On the variational principle.

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1: Variational principles: old and new

The classical variational principle

Definition

Let X be a Banach space. We shall say that $f : X \longrightarrow \mathbb{R}$ is *Gâteaux-differentiable* at x if there exists a continous linear map $Df(x) : X \longrightarrow X^*$ such that

$$\forall \xi \in X, \quad \lim_t \frac{1}{t}\left[f(x + t\xi) - f(x)\right] = \langle Df(x), \xi \rangle$$

In other words, the restriction of f to every line is differentiable (for instance, partial derivatives exist). If the map $x \longrightarrow Df(x)$ from X to X^* is norm-continuous, f is called C^1.

Theorem (Classical variational principle)

If f attains its minimum on X at a point $\bar{x}$, then $Df(\bar{x}) = 0$. If f is C^2, then the Hessian $D^2 f(x)$ is non-negative

Any point where $Df(\bar{x}) = 0$ is called a *critical point*.

Maximum, minimum or critical point ?

- Hero of Alexandria (3d century BCE) : light takes the shortest path from A to B. Deduces the laws of reflection
- Fermat (17th century): light takes the quickest path from A to B. Deduces the law of refraction
- Maupertuis (18th century): every mechanical system, when going from one configuration to another, minimizes the action. This is the least action principle.

In fact, nature is not interested in minimizing or maximizing. It is interested in critical points. The laws of physics are not

$$f(x) = \min$$

They are of the form

$$f'(x) = 0$$

The mathematical difficulty is that x is not a point, but a path, i.e. a map from R to R^n (ODE) or a map from R^p to R^n (PDE). The path space will be infinite-dimensional, and cannot be compact.

The billiard:

Theorem (Birkhoff)

Any convex billiard table has two diameters

The large diameter is obtained by maximizing the distance between two points on the boundary. But where is the small diameter ?

The extended variational principle (EVT)

Theorem (Ekeland, 1972)

Let (X, d) be a complete metric space, and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous map, bounded from below:

$$\{(x, a) \mid a \geq f(x)\} \text{ is closed in } X \times \mathbb{R} \quad \text{and} \quad f(x) \geq 0$$

Suppose $f(x_0) < \infty$. Then for every $r > 0$, there exists some $\bar{x}$ such that:

$$f(\bar{x}) \leq f(x_0)$$
$$d(\bar{x}, x_0) \leq r$$
$$f(x) \geq f(\bar{x}) - \frac{f(x_0)}{r} d(x, \bar{x}) \quad \forall x$$

So (a) $\bar{x}$ is better that x_0 (b) $\bar{x}$ can be chosen as close to x_0 as one wishes (c) $\bar{x}$ satisfies a cone condition

Local version

Form now on, X will be a Banach space with $d(x,y) = \|x-y\|$

Definition

$f : X \to \mathbb{R}$ is ε-supported at x if there exists some $\eta > 0$ and some $x^* \in X^*$ such that:

$$\|x-y\| \leq \eta \Longrightarrow f(y) - f(x) - \langle x^*, y-x \rangle \geq -\varepsilon \|y-x\|$$

Theorem

Suppose there is a C^1 function $\varphi : X \to \mathbb{R}$ such $\varphi(0) = 1$, $\varphi(x) = 0$ for $\|x\| \geq 1$ and $\varphi \geq 0$. Then , for every lower semi-continuous function $f : X \to \mathbb{R}$ and every $\varepsilon > 0$, the set of points x where f is ε-supported is dense in X.

Differentiability of convex functions.

Recall that, in a complete metric space, a countable intersection of open dense sets is dense (Baire).

Corollary

For every convex function f.on X, there is a subset $\Delta \subset X$, which is a countable intersection of open dense subsets, such that F is Fréchet-differentiable at every point $x \in \Delta$:

$$\|x - y\| \leq \eta \Longrightarrow \|f(y) - f(x) - \langle x^*, y - x \rangle\| \leq \varepsilon \|y - x\|$$

2: Optimization

First-order version of EVP

Theorem

Let X be a Banach space, and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous map, non-negative and G-differentiable. Then for every $x_0 \in X$ and $r > 0$, there exists some $\bar{x}$ such that:

$$f(\bar{x}) \leq f(x_0),$$
$$\|\bar{x} - x_0\| \leq r$$
$$\|Df(\bar{x})\|^* \leq \frac{f(x_0)}{r}$$

Corollary

There is a sequence x_n such that:

$$f(x_n) \to \inf f$$
$$Df(x_n) \to 0$$

Proof

For simplicity, take $x_0 = 0$ and $\inf f = 0$. Apply EVP to $x = \bar{x} + tu$ and let $u \longrightarrow 0$. We get:

$$f(\bar{x} + tu) \geq f(\bar{x}) - \frac{f(0)}{r} t \|u\| \quad \forall (t, u)$$

$$\lim_{t \to +0} \frac{1}{t} \left(f(\bar{x} + tu) - f(\bar{x}) \right) \geq -\frac{f(0)}{r} \|u\| \quad \forall u$$

$$\langle Df(x), u \rangle \geq -\frac{f(0)}{r} \|u\| \quad \forall u, \text{ or } \|Df(x)\|^* \leq \frac{f(0)}{r}$$

Existence problems

The compact case

Theorem

Suppose f is a lower semi-continuous function on a compact space. Then it attains its minimum

Proof.

There is a sequence x_n such that $f(x_n) \to \inf f$ and $Df(x_n) \to 0$. By compactness, it has a subsequence which converges to some $\bar{x}$, and by semi-continuity, $f(\bar{x}) \leq \liminf f(x_n) = \inf f$ □

Existence problems

The complete case

Definition

We shall say that f satisfies the Palais-Smale condition(PS) if every sequence x_n such that $f(x_n)$ converges and $Df(x_n) \to 0$ has a convergent subsequence

Theorem (Palais and Smale, 1964)

Any lower semi-continuous function on a Banach space, which is G-differentiable, bounded from below, and satisfies PS attains its minimum

Proof.

By EVP, there is a sequence x_n such that $f(x_n) \to \inf f$ and $Df(x_n) \to 0$. By PS, it has a subsequence which converges to some $\bar{x}$, and by semi-continuity, $f(\bar{x}) \leq \liminf f(x_n) = \inf f$ □

The EVP allows us to consider more restricted classes of minimizing sequences

Second-order version

Theorem (Borwein-Preiss, 1987)

Let X be a Hilbert space, and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a C^2 function, bounded from below. Then for every minimizing sequence x_n of f in X, there exists a sequence y_n of f such that:

$$\begin{aligned} f(y_n) &\to \inf f \\ \|x_n - y_n\| &\to 0 \\ Df(y_n) &\to 0 \\ \liminf_n \left(D^2 f(y_n) u, u\right) &\geq 0 \quad \forall u \in X \end{aligned}$$

So we can consider even more restricted classes of minimizing sequences, those along which $Df(x_n) \to 0$ and $\liminf Df(x_n) \geq 0$.

3: Critical point theory

Theorem (Ambrosetti and Rabinowitz 1973)

Let f be a continuous G-differentiable function on X. Assume that $f' : X \to X^*$ is norm-to-weak continuous and satisfies PS. Take two points x_0 and x_1 in X, and define:*

$$\Gamma := \left\{ c \in C^0\left([0,1];X\right) \mid c\left(0\right) = x_0,\ c\left(1\right) = x_1 \right\}$$
$$\gamma := \inf_{c \in \Gamma} \max_{0 \le t \le 1} f\left(c\left(t\right)\right)$$

Assume that $\gamma \ge \max\left(f\left(x_0\right), f\left(x_1\right)\right)$. *Then there exists some point* $\bar{x}$ *with*

$$f\left(\bar{x}\right) = \gamma$$
$$Df\left(\bar{x}\right) = 0$$

Theorem (Hofer 1983, Ghoussoub-Preiss 1989)

The critical point $\bar{x}$ in the preceding theorem must satisfies on of the following:

- *either there is a sequence x_n of (weak) local maxima such that $x_n \longrightarrow \bar{x}$*
- *or $\bar{x}$ has mountain-pass type, namely, there exists a neighbourhood $\mathcal{U}$ of x such that $\mathcal{U} \cap \{f < \gamma\}$ is neither empty nor connected.*

Proof

Consider the set of paths Γ and endow it with the uniform distance. It is a complete metric space. Consider the function $\varphi : \Gamma \longrightarrow \mathbb{R}$ defined by:

$$\varphi(c) := \max_{0 \le t \le 1} f(c(t))$$

It is bounded from below by γ, and lower semi-continuous. We can apply EVP, we get a path c_ε, take t_ε where $f(c_\varepsilon(t))$ attains its maximum, and set $x_\varepsilon = c_\varepsilon(t_\varepsilon)$. We show that $\|Df(x_\varepsilon)\|^* \le \varepsilon$, we let $\varepsilon \longrightarrow 0$ and we apply PS.

4: Inverse function theorems

A non-smooth inverse function theorem

Theorem (Ekeland, 2012)

Let X and Y be Banach spaces. Let $F : X \to Y$ be continuous and G-differentiable, with $F(0) = 0$. Assume that the derivative $DF(x)$ has a right-inverse $L(x)$, uniformly bounded in a neighbourhood of 0:

$$\forall v \in Y, \quad DF(x)\, L(x)\, v = v$$
$$\sup \{ \|L(x)\| \mid \|x\| \leq R \} < m$$

Then, for every $\bar{y}$ such that

$$\|\bar{y}\| \leq \frac{R}{m}$$

there is some $\bar{x}$ such that:

$$\|\bar{x}\| \leq m \|\bar{y}\|$$
$$F(\bar{x}) = \bar{y}$$

The standard inverse function theorem requires F to be C^1 and does not

Proof

Consider the function $f : X \to R$ defined by:

$$f(x) = \|F(x) - \bar{y}\|$$

It is continuous and bounded from below, so that we can apply EVP with $r = m\|\bar{y}\|$. We can find $\bar{x}$ with:

$$f(\bar{x}) \leq f(0) = \|\bar{y}\|$$
$$\|\bar{x}\| \leq m\|\bar{y}\| \leq R$$
$$\forall x, \quad f(x) \geq f(\bar{x}) - \frac{f(0)}{m\|\bar{y}\|}\|x - \bar{x}\| = f(\bar{x}) - \frac{1}{m}\|x - \bar{x}\|$$

I claim $F(\bar{x}) = \bar{y}$.

Proof (ct'd)

Assume $F(\bar{x}) \neq \bar{y}$. The last equation can be rewritten:

$$\forall t \geq 0,\ \forall u \in X, \qquad \frac{f(\bar{x}+tu) - f(\bar{x})}{t} \geq -\frac{1}{m}\|u\|$$

Simplify matters by assuming X is Hilbert. Then:

$$\left(\frac{F(\bar{x}) - \bar{y}}{\|F(\bar{x}) - \bar{y}\|}, DF(\bar{x})\,u\right) = \langle Df(\bar{x}), u\rangle \geq -\frac{1}{m}\|u\|$$

We now take $u = -L(\bar{x})(F(\bar{x}) - \bar{y})$, so that $DF(\bar{x})\,u = -(F(\bar{x}) - \bar{y})$. We get a contradiction:

$$\|F(\bar{x}) - \bar{y}\| \leq \frac{1}{m} L(\bar{x}) \|F(\bar{x}) - \bar{y}\| < \|F(\bar{x}) - \bar{y}\|$$

Global inverse function theorems

Taking $R = \infty$ gives:

Theorem

Let X and Y be Banach spaces. Let $F : X \rightarrow Y$ be continuous and Gâteaux-differentiable, with $F(0) = 0$. Assume that the derivative $DF(x)$ has a right-inverse $L(x)$, uniformly bounded on X

$$\forall v \in Y, \quad DF(x) L(x) v = v$$
$$\sup_x \|L(x)\| < m$$

Then, for every $\bar{y}$ there is some $\bar{x}$ such that:

$$\|\bar{x}\| \leq m \|\bar{y}\|$$
$$F(\bar{x}) = \bar{y}$$

The C^1 version is due to Hadamard.

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All papers can be found on my website:

http://www.ceremade.dauphine.fr/~ekeland/

THANK YOU VERY MUCH