From Ramsey to Thom: a classical problem in the calculus of variations leading to an implicit differential equation

Ivar Ekeland,
Canada Research Chair in Mathematical Economics, UBC

March 23, 2010

Abstract

In 1928, motivated by conversations with Keynes, Ramsey formulated an infinite-horizon problem in the calculus of variations. This problem is now classical in economic theory, and its solution lies at the heart of our understanding of economic growth. On the other hand, from the mathematical point of view, it was never solved in a satisfactory manner: In this paper, we give what we believe is the first complete mathematical treatment of the problem, and we show that its solution relies on solving an implicit differential equation. Such equations were first studied by Thom, and we use the geometric method he advocated. We then extend the Ramsey problem to non-constant discount rates, along the lines of Ekeland and Lazrak. In that case, there is time-inconsistency, meaning that optimal growth no longer is a relevant concept for economics, and has to be replaced with equlibrium growth. We briefly define what we mean by equilibrium growth, and proceed to prove that such a path actually exists, The problem, once again, reduces to solving an implicit differential equation, but this time the dimension is higher, and the analysis is more complicated: geometry is not enough, and we have to appeal to the central manifold theorem.

1 Introduction

In this paper, we show, as usual, that elementary problems in economic theory can lead to deep mathematical problems. Our example this time is the theory of economic growth. It was initiated by Frank Ramsey, in a seminal paper [15] written in 1928, at the tender age of 25. To quote the opening sentence of this paper, "the first problem I want to solve is this: how much of its income should a nation save ?". Ramsey's formulation of the problem, and the answer he gave, lie at the heart of the modern theory of economic growth. Most advanced textbooks in macroeconomics start with stating and solving some version of Ramsey's problem (see [1]. [4], [16], [3] for instance).

To the best of my knowledge and understanding, none of the solutions proposed for solving the Ramsey problem is correct - with one exception, of course, Ramsey himself, whose own statement was different than the one which is now in current use. This is because all these solutions rely on the so-called "transversality condition at infinity", an elusive result in optimal control, which simply does not hold in this context (more about this later). The first aim of this paper therefore is to give a complete a correct solution of Ramsey's problem, in the current form.

To do this, we have to resort to Caratheodory's method, that, is, to solve the Hamilton-Jacobi equation under suitable boundary conditions. The Hamilton-Jacobi equation is an ordinary differential equation, as was to be expected, but it is in *implicit form*, that is, it is written $\varphi\left(\dot{k}, k, t\right) = 0$, and our boundary condition $k(t_0) = k_0$ puts us precisely at a point $\left(\dot{k}_0, k_0, t_0\right)$where $\varphi\left(\dot{k}_0, k_0, t_0\right) = 0$ but $\frac{\partial \varphi}{\partial \dot{k}}\left(\dot{k}_0, k_0, t_0\right)$. In other words, the equation cannot be solved with respect to $\dot{k}$ and written in the standard form $\dot{k} = f(t, k)$.

To my knowledge, implicit differential equations were introduced into mathematics by René Thom [17] Strangely enough, in view of the rich geometry of the subject, it has not prospered: we do not know much more about implicit differential equations than Thom did in the seventies. We know the generic singularities of autonomous differential equations in the plane (see **??**), but this is not enough for our purposes. I am indebted to Francois Laudenbach [12] for showing me how to analyse the problem using Thom's approach. It turns out that the dynamics can be fully analysed, and lead to a solution of the Hamilton-Jacobi equation satisfying the required boundary condition, so that Ramsey's problem does have a solution. In my view, however, this is less interesting than the geometric picture which emerges, and which, alas, I do not have enough skill to draw.

In the second part of the paper, we extend Ramsey's problem to a new situation. It has now become important in economics to look at non-constant discount rates. These arise either from psychological considerations (individuals tend to be much more patient about the distant future than about the immediate one, see [10]) or from considerations of intergenerationa equity (see [7] or [8]). Such discount rates are known to give rise to time-inconsistency (more about this later), so that optimal trajectories, although they still exist mathematically, are economically irrelevant. One needs a new concept of solution, namely the equilibrium strategies, which we define according to [7] and [8]. To show that they exist, one again uses Carathedory's method, which now leads to a system of two differential equations, instead of a single one as in the case of constant discounting. This system is in implicit form, and we will apply Thom's geometric method again. This time, however, it does not by itself lead to a full solution, and we have to inject a more powerful tool, namely the central manifold theorem. The resulting dynamics look like a suspension in dimension three of the dynamis we found in dimension two.

To conclude, let me point out explicitly what is novel in this paper. The treatment of Ramsey's problem in the first part certainly is new, and in my view the only mathematically correct one. The definition of equilibrium strategies in the second part belongs to Ekeland and Lazrak, in their earlier paper, as well as the existence result in the case of logarithmic utility, $u(c) = \ln c$.

2 The Ramsey problem in the calculus of variations

2.1 Statement

Given functions u and f of one variable, and a number $\rho > 0$, find the function $c(t)$ which will solve:

$$\max_{c(\cdot)} \int_0^\infty u(c(t))\, e^{-\rho t} dt \tag{1}$$

$$\frac{dk}{dt} = f(k) - c(t) \tag{2}$$

$$k(0) = k_0 > 0, \quad k(t) \geq 0, \quad c(t) \geq 0 \tag{3}$$

This is stated as a control problem, where the control variable is c and the state variable is k. By expressing c in terms of k and dk/dt through equation (2), and substituting into the criterion (1), one can readily express the same problem as an infinite-horizon problem in the calculus of variations, namely:

$$\max_{k(\cdot)} \int_0^\infty u\left(f(k) - \frac{dk}{dt}\right) e^{-\rho t} dt, \quad k(0) = k_0 \tag{4}$$

$$k(0) = k_0 > 0, \quad k(t) \geq 0, \quad c(t) \geq 0 \tag{5}$$

Let us give some economic insight into this problem. Ramsey, under the influence of Keynes, is investigating optimal government policy. Society is reduced to a single, infinite-lived individual (the representative consumer). there is only one good in the economy, which can either be consumed (in which case it is denoted by c) or used as capital to produce more of the same (in which case it is denoted by k). If the level of capital at time t is k, the economy will produce $f(k)\, dt$ during the interval of time dt, part of which will be consumed and the rest saved for reinvestment. Equation (2) thus expresses budget balance: savings are equal to production minus consumption. Note that one can disinvest as well, ie consume more than one produces, which may eventually lead to $k = 0$, where production stops because there is no more capital.

Assumption 1. $u : (0, \ \infty) \longrightarrow R$ (the utility function) is concave, increasing, and C^2, with $u''(x) < 0$ and

$$\lim_{c \longrightarrow 0} u'(c) = -\infty, \qquad \lim_{c \longrightarrow \infty} u'(c) = 0$$

The typical examples to bear in mind are the power utilities, $u(c) = \frac{1}{1-\theta} c^{1-\theta}$ with $\theta > 0$. For $\theta = 1$, one sets $u(c) = \ln c$

Assumption 2. $f : [0, \ \infty] \longrightarrow R$ (the production function) is concave, increasing, and C^2, with$f"(x) < 0$ and:

$$f(0) = 0$$
$$\lim_{k \longrightarrow 0} f'(k) = -\infty, \quad \lim_{k \longrightarrow \infty} f'(k) = 0$$

Before I proceed, let me mention that Ramsey himself used a different (undiscounted) integral criterion than (1). The idea of discounting welfare by the factor $e^{-\rho t}$ came later and is now prevalent in economic modeling, so that the problem of maximizing (1) subject to (2) and (3) is still named after Ramsey.

2.2 The Euler-Lagrange equation

Suppose that the Ramsey problem in the form (4), (5) has a C^2 solution $k(t)$ with $k(t) > 0$ and $c(t) > 0$. The classical Euler-Lagrange equation then holds, yielding:

$$\frac{dc}{dt} \frac{u''(c)}{u'(c)} + f'(k) = \rho$$

with $c = f(k) - dk/dt$. This gives us a system of two ODEs for $k(t)$ and $c(t)$:

$$\frac{dk}{dt} = f(k) - (n+g)k - c \tag{6}$$
$$\frac{dc}{dt} = -\frac{1}{\theta} (\rho + \theta g + n - f'(k)) c \tag{7}$$

to which one must add the initial condition:

$$k(0) = k_0 \tag{8}$$

By inspection, one sees that there is a single fixed point (k_∞, c_∞), satisfying:

$$f'(k_\infty) = \rho \tag{9}$$
$$c_\infty = f(k_\infty) \tag{10}$$

The vertical line $f'(k) = \rho$ and the curve $f(k) = c$ divide the positive orthant R_+^2 in four regions. We can find out the sign of dk/dt and dc/dt in each region, and draw the phase diagram. It follows that the fixed point (k_∞, c_∞) is unstable, that there is a single trajectory $\left(\bar{k}(t), \bar{c}(t)\right)$ which converges to that fixed point:

$$\bar{k}(t) \longrightarrow k_\infty \text{ and } \bar{c}(t) \longrightarrow c_\infty \text{ when } t \longrightarrow \infty$$

while all the others go to one of the boundaries, $c = 0$ or $k = 0$.

It is well known that $\left(\bar{k}(t), \bar{c}(t)\right)$ is the optimal solution to the Ramsey problem. I have found no satisfactory proof in the literature. The traditional way to go at the problem is to point out that the Euler equation should be supplemented by two boundary conditions, one of which is known, namely $k(t_0) = k_0$, while the other should describe the behaviour of $k(t)$ when $t \longrightarrow \infty$. This is the celebrated "transversality condition at infinity", several versions of which have been given, either in discrete (see [[9]]) or in continuous (see [[2]], [[13]], [11]) time. Unfortunately, none of them applies to the present situation, where the candidate solution is the only one which does not go to the boundary, not to mention that the Euler equation, even with the appropriate boundary conditions, is necessary and not sufficient. So we will prove that $\left(\bar{k}(t), \bar{c}(t)\right)$ is the optimal solution by a different method, which is due to Caratheodory.

2.3 The royal road of Caratheodory.

Define a function $\tilde{u}(y)$ as follows:

$$\tilde{u}(y) := \max_x \{u(x) - xy\} \tag{11}$$

It follows from Assumption 1 that $\tilde{u} : (0, \infty) \longrightarrow R$ is convex and C^2, with $\tilde{u}''(y) > 0$. For instance,

$$u(x) = \frac{1}{1-\theta} x^{1-\theta}, \ \theta > 0 \implies \tilde{u}(y) = \frac{\theta}{1-\theta} y^{-\frac{1-\theta}{\theta}}$$

Formula (11) can be written in the following way:

$$\max_x \{u(x) - \tilde{u}(y) - xy\} = 0$$

from which we derive easily new relations:

$$u(c) = \min_x \{\tilde{u}(x) + cx\} \tag{12}$$

$$u'(x) = y \Longleftrightarrow y = -\tilde{u}'(x) \tag{13}$$

Now consider the following equation for some unknown function $V(k)$:

$$\tilde{u}(V') + fV' = \rho V \tag{14}$$

Let us define a *feasible path* $(k(t), c(t))$ as a solution of equation (2).

Theorem 1 *Suppose (14) has a C^2 solution $V(k)$ on open interval I containing k_∞ such that, for any $k_0 \in I$, the solution $\bar{k}(t)$ of the Cauchy problem:*

$$\frac{dk}{dt} = f(k) + \tilde{u}'(V'), \ k(0) = k_0 \tag{15}$$

stays in I and converges to k_∞ when $t \longrightarrow \infty$. Then, for any starting point $k_0 > 0$, the path given by $\bar{c}(t) = -\tilde{u}\left(V'\left(\bar{k}(t)\right)\right)$ is optimal among all feasible C^2 paths $(k(t), c(t))$ such that

$$k(0) = k_0, \quad \lim \sup_{T \longrightarrow \infty} e^{-\beta t} V((k(T))) \geq 0$$

and we have:

$$V(k_0) = \max\left\{\int_0^\infty u(c(t))\, e^{-\rho t} dt \mid \frac{dk}{dt} = f(k) - c,\ k(0) = k_0\right\} \tag{16}$$

Proof. Equation (14) can be rewritten as follows:

$$\max_x \{u(x) - xV'(k)\} + f(k)\, V'(k) - \rho V(k) = 0 \tag{17}$$

Consider any path $c(t), k(t)$ starting from k_0. Setting $k = k(t),\ x = c(t)$ in the preceding equation, and integrating, we get:

$$\int_0^T e^{-\rho t} \left[u(c(t)) - c(t)\, V'(k(t)) + f(k(t))\, V'(k(t)) - \rho V(k(t))\right] dt \leq 0 \text{ for every } T > 0$$

Since the path is feasible, we have $c = f - dk/dt$ and the left-hand side can be rewritten as follows:

$$\int_0^T e^{-\rho t} u(c(t))\, dt + \int_0^T e^{-\rho t} \left[\frac{dk}{dt} V'(k(t)) - \rho V(k(t))\right] dt \leq 0$$

$$\int_0^T e^{-\rho t} u(c(t))\, dt + e^{-\rho t} V(k(t)) \big|_0^T \leq 0$$

$$\int_0^T e^{-\beta t} u(c(t))\, dt + \lim \sup_{T \longrightarrow \infty} e^{-\beta t} V((k(T))) \leq V(k(0))$$

Letting $T \longrightarrow \infty$, we find:

$$\int_0^\infty e^{-\beta t} u(c(t))\, dt \leq V(k_0) \tag{18}$$

On the other hand, setting $x = \bar{c}(t) = \tilde{u}\left(V'\left(\bar{k}(t)\right)\right)$ achieves the maximum on the left-hand side of (17), leading to:

$$\int_0^\infty e^{-\beta t} u(\bar{c}(t))\, dt = V(k_0) \tag{19}$$

Comparing (18) and (19), we find that the feasible path $\left(\bar{k}(t), \bar{c}(t)\right)$ is optimal, and equation (16) then follows from (19). ■

Note that the C^2 regularity of $V(k)$ everywhere (including at k_∞) has played a crucial role in the proof.

Corollary 2 *If such a solution exists, it must satisfy:*

$$V(k_\infty) = \frac{1}{\rho} V(f(k_\infty))$$

Proof. Taking $k_0 = k_\infty$ in (16), we find $k(t) = k_\infty$, so that:

$$V(k_\infty) = \int_0^\infty e^{-\rho t} f(k_\infty)\, dt = \frac{1}{\rho} f(k_\infty) \tag{20}$$

■

Equation (14) is known in the literature as the Hamilton-Jacobi equation.

2.4 The dynamics of the Hamilton-Jacobi equation

The Ramsey problem is now reduced to finding a C^2 function $V(k)$ which satisfies the conditions of Theorem 1. It should be defined in a neighbourhood of k_∞, where k_∞ is defined by (9).

We thus have to solve the boundary-value problem:

$$\tilde{u}(V') + fV' = \rho V \tag{21}$$

$$V(k_\infty) = \frac{1}{\rho} f(k_\infty) \tag{22}$$

Equation (21) is an ODE in *implicit form*, that is, it is not solved with respect to V'.

As noted above (formula (12)), we have:

$$\min_x \{\tilde{u}(x) + xf(k)\} = u(f(k))$$

So equation (21) has:

- no solutions if $\rho V(k) < u(f(k))$,
- two solutions if $\rho V(k) > u(f(k))$
- one if $\rho V(k) = u(f(k))$.

In other works, the curve $\mathcal{C}$ given by $V = \frac{1}{\rho} u(f(k))$ splits the phase plane (k, V) in two regions. In the region above the curve, there are two possible directions of motion at every point, in the region below there are none. On the curve $\mathcal{C}$ itself, the two directions of the upper region come together, so there is a single (non-zero) vector of motion. If it points towards the curve, the trajectory cannot be continued after the hitting time, since it enters the region of non-existence, and if it points away from the curve, the trajectory is not defined before the hitting time. In other words, for every (k, V) such that $\rho V > u(f(k))$, there are two distinct solutions of (21) intersecting transversally at (k, V), and these solutions cannot be continued into the region where $\rho V < u(f(k))$, even though they hit the boundary $\mathcal{C}$ with non-zero velocity. Unfortunately, our initial condition (22) lies on $\mathcal{C}$, and we want the solution to exist both before and after the hitting time. Geometrically speaking, this means we are looking for a solution of (21) which hits the boundary tangentially.

We willl use a trick which goes back to René Thom, and which I learned from Francois Laudenbach. Rewrite (21) as a Pfaff system:

$$dV = pdk \tag{23}$$

$$\tilde{u}(p) + f(k)p = \rho V \tag{24}$$

and consider the initial-value problem:

$$V(k_0) = V_0 \tag{25}$$

Differentiating (24) leads to:

$$\left(\tilde{u}'(p) + f(k)\right) dp = p\left(\rho - f'(k)\right) dk \tag{26}$$

Case 1 $\rho V_0 > u\left(f\left(k_0\right)\right), \quad f'\left(k_0\right) \neq \rho$ In that case, we have:

$$\min_p \left\{\tilde{u}(p) + p f\left(k_0\right)\right\} < \rho V_0 \tag{27}$$

and the equation:

$$\tilde{u}(p) + f\left(k_0\right) p = \rho V_0$$

has two distinct solutions $p_1 \neq p_2$. Note that, since neither p_1 nor p_2 minimize the left-hand side of (27), we have $\tilde{u}'\left(p_i\right) + f(k) \neq 0$ for $i = 1, 2$.

Consider the initial-value problem:

$$\frac{dp}{dk} = \frac{p\left(\rho - f'(k)\right)}{\tilde{u}'(p) + f(k)}, \quad p\left(k_0\right) = p_i$$

It has a well-defined smooth solution $p_i(k)$, defined in a neighbourhood or $k = k_0$. We then consider the function:

$$V_i(k) := \frac{1}{\rho}\left(\tilde{u}\left(p(k)\right) + f(k)\, p_i(k)\right)$$

We have:

$$V_i\left(k_0\right) := \frac{1}{\rho}\left(\tilde{u}\left(p_i\right) + f\left(k_0\right) p_i\right) = V_0$$

so the function $V_i(k)$ solves the initial-value problem (**??**), (25). Taking $i = 1, 2$, we find two solutions $V_1(k)$ and $V_2(k)$ of that same initial-value problem, with $V_i'\left(k_0\right) = p_i$, for $i = 1, 2$. Since $p_1 \neq p_2$, the two solutions are not tangent at k_0.

Case 2 $\rho V_0 = u\left(f\left(k_0\right)\right), \quad f'\left(k_0\right) \neq \rho$ Since $\rho V_0 = u\left(f\left(k_0\right)\right)$, the equation $\tilde{u}(p) + f\left(k_0\right) p = \rho V_0$ has a single solution p_0, and we have:

$$\tilde{u}'\left(p_0\right) + f\left(k_0\right) = 0$$

In that case, we shall use the same system (23), (24), but we will take p instead of k as the independent variable. We consider the initial-value problem:

$$\frac{dk}{dp} = \frac{\tilde{u}'(p) + f(k)}{p\left(\rho - f'(k)\right)}, \quad k\left(p_0\right) = k_0 \tag{28}$$

It has a smooth solution $k(p)$, defined in a neighbourhood of $p = p_0$. We associate with it a curve in the phase space (k, V), defined in parametric form by the equations:

$$k = k(p)$$
$$V = \frac{1}{\rho}\left(\tilde{u}(p) + f\left(k(p)\right) p\right)$$

We have:

$$\frac{dk}{dp}(p_0) = 0$$
$$\frac{dV}{dp}(p_0) = \frac{1}{\rho}\left(\tilde{u}'(p_0) + p_0 f'(k_0)\frac{dk_0}{dp} + f(k_0)\right) = 0$$

So the curve has a cusp at (k_0, V_0). The tangent at the cusp is given by:

$$\frac{d^2k}{dp^2}(p_0) = \frac{\tilde{u}''(p_0)}{p_0(\rho - f'(k_0))} \neq 0$$
$$\frac{d^2V}{dp^2}(p_0) = \frac{1}{\rho}\left(\tilde{u}''(p_0) + f'(k_0)\frac{d^2k}{dp^2}(p_0)\right) = \frac{\tilde{u}''(p_0)}{\rho - f'(k_0)} \neq 0$$

The slope m of the tangent is given by:

$$m = \frac{d^2V}{dp^2}(p_0) / \frac{d^2k}{dp^2}(p_0) = p_0$$

and the cusp lies on both sides of its tangent. The smooth solution $k(p)$ of the initial-value problem (28) does not give rise to a function $V(k)$ with $V(k_0) = V_0$ and satisfying the HJB equation (**??**) in some neighbourhood of k_0. It gives rise to two smooth functions $V_1(k)$ and $V_2(k)$, with $V_1(k_0) = V_2(k_0) = V_0$, both of which are defined on the same interval $(k_0 - \varepsilon,\ k_0]$ (or $[k_0,\ k_0 + \varepsilon)$), and which satisfy the HJB equation on that semi-closed interval.

In other words, the two trajectories of Case 1 come together tangentially at the boundary of the domain, and stop there: they cannot be continued into the region under the curve $\rho V = u(f(k))$, even though they hit the curve with non-zero velocity.

Case 3 $\rho V_\infty = u(f(k_\infty))$, $\quad f'(k_\infty) = \rho$ We have to solve (26) with the initial condition, $k(p_\infty) = k_\infty$, where k_∞ is as above and p_∞ is given by:

$$\tilde{u}(p_\infty) + f(k_\infty)p_\infty = \rho V(k_\infty) \Longleftrightarrow p_\infty = u'(f(k_\infty)) \Longleftrightarrow f(k_\infty) = -\tilde{u}'(p_\infty)$$

Consider the Cauchy problem, for two functions $k(t)$ and $p(t)$:

$$\frac{dk}{dt} = \tilde{u}'(p) + f(k), \quad k(0) = k_\infty$$
$$\frac{dp}{dt} = p(\rho - f'(k)), \quad p(0) = p_\infty$$

The linearized equation is:

$$\begin{pmatrix} f'(k_\infty) & \tilde{u}''(p_\infty) \\ -p_\infty f''(k_\infty) & \rho - f'(k) \end{pmatrix} = \begin{pmatrix} \rho & \tilde{u}''(p_\infty) \\ -u'(f(k_\infty))f''(k_\infty) & 0 \end{pmatrix}$$

and the characteristic equation:

$$\lambda^2 - \rho\lambda + u'\left(f\left(k_\infty\right)\right) f''\left(k_\infty\right) \tilde{u}''\left(p_\infty\right) = 0$$
$$\lambda = \rho \pm \sqrt{\rho^2 - 4\tilde{u}''\left(p_\infty\right) u'\left(f\left(k_\infty\right)\right) f''\left(k_\infty\right)}$$

The corresponding eigenvectors are given by:

$$-u'\left(f\left(k_\infty\right)\right) f''\left(k_\infty\right) dk + \lambda dp = 0$$

This is a hyperbolic fixed point, with a stable and unstable manifold, $\mathcal{S}$ and $\mathcal{U}$. The tangents at the origin are:

$$\frac{dp_s}{dk_s}\left(k_\infty\right) = \frac{\lambda_- - \rho}{\tilde{u}''\left(p_\infty\right)} = \frac{u'\left(f\left(k_\infty\right)\right) f''\left(k_\infty\right)}{\lambda_-}$$
$$\frac{dp_u}{dk_u}\left(k_\infty\right) = \frac{\lambda_+ - \rho}{\tilde{u}''\left(p_\infty\right)} = \frac{u'\left(f\left(k_\infty\right)\right) f''\left(k_\infty\right)}{\lambda_+}$$

Choose a smooth parametrization $\left(k_s\left(x\right), p_s\left(x\right)\right)$ and $\left(k_u\left(y\right), p_u\left(y\right)\right)$ for these manifolds. Substituting, we get two curves:

$$k = k_s\left(x\right), \quad V = \frac{\tilde{u}\left(p_s\left(x\right)\right) + f\left(k_s\left(x\right)\right) p_s\left(x\right)}{\rho}$$
$$k = k_u\left(y\right), \quad V = V = \frac{\tilde{u}\left(p_u\left(y\right)\right) + f\left(k_u\left(y\right)\right) p_s\left(y\right)}{\rho}$$

Both curves are graphs, of functions $V_s\left(k\right)$ and $V_u\left(k\right)$ respectively. Both of them satisfy the HJB equation in a neighbourhood of k_∞, with $V_s\left(k_\infty\right) = V_u\left(k_\infty\right) = V_\infty$. We have:

$$\frac{dV_s}{dk}\left(k_\infty\right) = \frac{1}{\rho}\left(\tilde{u}'\left(p_\infty\right)\frac{dp}{dk} + p_\infty f'\left(k_\infty\right) + f\left(k_\infty\right)\frac{dp}{dk}\right)$$
$$= \frac{1}{\rho}\left(p_\infty\left(\rho - f'\left(k_\infty\right)\right) + p_\infty f'\left(k_\infty\right)\right) = p_\infty$$
$$\frac{d^2V_s}{dk^2}\left(k_\infty\right) = \frac{dp_s}{dk_s}\left(k_\infty\right) = \frac{u'\left(f\left(k_\infty\right)\right) f''\left(k_\infty\right)}{\lambda_-} = \frac{\lambda_+ - \rho}{\tilde{u}''\left(p_\infty\right)}$$

Similarly, we have:

$$\frac{dV_u}{dk}\left(k_\infty\right) = p_\infty$$
$$\frac{d^2V_s}{dk^2}\left(k_\infty\right) = \frac{dp_s}{dk_s}\left(k_\infty\right) = \frac{u'\left(f\left(k_\infty\right)\right) f''\left(k_\infty\right)}{\lambda_+} = \frac{\lambda_+ - \rho}{\tilde{u}''\left(p_\infty\right)}$$

Note that at the point $\left(k_\infty, V_\infty\right)$, both curves are tangent to each other, and to the boundary $V = \frac{1}{\rho} f\left(u\left(k\right)\right)$

2.5 Solving Ramsey's problem

We have two C^2 solutions of the Hamilton-Jacobi equation satisfying the desired condition $V(k_\infty) = \frac{1}{\rho} u(f(k_\infty))$ with $f'(k_\infty) = \rho$. To apply Theorem 1, we have to show that the solution $k(t)$ of (15) converges to k_∞.

Let us linearize the equation:

$$\frac{dk}{dt} = f(k) + \tilde{u}'(V'(k))$$
$$\frac{dk}{dt} = \left[f'(k_\infty) + \tilde{u}''(p_\infty) \frac{d^2V}{dk^2} \right]$$

Convergence to k_∞ requires that

$$f'(k_\infty) + \tilde{u}''(p_\infty) \frac{d^2V}{dk^2} \leq 0$$
$$\rho + \tilde{u}''(p_\infty) \frac{\lambda_\pm - \rho}{\tilde{u}''(p_\infty)} = \lambda_\pm \leq 0$$

This happens if and only if $\lambda = \lambda_-$. So, of the two solutions, one only satisfies the stability condition, namely $V_s(k)$. Let us conclude:

Theorem 3 *Problem (4), (5) has a solution $k(t)$ for any initial capital $k_0 > 0$, and we have:*

$$k(t) \to k_\infty \text{ when } t \to \infty$$

where k_∞ is independent of k_0 and u. It is characterized by:

$$f'(k_\infty) = \rho$$

3 Non-constant discount rate and time inconsistency

We consider the classical Ramsey model with a non-constant discount rate. The planner, facing at time t a future consumption schedule $c(s)$, with $t \leq s_1 \leq s \leq s_2$, derives from it the utility

$$U_t(c) = \int_{s_1}^{s_2} R(s-t)\, u(c(s))\, ds \tag{29}$$

for discount function R. We assume that it is continuously differentiable, with

$$R(0) = 1, \quad R(s) > 0 \ \ \forall s$$
$$\int_0^\infty R(s)ds < \infty$$

The other assumptions we have made regarding the utility function $u(c)$ and the production function $f(k)$ are unchanged, as well as the balance equation between savings and consumption. So the decision maker at time t strives to maximize the objective:

$$\int_t^{\infty} R(s-t)\, u(c(s))\, ds \tag{30}$$

under the constraint:

$$\frac{dk(s)}{ds} = f(k(s)) - c(s), \quad k(t) = k_t, \tag{31}$$

where $k(s)$ is the amount of capital available at time s, and k_t is the initial amount available at time t.

It is by now well-known that using non-exponential discount rates creates time inconsistency, in the sense that the relative ordering of two consumptions schedules $c^1(s)$ and $c^2(s)$, both defined on $t_1 \leq s \leq t_2$, can be reversed by the mere passage of time. To see this, take $t_1 < t_2 \leq s_1$, and say $U_{t_1}(c_1) < U_{t_1}(c_2)$, so that, at time t_1, the decision-maker prefers c_1 to c_2. If $R(t) = \exp(-rt)$, then $U_{t_2}(c) = \exp(r(t_2 - t_1))\, U_{t_1}(c)$, so we have $U_{t_2}(c_1) < U_{t_2}(c_2)$ and the relative ordering persists. In the non-exponential case, however, $U_{t_2}(c)$ is no longer proportional to $U_{t_1}(c)$, and we may have $U_{t_2}(c_1) > U_{t_2}(c_2)$. As a consequence, the successive decision-makers cannot agree on a common optimal policy: each of them will have his own. Optimizing the constraint (30) with $t = t_1$ under the constraint (31) will lead to a t_1-optimal policy c^1 defined on $s \geq t_1$. At any subsequent time $t_2 > 0$, the decision-maker will revisit the problem, setting $t = t_2$ in (30) and (31). He will then find an optimal solution c^2 on $s \geq t_2$ that is different from c^1.

In other words, for general discount functions, there are a plethora of *temporary* optimal policies: each of them will be optimal when evaluated from one particular point in time, but will cease to be so when time moves forward. In the absence of a commitment technology, there is no way for the planner at time t to implement his preferred (first-best) solution. He will then consider the problem as a leader-follower game between successive players, and seek a second-best solution.

3.1 Equilibrium strategies: construction and definition

We introduce equilibrium paths, along the lines of Ekeland and Lazrak [7], [8], [**?**]. We restrict our analysis to *stationary* and *smooth Markov* strategies, in the sense that the policy depends only on the current capital stock and not on past history, current time or some extraneous factors. Such a strategy is given by $c = \sigma(k)$, where $\sigma : R \to R$ is a continuously differentiable function. Note that this excludes all the first-best strategies, that is, those which are optimal at some time t. Indeed, maximising (30) under the constraint (31) will lead to a strategy $\sigma(t, k)$, where the initial time t comes in explicitly, so it is not Markov in the sense of Ekeland and Lazrak.

If we apply the strategy σ, the dynamics of capital accumulation from $t = 0$ are given by:

$$\frac{dk}{ds} = f(k(s)) - \sigma(k(s)), \ k(0) = k_0$$

We shall say σ *converges to* $\bar{k}$ if $k(s) \longrightarrow \bar{k}$ when $s \longrightarrow \infty$ and the initial value k_0 is sufficiently close to $\bar{k}$; we will say that $\bar{k}$ is a *steady* state of σ. A strategy σ is *convergent* if there is some $\bar{k}$ such that σ converges to $\bar{k}$. In that case, the integral (30) is obviously convergent, and its successive derivatives can be computed by differentiating under the integral. Note that if σ converges to $\bar{k}$, then we must have

$$f(\bar{k}) = \sigma(\bar{k})$$

Suppose the planner at time 0 picks a stationary smooth Markov strategy $\bar{\sigma}$ (which, again, cannot be his preferred one) converging to $\bar{k}$ and follows it up to time $t > 0$. He then stops to reassess, and asks himself whether he should change to another candidate strategy $\sigma(k)$ (still stationary, smooth, Markov, and convergent, but the new limit k is allowed to be different from $\bar{k}$. More specifically, he will compare two possibilities:

- continuing with $\bar{\sigma}$, that is, consuming $c(s) = \sigma(k(s))$ for $s \geq t$
- switching to σ on $t \leq s \leq \varepsilon$ and back to $\bar{\sigma}$ on $t + \varepsilon \leq s$

We shall say that $\bar{\sigma}$ is an equilibrium strategy if, for any time $t > 0$, there is some $\varepsilon > 0$ such that continuing with $\bar{\sigma}$ is the better option. This definition will be make precise in a moment.

The decision maker begins at time $s = t$ with capital stock k_t. If all future decision-makers apply the strategy $\bar{\sigma}$, the resulting path $k_t(s)$ obeys

$$\frac{dk_t}{ds} = f(k_t(s)) - \bar{\sigma}(k_t(s)), \quad s \geq t \tag{32}$$

$$k_t(t) = k_t. \tag{33}$$

The planner at time t applies the new strategy σ on $[t, \ t+\varepsilon]$, and reverts to $\bar{\sigma}$ after that. Since $\sigma(k)$ is smooth, the immediate utility flow during $[t, \ t+\varepsilon]$ is $u(c)\varepsilon$, with $c = \sigma(k)$. At time $t+\varepsilon$, the resulting capital will be $k + (f(k) - c)\varepsilon$, and from then on, the strategy $\bar{\sigma}$ will be applied which results in a capital stock $k(s)$ satisfying

$$\frac{dk}{ds} = f(k(s)) - \bar{\sigma}(k(s)), \quad s \geq \varepsilon \tag{34}$$

$$k(\varepsilon) = k + (f(k) - c)\varepsilon. \tag{35}$$

The capital stock $k(t)$ can be written as $k(s) = k_t(s) + k_c(s)\varepsilon$ where

$$\frac{dk_c}{ds} = (f'(k_t(s)) - \bar{\sigma}'(k_t(s)))\, k_c(s), \ s \geq \varepsilon \tag{36}$$

$$k_c(0) = \bar{\sigma}(k) - c \tag{37}$$

where f' and $\bar{\sigma}'$ stand for the derivatives of f and $\bar{\sigma}$. Summing up, we find that the discounted utility for the planner at time t if he switches is

$$u(c)\,\varepsilon + \int_{t+\varepsilon}^{\infty} R(s-t)\,u\left(\bar{\sigma}\left(k_t(s) + \varepsilon k_c(s)\right)\right)ds \tag{38}$$

plus higher-order terms in ε. This it to be compared to his discounted utility if he does not switch, namely:

$$\int_{t}^{\infty} R(s-t)\,u\left(\bar{\sigma}\left(k_t(s)\right)\right)ds, \tag{39}$$

We want (38) this be inferior to (39), at least when $\varepsilon > 0$ is small enough.The difference is equal to:

$$\varepsilon\left[u(c) - u(\bar{\sigma}(k)) + \int_{t}^{\infty} R(s-t)\,u'\left(\bar{\sigma}\left(k_t(s)\right)\right)\bar{\sigma}'\left(k_t(s)\right)k_c(s)\,ds\right] \tag{40}$$

where k_c solves the linear equation

$$\frac{dk_c}{ds} = \left(f'\left(k_t(s)\right) - \bar{\sigma}'\left(k_t(s)\right)\right)k_c(s), \quad s \geq t \tag{41}$$

$$k_c(t) = \bar{\sigma}(k) - c. \tag{42}$$

In formula (40), the free variable is $c = \sigma(k)$. We want:

$$\forall c, \quad \varepsilon\left[u(c) - u(\bar{\sigma}(k)) + \int_{t}^{\infty} R(s-t)\,u'\left(\bar{\sigma}\left(k_t(s)\right)\right)\bar{\sigma}'\left(k_t(s)\right)k_c(s)\,ds\right] \leq 0 \tag{43}$$

Note that setting $c = \bar{\sigma}(k)$ gives $k_c = 0$ and hence yields 0 on the left-hand side of (43). Note also that it is enough to check the relation for $t = 0$.

Definition 4 *A convergent Markov strategy $\sigma : R \to R$ is an* equilibrium *for the intertemporal decision model (30) under the constraint (31) if, for every $k \in R$, we have:*

$$\bar{\sigma}(k) = \arg\max_{c}\left[u(c) - u(\bar{\sigma}(k)) + \int_{0}^{\infty} R(s)\,u'\left(\bar{\sigma}\left(k(s)\right)\right)\bar{\sigma}'\left(k(s)\right)k_c(s)\,ds\right] \tag{44}$$

with:

$$\frac{dk_0}{ds} = f\left(k_0(s)\right) - \sigma\left(k_0(s)\right), \quad s \geq 0$$

$$k_0(t) = k.$$

and:

$$\frac{dk_c}{ds} = \left(f'\left(k_0(s)\right) - \sigma'\left(k_0(s)\right)\right)k_c(s), \quad s \geq 0$$

$$k_c(t) = \sigma(k) - c.$$

3.2 Characterization of the equilibrium strategies

The main result of Ekeland and Lazrak is that equilibrium strategies can be fully specified by a single function, the *value function* $v(k)$, which is reminiscent of-although different from - the value function in optimal control. We will state their result without proof, referring to the original papers for details.

Given a Markov strategy $\sigma(k)$, continuously differentiable and convergent, we shall be dealing with the Cauchy problem (32), (33). We shall denote by $k_0(t) = \mathcal{K}(\sigma; t, k)$ the *flow* associated with the differential equation:

$$\frac{\partial \mathcal{K}(\sigma; t, k)}{\partial t} = f(\mathcal{K}(\sigma; t, k)) - \sigma(\mathcal{K}(\sigma; t, k)) \tag{45}$$

$$\mathcal{K}(\sigma; 0, k) = k. \tag{46}$$

Theorem 5 *Suppose there is a C^2 function $v(k)$ such that $\sigma(k) := -\tilde{u}'(v(k))$ is a stationary Markov stragegy which converges to some $\bar{k}$. Suppose that $v(k)$ solves the equation:*

$$-\int_0^\infty R'(t) u\left(-\tilde{u}'\left(v'(\mathcal{K}\left(-\tilde{u}'\left(v'\right); t, k\right)\right)\right) dt = \tilde{u}(v'(k)) + v'(k) f(k) \tag{DE}$$

with the boundary condition:

$$v(\bar{k}) = u\left(f\left(\bar{k}\right)\right) \int_0^\infty R(t)\, dt \tag{BC}$$

Then $\sigma(k)$ is an equilibrium strategy

With exponential discounting, $R(t) = e^{-\delta t}$ (DE) is the ordinary Hamilton-Jacobi equation

$$\delta v(k) = \sup_c \left[u(c) + v'(k)\left(f(k) - c\right)\right] = \tilde{u}(v'(k)) + v'(k) f(k). \tag{47}$$

Equation (DE) is not of a classical mathematical type. The integral term, which is non-local (an integral along the trajectory of the flow (45) associated with the solution v) creates a loss of regularity in the functional equation that generates mathematical complications. As a result, the question whether equilibrium solutions exist for general discount rates $R(t)$ is still open. We propose to solve it in a particular case of economic interest.

3.3 Quasi-exponential discount

We shall use the following specification:

$$R(t) = \lambda \exp(-\delta t) + (1 - \lambda) \exp(-\rho t) \tag{48}$$

with $\rho > \delta$ and $\lambda > 0$ (note that we do not require this is a convex combination). We refer to [**?**] for the economic background.

To simplify notations, we set:

$$\frac{\delta+\rho}{2}=a, \quad \frac{\delta-\rho}{2}=b$$

Note that:

$$a>0>b$$

Theorem 6 *Let $v(k)$ be a C^2 function such that the strategy $\sigma(k)=-\tilde{u}'(v'(k))$ converges to k_∞. Then v satisfies (DE) and (BC) if and only if there exists a C^1 function $w(k)$, such that (v,w) satisfies the system:*

$$\tilde{u}(v')+v'f=av+bw \tag{49}$$

$$(\tilde{u}'(v')+f)w'+(2\lambda-1)[\tilde{u}(v')-v'\tilde{u}'(v')]=bv+aw \tag{50}$$

with the boundary conditions:

$$v(k_\infty)=\left(\frac{\lambda}{\delta}+\frac{1-\lambda}{\rho}\right)u(f(k_\infty))=\left(\frac{\lambda}{a+b}+\frac{1-\lambda}{a-b}\right)u(f(k_\infty)) \tag{51}$$

$$w(k_\infty)=\left(\frac{\lambda}{\delta}-\frac{1-\lambda}{\rho}\right)u(f(k_\infty))=\left(\frac{\lambda}{a+b}-\frac{1-\lambda}{a-b}\right)u(f(k_\infty)) \tag{52}$$

Proof. Set, with $c(t)=-\tilde{u}'(v'(k(t)))$ as usual

$$v(k)=\lambda\int_0^\infty e^{-\delta t}u(c(t))\,dt+(1-\lambda)\int_0^\infty e^{-\rho t}u(c(t))\,dt$$

$$w(k)=\lambda\int_0^\infty e^{-\delta t}u(c(t))\,dt-(1-\lambda)\int_0^\infty e^{-\rho t}u(c(t))\,dt$$

Then

$$\lambda\int_0^\infty e^{-\delta t}u(c(t))\,d=\frac{v(k)+w(k)}{2}$$

$$(1-\lambda)\int_0^\infty e^{-\rho t}u=\frac{v(k)-w(k)}{2}$$

and (DE) becomes:

$$\delta\frac{v+w}{2}+\rho\frac{v-w}{2}=\tilde{u}(v')+v'f$$

as desired. This gives (ED1). For (ED2), we use Lemma 10 of [**?**] (which we restate here[1] for the reader's convenience) with

$$\sigma(k) = -\tilde{u}'(v'(k))$$
$$R(t) = \lambda e^{-\delta t} - (1-\lambda) e^{-\rho t}$$

so that $R(0) = (2\lambda - 1)$. We get an equation for $w(k)$:

$$w'(k)(f(k) + \tilde{u}'(v'(k))) + (2\lambda - 1) u(-\tilde{u}'(v'(k))) = \int_0^\infty \left(\lambda\delta e^{-\delta t} - (1-\lambda)\rho e^{-\rho t}\right) u(c(t))\, dt$$
$$= \delta \frac{v+w}{2} - \rho \frac{v-w}{2} = \frac{\delta - \rho}{2} v + \frac{\delta + \rho}{2} w$$

which is precisely (50) ■

The boundary-value problem (49), (50), (51), (52) replaces the Hamilton-Jacobi equation (21), (22) of the Ramsey problem.

3.4 Solving the boundary-value problem

3.4.1 The geometry

Problem (49), (50) is an implicit differential system. Indeed, as in the preceding section, the equation $\tilde{u}(v') + v'f = av + bw$ has no solutions if $av + bw < u(f(k))$ and two solutions if $av + bw > u(f(k))$. In other words, the surface:

$$av + bw = u(f(k))$$

separates the phase space $R_+ \times R_+ \times R$, with coordinates (k, v, w), into two regions. In the upper one, there are two dynamical systems, that is, at every point there are two possible velocities. As the point approaches the boundary, the two velocities get closer, and at the boundary they coincide, without vanishing. When one crosses the boundary, both vector fields disappear. In other words, the two dynamical systems in the upper half hit the boundary together (or emerge from the boundary together), and there is nothing in the lower half.

On can recognize the situation we have already seen in the preceding section, and hope that this is just a suspension of the 2-dimensional situation, the third variable being inessential. We will indeed prove this, but this is a more complicated situation, simple geometrical tools will not be enough, and we will have to resort to the central manifold theorem.

[1]

Lemma 7 *Let $\sigma(k)$ be any smooth convergent Markov strategy. Denote its steady state by k_∞. Let $R : [0, \infty] \longrightarrow R$ be any C^1 function with exponential decay at infinity. The following are equivalent:*

$$I(k) = \int_0^\infty R(t)\, u(\sigma(\mathcal{K}(\sigma; t, k)))\, dt$$

$$I'(k)(f(k) - \sigma(k)) + \int_0^\infty R'(t)\, u(\sigma(\mathcal{K}(\sigma; t, k)))\, dt + R(0)\, u(\sigma(k)) = 0$$
$$I(k_\infty) = \int_0^\infty R(t) \ln f(k_\infty)$$

3.4.2 Enlarging the phase space

Enlarge the phase space, as in the preceding section. The new phase space is $(k, v, w, p, q) = R^5$. In that space, we consider the 3-dimensional manifold M^3 given by:

$$\tilde{u}(p) + pf(k) = av + bw \tag{53}$$

$$q(\tilde{u}'(p) + f(k)) + (2\lambda - 1)(\tilde{u}(p) - p\tilde{u}'(p)) = bv + aw \tag{54}$$

Clearly (k, p, q) is a coordinate system. We derive v and w from (53) and (54) by:

$$\left(a^2 - b^2\right) v = \tilde{u}(p)(a - (2\lambda - 1)b) + \tilde{u}'(p)((2\lambda - 1)p - q)b + f(k)(ap - bq) \tag{55}$$

$$\left(a^2 - b^2\right) w = -\tilde{u}(p)(b - (2\lambda - 1)a) - \tilde{u}'(p)((2\lambda - 1)p - q)b + f(k)(aq - bp) \tag{56}$$

On this manifold we consider the Pfaff system:

$$dv = pdk \tag{57}$$

$$dw = qdk \tag{58}$$

Differentiating (53) and (54) yields two further equations:

$$(\tilde{u}'(p) + f(k))dp = (bq + (a - f'(k))p)dk \tag{59}$$

$$(\tilde{u}'(p) + f(k))dq + (q - (2\lambda - 1)p)\tilde{u}''(p)dp = (bp + (a - f'(k))q)dk \tag{60}$$

The initial condition for the Pfaff system (57), (58), (59), (60) is given by (51) and (52), namely:

$$v_0 = \left(\frac{\lambda}{a+b} + \frac{1-\lambda}{a-b}\right) u(f(k_0)) \tag{61}$$

$$w_0 = \left(\frac{\lambda}{a+b} - \frac{1-\lambda}{a-b}\right) u(f(k_0)) \tag{62}$$

Writing this into the equations defining into the equation (53) and (54) defining M^3 yields:

$$\tilde{u}(p_0) + pf(k_0) = av_0 + bw_0 = u(f(k_0)) \tag{63}$$

$$q_0(\tilde{u}'(p_0) + f(k_0)) + (2\lambda - 1)(\tilde{u}(p_0) - p\tilde{u}'(p_0)) = bv_0 + aw_0 = (2\lambda - 1)u(f(k_0)) \tag{64}$$

Equation (63) determines p_0 uniquely:

$$p_0 = u'(f(k_0)) \tag{65}$$

Equation (64) then is automatically satisfied. On the other hand, substituting $p_0 = u'(f(k_0))$ in equation (59) determines q_0 by the equation:

$$bq_0 + (a - f'(k_0))\, p_0 = 0 \tag{66}$$

Note for future use that:

$$q_0 - (2\lambda - 1)\, p_0 = \frac{a - (2\lambda - 1)\, b - f'(k_0)}{b} u'(f(k_0)) \tag{67}$$

$$bp_0 + (a - f'(k_0))\, q_0 = \frac{b^2 - (a - f'(k_0))^2}{b} u'(f(k_0)) \tag{68}$$

Let us now summarize. We have to solve we have to solve the Pfaff system (59), (60) under the initial condition(65), (66):

$$\begin{array}{c} (\tilde{u}'(p) + f(k))\, dp - (bq + (a - f'(k))\, p)\, dk = 0 \\ (\tilde{u}'(p) + f(k))\, dq + (q - (2\lambda - 1)\, p)\, \tilde{u}''(p)\, dp - (bp + (a - f'(k))\, q)\, dk = 0 \\ p_0 = u'(f(k_0)) \\ q_0 = -\frac{1}{b}(a - f'(k_0))\, u'(f(k_0)) \end{array} \tag{69}$$

Once a solution is found (that is, once p, q, k have been found as smooth functions $p(s), q(s), k(s)$ of a parameter s), we will derive $v(p, q, k)$ and $w(p, q, k)$ by the formulas (55) and (56).

3.4.3 Solving the problem

Introduce the additional variable s, and consider the system:

$$\begin{array}{c} \frac{dk}{ds} = (\tilde{u}'(p) + f(k))^2 \\ \frac{dp}{ds} = (bq + (a - f'(k))\, p)\, (\tilde{u}'(p) + f(k)) \\ \frac{dq}{ds} = -(q - (2\lambda - 1)\, p)\, \tilde{u}''(p)\, (bq + (a - f'(k))\, p) + (\tilde{u}'(p) + f(k))\, (bp + (a - f'(k))\, q) \end{array} \tag{70}$$

The trajectories of this system satisfying $p(0) = p_0,\ q(0) = q_0,\ k(0) = k_0$ are solutions of problem (69).

Let us linearize the system (70) around the fixed point. We get:

$$\begin{array}{c} \frac{d\bar{k}}{ds} = 0 \\ \frac{d\bar{p}}{ds} = 0 \\ \frac{d\bar{q}}{ds} = A\bar{q} + B\bar{p} + C\bar{k} \end{array} \tag{71}$$

where

$$A = -(q_0 - (2\lambda - 1)\, p_0)\, \tilde{u}''(p_0)\, b = [f'(k_0) - a + (2\lambda - 1)\, b]\, u'(f(k_0))\, \tilde{u}''(p_0)$$

$$B = [(a - f'(k_0))\, (2\lambda - 1) + b]\, \tilde{u}''(p_0)\, p_0 = [(a - f'(k_0))\, (2\lambda - 1) + b]\, u'(f(k_0))\, \tilde{u}''(p_0)$$

$$C = (bp_0 + (a - f'(k_0))\, q_0)\, f'(k_0) = \frac{b^2 - (a - f'(k_0))^2}{b} u'(f(k_0))\, f'(k_0)$$

If $(q_0 - (2\lambda - 1) p_0) \neq 0$, that is, by equation (67), if:

$$a - (2\lambda - 1) b - f'(k_0) \neq 0 \tag{72}$$

then the linearized system has eigenvalues $(A, 0, 0)$, with $A \neq 0$. By the central manifold theorem, there is a one-dimensional invariant manifold tangent to $\bar{k} = \bar{p} = 0$, and a two-dimensional invariant manifold tangent to $A\bar{q} + B\bar{p} + C\bar{k} = 0$. The equation of the invariant manifold is $q = h(k, p)$, with

$$h(k_0, p_0) = q_0, \quad \frac{\partial h}{\partial k}(k_0, p_0) = -\frac{C}{A}, \quad \frac{\partial h}{\partial p}(k_0, p_0) = -\frac{B}{A}$$

Invariance means that the function h has to satisfies the system:

$$(\tilde{u}'(p) + f(k)) \frac{dp}{ds} - (bh(k,p) + (a - f'(k)) p) \frac{dk}{ds} = 0$$

$$(\tilde{u}'(p) + f(k)) \left(\frac{\partial h}{\partial k} \frac{dk}{ds} + \frac{\partial h}{\partial p} \frac{dp}{ds} \right) + (q - (2\lambda - 1) p) \tilde{u}''(p) \frac{dp}{ds} - (bp + (a - f'(k)) h(k,p)) \frac{dk}{ds} = 0$$

Let us rewrite the last equation as follows:

$$\frac{dp}{dk} = \frac{(bp + (a - f'(k)) h(k,p)) - \frac{\partial h}{\partial k} (\tilde{u}'(p) + f(k))}{(\tilde{u}'(p) + f(k)) \frac{\partial h}{\partial p} + (q - (2\lambda - 1) p) \tilde{u}''(p)} \tag{73}$$

At (k_0, p_0), we have:

$$\frac{dp}{dk} = \frac{bp_0 + (a - f'(k_0)) q_0}{(q_0 - (2\lambda - 1) p_0) \tilde{u}''(p_0)} = \frac{b^2 - (a - f'(k_0))^2}{a - (2\lambda - 1) b - f'(k_0)} \tag{74}$$

So equation (73) is a simple Cauchy problem, which yields a smooth function $p(k)$. We find $q(k), v(k)$ and $w(k)$ by setting $q(k) = h(k, p(k))$ and by substituting in formulas (55) and (56). We have thus found a solution of the boundary-value problem (49), (50), (51), (52).

3.5 Existence of an equilibrium strategy

We have found a candidate equilibrium strategy, namely $\sigma(k) = -\tilde{u}'(v'(k))$, and by Theorem 6, all that remains to show is that this strategy converges to k_0.

The linearized dynamics at k_0 are given by:

$$\frac{d\bar{k}}{dt} = (f'(k_0) - \tilde{u}''(v'(k_0)) v''(k_0)) \bar{k}$$

with $\bar{k} = k - k_0$. It is sufficient to show that:

$$f'(k_0) - \tilde{u}''(v'(k_0)) v''(k_0) < 0 \tag{75}$$

We have, by equation (74)

$$v'(k_0) = p_0, \quad v''(k_0) = \frac{dp}{dk}(k_0, p_0) = \frac{b^2 - (a - f'(k_0))^2}{a - (2\lambda - 1)b - f'(k_0)}$$

and we easily check that:

$$\begin{aligned} f'(k_0) + \tilde{u}''(v'(k_0))\, v''(k_0) &= f'(k_0) - \frac{b^2 - (a - f'(k_0))^2}{a + (2\lambda - 1)b - f'(k_0)} \\ &= \frac{(a - (2\lambda - 1)b) f'(k_\infty) + b^2 - a^2}{f'(k_\infty) - a - (2\lambda - 1)b} \end{aligned}$$

Substituting:

$$\begin{aligned} \frac{a^2 - b^2}{(a - (2\lambda - 1)b)} &= \frac{1}{\frac{\lambda}{\delta} + \frac{1-\lambda}{\rho}} \\ a + (2\lambda - 1)b &= \lambda\delta + (1 - \lambda)\rho \end{aligned}$$

we find that inequality (75) can be rewritten as follows:

$$\frac{f'(k_0) - \frac{1}{\frac{\lambda}{\delta} + \frac{1-\lambda}{\rho}}}{f'(k_0) - \lambda\delta - (1 - \lambda)\rho} \leq 0$$

which will be satisfied provided $f'(k_\infty)$ lies between the roots of the numerator and denominator.

We summarize:

$$R(t) = \lambda \exp(-\delta t) + (1 - \lambda)\exp(-\rho t) \tag{76}$$

with $\rho > \delta$ and $\lambda > 0$ (note that we do not require this is a convex combination). We refer to [**?**] for the economic background.

Theorem 8 *Suppose $\rho > \delta$ and $\lambda > 0$. Define $\underline{k}$ and $\bar{k}$ by:*

$$\begin{aligned} f'(\underline{k}) &= \lambda\delta + (1 - \lambda)\rho \\ f'(\bar{k}) &= \frac{1}{\frac{\lambda}{\delta} + \frac{1-\lambda}{\rho}} \end{aligned}$$

Suppose $a - (2\lambda - 1)b - f'(k_0) \neq 0$ and assume:

$$\begin{aligned} &\underline{k} < k_\infty < \bar{k} \quad \text{if} \quad 0 < \lambda < 1 \\ &\bar{k} < k_\infty < \underline{k} \ \text{if} \ 1 < \lambda < \frac{\rho}{\rho - \delta} \\ &\bar{k} < k_\infty \ \text{if} \ \lambda > \frac{\rho}{\rho - \delta} \end{aligned}$$

Then there is a Markov equilibrium strategy converging to k_∞

References

[1] Aghion, Philippe and Howitt, Peter (1998) "Endogeneous growth theory", MIT Press

[2] Aubin, Jean-Pierre and Clarke, Frank (1979) "Shadow prices and duality for a class of optimal control problems", SIAM J. Control Optim. 17 , pp. 567–586.

[3] Barro, Robert and Sala-i-Martin, Xavier (1995) "Economic growth", McGraw and Hill

[4] Blanchard, Olivier and Fisher, Stanley (1994) "Lectures on macroeconomics", MIT Press64), p. 39-50

[5] Carr, Jack , "Applications of Centre Manifold Theory" Springer-Verlag, 1981.

[6] Davydov (2008) "Generic singularities of implicit systems of first order differential equations in the plane", Japanese Journal of Mathematics 3 (1),.

[7] Ekeland, Ivar and Lazrak, Ali (2005) "Being serious about non-commitment", http://arxiv.org/abs/math/0604264

[8] Ekeland, Ivar and Lazrak, Ali (2008) "Equilibrium policies when preferences are time inconsistent", http://arxiv.org/abs/0808.3790

[9] Ekeland, Ivar and Scheinkman, Jose (1986) "Transversality conditions for some infinite horizon discrete time opimization problems", Mathematics of Operations Research 11, p. 216-229

[10] Frederick, Shane, Loewenstein, George, and O'Donoghue, Ted "Time discounting and time preference: a critical review", Journal of Economic Literature 40 (2002), 351-401

[11] Kamihigashi, Takashi (2001) Necessity of Transversality Conditions for Infinite Horizon Problems. Econometrica 69 (4), 995-1012.

[12] Laudenbach, Francois, personal communication

[13] Michel, Philippe (1982) "On the Transversality Condition in Infinite Horizon Optimal Problems." Econometrica 50, p. 975-985

[14] Samuelson, Paul (1937) ""A note on measurement of utility", Review of Economic Studies 4, p. 155-61

[15] Ramsey, Frank (1928) "A mathematical theory of saving", Economic Journal 38, p. 543-559

[16] Romer, David (1996), "Advanced macroeconomics", McGraw and Hill

[17] Thom, René, "Sur les équations différentielles multiformes et leurs intégrales singulières", Bol. Soc. Mat. Bras. 3 (1972), p. 1-11