right\|_{L^2}^2 \leq C(m)$$

So that all the derivatives $\partial_x^k\partial_y u$ and $\partial_x^k u$ are bounded in $L^2\left(R_+^2\right)$. Switching the role of x and y in the right-hand side of formula (18), we get the derivatives $\partial_y^k\partial_x u$ and $\partial_x^k u$. The missing derivatives are provided by the Euler equation. The regularity result is proved in the case $n=2$.

Remark 7 *The crucial point in this proof is that the cutoff function ξ_s is never differentiated.*

4 The proof in the general case.

It is long and rather tedious. There are two additional difficulties to face:

- the method gives some derivatives, but the other ones do not follow from the Euler equation.
- the solution u_ε to the penalized problem need not be smooth at the edges $E_{ij} = \{x_i = x_j = 0\}, i \neq j$.

Nevertheless, we still consider the penalized functional:

$$J_\varepsilon(u) = \int_{R_+^n}\left[\left(\partial_{x_1}\partial_{x_2}...\partial_{x_n}u\right)^2 + \left(Q\tilde{D}_{n-1}^n u, \tilde{D}_{n-1}^n u\right)\right]_n + \varepsilon\int_{R_+^n}\sum_{i=1}^n\left|\partial_i^{n+1}u\right|^2$$

and the associated problem:

$$(\mathrm{P}_\varepsilon)\ \left\{\begin{array}{c}\inf\ J_\varepsilon(u)\\ u\in E_n, u(0)=1\end{array}\right.$$

Denote by u_ε the minimizer of (P_ε). The problem is elliptic so, of course, u_ε is smooth except possibly near the E_{ij}. However, there is some regularity even there. Namely, for an elliptic problem with locally smooth (flattened) boundary, one proves regularity near the boundary in some directions by taking difference quotients of the equation in that direction, and then multiplying by the corresponding difference quotients of the functions and integrating. In fact, this method works just as well at nonsmooth boundary points provided, near such a point, the domain is translation invariant in the chosen direction. This idea was used, for example, in [3].

Applying this in our situation, we find that for every i, derivatives of the form $D^{n+1}\partial_{x_i}^m u_\varepsilon$ are square integrable in the region $x_i > s$ for any $s > 0$. The same is true in general for:

$$D^{n+1}\partial_{x_{i_1}}\partial_{x_{i_2}}...\partial_{x_{i_l}}u_\varepsilon$$

in the regions where $x_{i_k} \geq s > 0,\ 1 \leq k \leq l$. Hence, in the optimality conditions for (P_ε) we may insert test functions involving many derivatives in any one x_i.

Because of the length of the formulas, we will carry out the proof only in the case $n = 3$. The optimality condition for (P_ε) then takes the form:

$$\int_{R^3_+} \left[\left(\partial_x \partial_y \partial_z u_\varepsilon\right) \left(\partial_x \partial_y \partial_z \varphi\right) + \left(Q \tilde{D}^3_2 u_\varepsilon, \tilde{D}^3_2 \varphi \right) \right] +$$
$$\varepsilon \int_{R^3_+} \left[\left(\partial^4_x u_\varepsilon\right) \left(\partial^4_x \varphi\right) + \left(\partial^4_y u_\varepsilon\right) \left(\partial^4_x \varphi\right) + \left(\partial^4_z u_\varepsilon\right) \left(\partial^4_x \varphi\right) \right] = 0 \qquad (21)$$

for all φ in E_3 such that $\varphi(0) = 0$.

4.1 First test functions.

We start with:

$$\varphi(x, y, z) = (-1)^{k+1} \partial^k_x \left(\xi^{10+2k}_s (x) \, \partial^{k+2}_x u_\varepsilon \right) \qquad (22)$$

for $k \geq 0$. The exponent for ξ comes from the application of Lemma 4. For general n, it should be $2n + 4$, yielding 8 for $n = 2$ and 10 for $n = 3$.

Writing (22) into (21), we get, as before, two terms which will be estimated separately:

$$M(u_\varepsilon) + \varepsilon H(u_\varepsilon)$$

The first term $M(u_\varepsilon)$ is a linear combination of the following (here, as before, for simplification, we are ignoring certain product terms in $(Q\bullet, \bullet)$, like $u_x u$).

$$(-1)^{k+1} \int \left(\partial_x \partial_y \partial_z u_\varepsilon\right) \partial_x \partial_y \partial_z \left[\partial^k_x \left(\xi^{10+2k}_s \partial^{k+2}_x u_\varepsilon \right) \right]$$
$$(-1)^{k+1} \int \left(\partial_x \partial_y u_\varepsilon\right) \partial_x \partial_y \left[\partial^k_x \left(\xi^{10+2k}_s \partial^{k+2}_x u_\varepsilon \right) \right]$$
$$(-1)^{k+1} \int \left(\partial_x \partial_z u_\varepsilon\right) \partial_x \partial_z \left[\partial^k_x \left(\xi^{10+2k}_s \partial^{k+2}_x u_\varepsilon \right) \right]$$
$$(-1)^{k+1} \int \left(\partial_x u_\varepsilon\right) \partial_x \left[\partial^k_x \left(\xi^{10+2k}_s \partial^{k+2}_x u_\varepsilon \right) \right]$$
$$(-1)^{k+1} \int \left(\partial_y \partial_z u_\varepsilon\right) \partial_y \partial_z \left[\partial^k_x \left(\xi^{10+2k}_s \partial^{k+2}_x u_\varepsilon \right) \right]$$
$$(-1)^{k+1} \int \left(\partial_y u_\varepsilon\right) \partial_y \left[\partial^k_x \left(\xi^{10+2k}_s \partial^{k+2}_x u_\varepsilon \right) \right]$$
$$(-1)^{k+1} \int \left(\partial_z u_\varepsilon\right) \partial_z \left[\partial^k_x \left(\xi^{10+2k}_s \partial^{k+2}_x u_\varepsilon \right) \right]$$
$$(-1)^{k+1} \int \left(u_\varepsilon\right) \left[\partial^k_x \left(\xi^{10+2k}_s \partial^{k+2}_x u_\varepsilon \right) \right]$$

which, after integration by parts, turn out to be equivalent to the following:

$$\int \left|\xi_s^{5+k}\partial_x^{k+2}\partial_y\partial_z u_\varepsilon\right|^2$$
$$\int \left|\xi_s^{5+k}\partial_x^{k+2}\partial_y u_\varepsilon\right|^2$$
$$\int \left|\xi_s^{5+k}\partial_x^{k+2}\partial_z u_\varepsilon\right|^2$$
$$\int \left|\xi_s^{5+k}\partial_x^{k+2} u_\varepsilon\right|^2$$
$$-\int \xi_s^{10+2k}\left(\partial_x^k\partial_y\partial_z u_\varepsilon\right)\left(\partial_x^{k+2}\partial_y\partial_z u_\varepsilon\right)$$
$$-\int \xi_s^{10+2k}\left(\partial_x^k\partial_y u_\varepsilon\right)\left(\partial_x^{k+2}\partial_y u_\varepsilon\right)$$
$$-\int \xi_s^{10+2k}\left(\partial_x^k\partial_z u_\varepsilon\right)\left(\partial_x^{k+2}\partial_z u_\varepsilon\right)$$
$$-\int \xi_s^{10+2k}\left(\partial_x^k u_\varepsilon\right)\left(\partial_x^{k+2} u_\varepsilon\right)$$

They can all be estimated in terms of $\left\|\xi^{5+k}\partial_x^{k+2}\partial_y\partial_z u_\varepsilon\right\|_{L^2}$, $\left\|\xi^{5+k}\partial_x^{k+2}\partial_y u_\varepsilon\right\|_{L^2}$, $\left\|\xi^{5+k}\partial_x^{k+2}\partial_z u_\varepsilon\right\|_{L^2}$, $\left\|\xi^{5+k}\partial_x^{k+2} u_\varepsilon\right\|_{L^2}$, $\left\|\xi^{5+k}\partial_x^{k}\partial_y\partial_z u_\varepsilon\right\|_{L^2}$, $\left\|\xi^{5+k}\partial_x^{k}\partial_y u_\varepsilon\right\|_{L^2}$, $\left\|\xi^{5+k}\partial_x^{k}\partial_z u_\varepsilon\right\|_{L^2}$, $\left\|\xi^{5+k}\partial_x^{k} u_\varepsilon\right\|_{L^2}$.

We also have terms like:

$$H\left(u_\varepsilon\right)=(-1)^{k+1}\int\left(\partial_x^4 u_\varepsilon\right)\partial_x^4\partial_x^k\left(\xi_s^{10+2k}\partial_x^{k+2}u_\varepsilon\right)$$

which, after integration by parts, becomes:

$$H\left(u_\varepsilon\right)=\int\left(\partial_x^{k+5}u_\varepsilon\right)\partial_x^3\left(\xi_s^{10+2k}\partial_x^{k+2}u_\varepsilon\right)+\ldots$$

As in the case $n=2$, we use lemma 4 to estimate $H\left(u_\varepsilon\right)$ and then argue by induction to obtain the analogue of Proposition 3, namely:

$$\int \xi_s^{10+2k}\left[\left|\partial_x^{k+2}\partial_y\partial_z u_\varepsilon\right|^2+\left|\partial_x^{k+2}\partial_y u_\varepsilon\right|^2+\left|\partial_x^{k+2}\partial_z u_\varepsilon\right|^2+\left|\partial_x^{k+2}u_\varepsilon\right|^2\right]\leq C\left(k,s\right) \tag{23}$$

Then, as before, we may let $\varepsilon\to 0$ through a sequence; the limit of u_ε is our function u, so that the preceding inequality holds for u.

We now repeat the process: we use the test function (22) in the original problem, without ε, and we find that the estimate holds for u, with a constant $C\left(k\right)$ independent of s. Hence, letting $s\to 0$, we find:

$$\int\left[\left|\partial_x^{k+2}\partial_y\partial_z u\right|^2+\left|\partial_x^{k+2}\partial_y u\right|^2+\left|\partial_x^{k+2}\partial_z u\right|^2+\left|\partial_x^{k+2}u\right|^2\right]\leq C\left(k\right)$$

Similarly, using y, and then z, in place of x, we obtain:

$$\int \left[\left|\partial_y^{k+2}\partial_z\partial_x u\right|^2 + \left|\partial_y^{k+2}\partial_z u\right|^2 + \left|\partial_y^{k+2}\partial_x u\right|^2 + \left|\partial_y^{k+2} u\right|^2\right] \leq C\left(k\right)$$

$$\int \left[\left|\partial_z^{k+2}\partial_x\partial_y u\right|^2 + \left|\partial_z^{k+2}\partial_x u\right|^2 + \left|\partial_z^{k+2}\partial_y u\right|^2 + \left|\partial_z^{k+2} u\right|^2\right] \leq C\left(k\right)$$

4.2 Second test functions.

We now take, for $k, m \geq 0$ and $s > 0$:

$$\varphi\left(x,y,z\right) = \left(-1\right)^{m+k}\partial_x^k\partial_y^m\left(\xi_s^{10+2k}\left(x\right)\xi_s^{10+2m}\left(y\right)\partial_x^{k+2}\partial_y^{m+2}u_\varepsilon\right) \tag{24}$$

and insert this in (21). As before, we get an expression of the form $M\left(u_\varepsilon\right) + \varepsilon H\left(u_\varepsilon\right) = 0$. The first term is a linear combination of the following:

$$\left(-1\right)^{m+k}\int\left(\partial_x\partial_y\partial_z u_\varepsilon\right)\partial_x\partial_y\partial_z\left[\partial_x^k\partial_y^m\left(\xi_s^{10+2k}\left(x\right)\xi_s^{10+2m}\left(y\right)\partial_x^{k+2}\partial_y^{m+2}u_\varepsilon\right)\right]$$

$$\left(-1\right)^{m+k}\int\left(\partial_x\partial_y u_\varepsilon\right)\partial_x\partial_y\left[\partial_x^k\partial_y^m\left(\xi_s^{10+2k}\left(x\right)\xi_s^{10+2m}\left(y\right)\partial_x^{k+2}\partial_y^{m+2}u_\varepsilon\right)\right]$$

$$\left(-1\right)^{m+k}\int\left(\partial_x\partial_z u_\varepsilon\right)\partial_x\partial_z\left[\partial_x^k\partial_y^m\left(\xi_s^{10+2k}\left(x\right)\xi_s^{10+2m}\left(y\right)\partial_x^{k+2}\partial_y^{m+2}u_\varepsilon\right)\right]$$

$$\left(-1\right)^{m+k}\int\left(\partial_y\partial_z u_\varepsilon\right)\partial_y\partial_z\left[\partial_x^k\partial_y^m\left(\xi_s^{10+2k}\left(x\right)\xi_s^{10+2m}\left(y\right)\partial_x^{k+2}\partial_y^{m+2}u_\varepsilon\right)\right]$$

$$\left(-1\right)^{m+k}\int\left(\partial_z u_\varepsilon\right)\partial_z\left[\partial_x^k\partial_y^m\left(\xi_s^{10+2k}\left(x\right)\xi_s^{10+2m}\left(y\right)\partial_x^{k+2}\partial_y^{m+2}u_\varepsilon\right)\right]$$

$$\left(-1\right)^{m+k}\int\left(\partial_y u_\varepsilon\right)\partial_y\left[\partial_x^k\partial_y^m\left(\xi_s^{10+2k}\left(x\right)\xi_s^{10+2m}\left(y\right)\partial_x^{k+2}\partial_y^{m+2}u_\varepsilon\right)\right]$$

$$\left(-1\right)^{m+k}\int\left(\partial_x u_\varepsilon\right)\partial_x\left[\partial_x^k\partial_y^m\left(\xi_s^{10+2k}\left(x\right)\xi_s^{10+2m}\left(y\right)\partial_x^{k+2}\partial_y^{m+2}u_\varepsilon\right)\right]$$

$$\left(-1\right)^{m+k}\int u_\varepsilon\left[\partial_x^k\partial_y^m\left(\xi_s^{10+2k}\left(x\right)\xi_s^{10+2m}\left(y\right)\partial_x^{k+2}\partial_y^{m+2}u_\varepsilon\right)\right]$$

which, after integration by parts, turn out to be equal to the following:

$$\int \left|\xi_s^{5+k}(x)\,\xi_s^{5+m}(y)\,\partial_x^{k+2}\partial_y^{m+2}\partial_z u_\varepsilon^\eta\right|^2$$
$$\int \left|\xi_s^{5+k}(x)\,\xi_s^{5+m}(y)\,\partial_x^{k+2}\partial_y^{m+2} u_\varepsilon^\eta\right|^2$$
$$-\int \xi_s^{10+2k}(x)\,\xi_s^{10+2m}(y)\left(\partial_x^{k+2}\partial_y^{m}\partial_z u_\varepsilon^\eta\right)\left(\partial_x^{k+2}\partial_y^{m+2}\partial_z u_\varepsilon^\eta\right)$$
$$-\int \xi_s^{10+2k}(x)\,\xi_s^{10+2m}(y)\left(\partial_x^{k}\partial_y^{m+2}\partial_z u_\varepsilon^\eta\right)\left(\partial_x^{k+2}\partial_y^{m+2}\partial_z u_\varepsilon^\eta\right)$$
$$\int \xi_s^{10+2k}(x)\,\xi_s^{10+2m}(y)\left(\partial_x^{k}\partial_y^{m}\partial_z u_\varepsilon^\eta\right)\left(\partial_x^{k+2}\partial_y^{m+2}\partial_z u_\varepsilon^\eta\right)$$
$$-\int \xi_s^{10+2k}(x)\,\xi_s^{10+2m}(y)\left(\partial_x^{k}\partial_y^{m+1} u_\varepsilon^\eta\right)\left(\partial_x^{k+2}\partial_y^{m+2} u_\varepsilon^\eta\right)$$
$$-\int \xi_s^{10+2k}(x)\,\xi_s^{10+2m}(y)\left(\partial_x^{k+1}\partial_y^{m} u_\varepsilon^\eta\right)\left(\partial_x^{k+2}\partial_y^{m+2} u_\varepsilon^\eta\right)$$
$$\int \xi_s^{10+2k}(x)\,\xi_s^{10+2m}(y)\left(\partial_x^{k}\partial_y^{m} u_\varepsilon^\eta\right)\left(\partial_x^{k+2}\partial_y^{m+2} u_\varepsilon^\eta\right)$$

The term $H(u_\varepsilon)$, after integration by parts, is:

$$H(u_\varepsilon) = \int \xi_s^{10+2k}(x)\,\xi_s^{10+2m}(y)\left\{\left|\partial_x^{k+5}\partial_y^{m+1}u_\varepsilon\right|^2 + \left|\partial_x^{k+1}\partial_y^{m+5}u_\varepsilon\right|^2 + \left|\partial_x^{k+1}\partial_y^{m+1}\partial_z^4 u_\varepsilon\right|^2\right\} + \ldots$$

As usual, this term is handled by Lemma 4. For the terms in $M(u_\varepsilon)$, we use induction on m, starting with $m = 0$, which is a term controlled by (23). Thus we find:

$$\int \xi_s^{10+2k}(x)\,\xi_s^{10+2m}(y)\left\{\left|\partial_x^{k+2}\partial_y^{m+2}\partial_z u_\varepsilon\right|^2 + \left|\partial_x^{k+2}\partial_y^{m+2}u_\varepsilon\right|^2\right\} \le C(k,m,s) \tag{25}$$

As before, using this information, we insert (24), with u_ε replaced by u, in the optimality condition for (P). Repeating the procedure, we find that the preceding estimate holds for u with C independent of s. Letting $s \to 0$, we get:

$$\int \left\{\left|\partial_x^{k+2}\partial_y^{m+2}\partial_z u\right|^2 + \left|\partial_x^{k+2}\partial_y^{m+2}u\right|^2\right\} \le C(k,m) \tag{26}$$

Similarly, interchanging x, y, and z, we obtain also:

$$\int \left\{\left|\partial_y^{k+2}\partial_z^{m+2}\partial_x u\right|^2 + \left|\partial_y^{k+2}\partial_z^{m+2}u\right|^2\right\} \le C(k,m) \tag{27}$$

$$\int \left\{\left|\partial_z^{k+2}\partial_x^{m+2}\partial_y u\right|^2 + \left|\partial_z^{k+2}\partial_x^{m+2}u\right|^2\right\} \le C(k,m) \tag{28}$$

4.3 Third test functions.

For $k, m, p \ge 0$ and $s \ge 0$, we now take as test function:

$$\varphi(x,y,z) = (-1)^{k+m+p+1}\,\partial_x^k\partial_y^m\partial_z^p\left(g^2\partial_x^{k+2}\partial_y^{m+2}\partial_z^{p+2}u_\varepsilon\right) \tag{29}$$

where:

$$g(x,y,z) = \xi_s^{5+k}(x)\,\xi_s^{5+m}(y)\,\xi_s^{5+p}(y)$$

Insert this φ into the optimality conditions (21). As before, we get an expression of the form $M(u_\varepsilon)+\varepsilon H(u_\varepsilon)=0$. The first term is a linear combination of the following:

$$(-1)^{k+m+p+1}\int (\partial_x\partial_y\partial_z u_\varepsilon)\,\partial_x\partial_y\partial_z\left[\partial_x^k\partial_y^m\partial_z^p\left(g^2\partial_x^{k+2}\partial_y^{m+2}\partial_z^{p+2}u_\varepsilon\right)\right]$$
$$(-1)^{k+m+p+1}\int (\partial_x\partial_y u_\varepsilon)\,\partial_x\partial_y\left[\partial_x^k\partial_y^m\partial_z^p\left(g^2\partial_x^{k+2}\partial_y^{m+2}\partial_z^{p+2}u_\varepsilon\right)\right]$$
$$(-1)^{k+m+p+1}\int (\partial_x\partial_z u_\varepsilon)\,\partial_x\partial_z\left[\partial_x^k\partial_y^m\partial_z^p\left(g^2\partial_x^{k+2}\partial_y^{m+2}\partial_z^{p+2}u_\varepsilon\right)\right]$$
$$(-1)^{k+m+p+1}\int (\partial_y\partial_z u_\varepsilon)\,\partial_y\partial_z\left[\partial_x^k\partial_y^m\partial_z^p\left(g^2\partial_x^{k+2}\partial_y^{m+2}\partial_z^{p+2}u_\varepsilon\right)\right]$$
$$(-1)^{k+m+p+1}\int (\partial_x u_\varepsilon)\,\partial_x\left[\partial_x^k\partial_y^m\partial_z^p\left(g^2\partial_x^{k+2}\partial_y^{m+2}\partial_z^{p+2}u_\varepsilon\right)\right]$$
$$(-1)^{k+m+p+1}\int (\partial_y u_\varepsilon)\,\partial_y\left[\partial_x^k\partial_y^m\partial_z^p\left(g^2\partial_x^{k+2}\partial_y^{m+2}\partial_z^{p+2}u_\varepsilon\right)\right]$$
$$(-1)^{k+m+p+1}\int (\partial_z u_\varepsilon)\,\partial_z\left[\partial_x^k\partial_y^m\partial_z^p\left(g^2\partial_x^{k+2}\partial_y^{m+2}\partial_z^{p+2}u_\varepsilon\right)\right]$$
$$(-1)^{k+m+p+1}\int u_\varepsilon\left[\partial_x^k\partial_y^m\partial_z^p\left(g^2\partial_x^{k+2}\partial_y^{m+2}\partial_z^{p+2}u_\varepsilon\right)\right]$$

which, after integration by parts, turns out to be equal to the following:

$$\int \left|g\partial_x^{k+2}\partial_y^{m+2}\partial_z^{p+2}u_\varepsilon\right|^2$$
$$-\int g^2\left(\partial_x^{k+2}\partial_y^{m+2}\partial_z^{p}u_\varepsilon\right)\left(\partial_x^{k+2}\partial_y^{m+2}\partial_z^{p+2}u_\varepsilon\right)$$
$$-\int g^2\left(\partial_x^{k+2}\partial_y^{m}\partial_z^{p+2}u_\varepsilon\right)\left(\partial_x^{k+2}\partial_y^{m+2}\partial_z^{p+2}u_\varepsilon\right)$$
$$-\int g^2\left(\partial_x^{k}\partial_y^{m+2}\partial_z^{p+2}u_\varepsilon\right)\left(\partial_x^{k+2}\partial_y^{m+2}\partial_z^{p+2}u_\varepsilon\right)$$
$$\int g^2\left(\partial_x^{k+2}\partial_y^{m}\partial_z^{p}u_\varepsilon\right)\left(\partial_x^{k+2}\partial_y^{m+2}\partial_z^{p+2}u_\varepsilon\right)$$
$$\int g^2\left(\partial_x^{k}\partial_y^{m+2}\partial_z^{p}u_\varepsilon\right)\left(\partial_x^{k+2}\partial_y^{m+2}\partial_z^{p+2}u_\varepsilon\right)$$
$$\int g^2\left(\partial_x^{k}\partial_y^{m}\partial_z^{p+2}u_\varepsilon\right)\left(\partial_x^{k+2}\partial_y^{m+2}\partial_z^{p+2}u_\varepsilon\right)$$
$$-\int g^2\left(\partial_x^{k}\partial_y^{m}\partial_z^{p}u_\varepsilon\right)\left(\partial_x^{k+2}\partial_y^{m+2}\partial_z^{p+2}u_\varepsilon\right)$$

We don't bother to write down the expression for $H(u_\varepsilon)$ .Arguing as before, we use Lemma 4 an induction on p, starting the induction with the aid of (25).

We thus obtain estimates for $\partial_x^{k+2}\partial_y^{m+2}\partial_z^{p+2}u_\varepsilon$. Then we let $\varepsilon \to 0$ and apply the argument again to the original problem using the test function (29), with u_ε replaced by u. Letting $s \to 0$, we obtain finally the desired result:

$$\int \left|\partial_x^{k+2}\partial_y^{m+2}\partial_z^{p+2}u\right|^2 \leq C\,(k,m,p) \text{ for } m,k,p \geq 0$$

5 Remarks.

Is u positive ? This is a natural question, and we don't know the answer. It is easy to see that in R_+^n, we have:

$$-1 < u < 1$$

Namely, from the estimates we know that $u \to 0$ as $|x| \to \infty$. It there was a point $x_0 \in R_+^n$ where $u = 1$, then, denoting by $I\,(x)$ the integrand in $J\,(u)$, which is positive, we would have:

$$\int_{x_0+R_+^n} I \geq J\,(u) > \int_{x_0+R_+^n} I$$

which is impossible. By considering $-u$, we find that $u > -1$.

One may ask whether the regularity result holds in case the quadratic form Q has variable coefficients. In case it is uniformly positive definite, with bounded coefficients, and, for any m, all derivatives of the coefficients up to order m are bounded by constants $A\,(m)$, then the answer is yes. The proof we have given works also in that case. During the integration by parts, derivatives of the coefficients will appear, but only in lower order terms, which are handled using Lemma 4 and induction.

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