Math. Ann. 288, 133–160 (1990)

Mathematische
Annalen
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A variational approach to homoclinic orbits in Hamiltonian systems

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Received November 7, 1989; in revised form February 19, 1990

1. Homoclinic orbits

We are given a C^2 map $H:\mathbb{R}\times\mathbb{R}^{2N}\to\mathbb{R}$, and we consider the associated system of ordinary differential equations

$$x' = J\nabla_x H(t,x), \tag{1.1}$$

where J denotes the $2N\times 2N$ matrix

$$J=\begin{pmatrix} 0 & I \\ -I & 0\end{pmatrix}$$

with $J^* = J^{-1} = -J$, and $\nabla_x H(t,x)$ denotes the vector

$$\frac{\partial H}{\partial x_i}(t,x), \qquad 1\leqq i\leqq 2N.$$

Systems of the form (1.1) are called Hamiltonian, and the function H is referred to as the Hamiltonian of the system. Throughout this paper, it will be assumed that H is periodic with respect to time:

$$\exists T>0: H(t+T,x)=H(t,x) \qquad \forall (t,x).$$

A lot of attention has been devoted in recent years to finding periodic solutions of system (1.1) under convexity assumptions. We refer to the forthcoming book [5] for a survey. It is our purpose in the present work to find other types of solutions, namely the doubly asymptotic solutions, first discovered by Poincaré [13].

If $\bar{x}$ is a periodic solution of system (1.1), that is $\bar{x}(t+T)=\bar{x}(t)$ for all t, another solution z will be called doubly asymptotic to $\bar{x}$ if $|z(t)-\bar{x}(t)|\to 0$ when $t\to\pm\infty$.

Assume for instance that the periodic solution $\bar{x}$ is hyperbolic. This means that the matrix $M(T)$ has no eigenvalues of modulus 1. Here $M(t)$ is the resolvent of the

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** Supported by U.S. Army contract DAJA45-88-C-0009

linearized system around $\bar{x}$, which is defined by

$$\begin{cases} \dfrac{dM}{dt} = JH''(t, \bar{x}(t))M \\ M(0) = I . \end{cases}$$

Then there are two smooth n-dimensional sub-manifolds W_u and W_s, called respectively the *unstable* and the *stable* manifold, defined by

$$\xi \in W_u \quad \text{iff} \quad |\varphi_0^t x - \bar{x}(t)| \to 0 \quad \text{when} \quad t \to -\infty$$

$$\xi \in W_s \quad \text{iff} \quad |\varphi_0^t x - \bar{x}(t)| \to 0 \quad \text{when} \quad t \to +\infty .$$

Here we have denoted by $\varphi_{t_1}^{t_2}$ the flow of (1.1), and assumed that it is globally defined. We also have

$$\bar{x}(0) \in W_u \cap W_s .$$

It follows from the definitions that a solution $z(t)$ is doubly asymptotic to $\bar{x}$ if and only if

$$z(0) \in W_u \cap W_s .$$

In other words, a doubly asymptotic solution exists if and only if the stable and unstable manifold intersect away from $\bar{x}(0)$. If they intersect transversally, Poincaré ([13, Chap. XXXIII]; see [11] for a modern exposition) showed that there must be infinitely many doubly asymptotic solutions.

The drawback in this approach is twofold: one must show (a) that W_u and W_s intersect; (b) that the intersection is transversal. This can be done in certain situations, the most notorious of which is the so-called Melnikov theory, [10] or [6] which depends on the presence of a small parameter ε. But it is not an easy task in general.

This is why we are trying another, variational, approach. We first simplify the problem by applying Floquet theory to the linearized system around $\bar{x}$

$$y' = JH''(t, \bar{x}(t))y . \tag{1.2}$$

If, for instance, the eigenvalues of $M(T)$ are simple (so that neither 1 or -1 is an eigenvalue), by a suitable T-periodic change of variables

$$y = P(t)z, \quad \text{with} \quad P(t+T) = P(t)$$

we can bring system (1.2) in the form

$$z' = Ez , \tag{1.3}$$

where E is a real matrix with constant coefficients, all eigenvalues of which have non-zero real part. In addition, $P(t)$ will be symplectic, [12] so that the Hamiltonian form of the system will be preserved:

$$E = JA, \quad \text{with} \quad A^* = A .$$

Write $x = \bar{x}(t) + y$, and separate the linear terms in system (1.1):

$$\begin{cases} y' = JH''(t, \bar{x}(t))y + J\nabla_y K(t, y) \\ K(t, y) \equiv H(t, \bar{x}(t) + y) - (\nabla_x H(t, \bar{x}(t)), y) - \frac{1}{2}(H''(t, \bar{x}(t))y, y) . \end{cases}$$

Performing the symplectic change of variables $y=P(t)z$, we get another Hamiltonian system

$$z'=JAz+J\nabla_z R(t,z) \tag{1.4}$$

with $R(t,z)\equiv K(t,P(t)z)$. It follows from our construction that

$$\frac{R(t,z)}{|z|^2}\to 0 \quad \text{when} \quad |z|\to 0 \tag{1.5}$$

$$R(t+T,z)=R(t,z)\ \forall(t,z). \tag{1.6}$$

We shall study Hamiltonian systems of type (1.4), satisfying conditions (1.5) and (1.6). Of course, more stringent assumptions will be needed if we are to prove the existence of doubly asymptotic orbits. We shall assume that

$$\forall t\in\mathbb{R}\ R(t,\cdot) \quad \text{is strictly convex} \tag{1.7}$$

and that, for some $\alpha>2$ and $k_1,k_2>0$, we have

$$R(t,x)\leqq\frac{1}{\alpha}(\nabla_x R(t,x),x)\ \forall t,\ \forall x\neq 0 \tag{1.8}$$

$$k_1|x|^\alpha\leqq R(t,x)\leqq k_2|x|^\alpha. \tag{1.9}$$

Condition (1.8) is equivalent to the following:

$$\forall(t,x),\ \forall\lambda\geqq 1, \qquad R(t,\lambda x)\geqq\lambda^\alpha R(t,x) \tag{1.10}$$

which implies in particular condition (1.5).

Then we prove

Theorem 1. *Assume $E=JA$ has no eigenvalues with zero real part, and that R is a C^2 function satisfying* (1.6) *to* (1.9). *Then* (1.4) *has at least two solutions z_1 and z_2 doubly asymptotic to* 0:

$$z_i'\in L^\beta(\mathbb{R}) \quad \text{and} \quad z_i\in L^\gamma(\mathbb{R})\ \forall\gamma\geqq\beta$$

$\left(\textit{where } \beta \textit{ is the conjugate exponent of } \alpha, \textit{ defined by } \frac{1}{\alpha}+\frac{1}{\beta}=1\right)$, *and which are non-trivial and distinct in the following sense:*

$$\forall k\in\mathbf{Z}, k\neq 0,\ \forall t \qquad z_1(t)\neq z_2(t+kT)\neq 0.\ \square \tag{1.11}$$

In the autonomous case, this result simplifies.

Theorem 2. *Assume that $R\in C^2(\mathbb{R}^{2N},\mathbb{R})$ satisfies:*

$$R \text{ is strictly convex, with } R(0)=0 \quad \text{and} \quad R'(0)=0, \tag{1.12}$$

$$\exists\varrho>0 \quad \text{such that} \quad \|x\|\geqq\varrho\Rightarrow\tfrac{1}{2}(Ax,x)+R(x)>0, \tag{1.13}$$

$$\exists k>0,\ \exists\alpha>2 \quad \text{such that} \quad \|x\|\leqq\varrho\Rightarrow\begin{cases}(x,R'(x))\geqq\alpha R(x)\\ k|x|^\alpha\leqq R(x).\end{cases} \tag{1.14}$$

Then the equation

$$z'=JAz+JR'(z) \tag{1.15}$$

has at least one solution, which is non-trivial and doubly asymptotic to 0

$$z \in W^{1,\beta}(\mathbb{R}) \quad \textit{and} \quad z \neq 0. \tag{1.16}$$

Proof. Modify R outside the ball $\|x\| \leqq \varrho$ to get a strictly convex function $\tilde{R} \in C^2(\mathbb{R}^{2n}, \mathbb{R})$ such that:

$$\|x\| \leqq \varrho \Rightarrow \tilde{R}(x) = R(x); \tag{1.17}$$

$$\tfrac{1}{2}(Ax, x) + \tilde{R}(x) = 0 \Rightarrow \|x\| < \varrho; \tag{1.18}$$

$$\forall x, \ \forall \lambda \geqq 1, \qquad \tilde{R}(\lambda x) \geqq \lambda^\alpha R(x); \tag{1.19}$$

$$\exists k, k' > 0 \quad \text{such that} \quad \forall x, \ k|x|^\alpha \leqq \tilde{R}(x) \leqq k'|x|^\alpha. \tag{1.20}$$

This is possible by the conditions on R. By Theorem 1, the equation

$$z' = JAz + J\tilde{R}'(z) \tag{1.21}$$

has a solution $\bar{z} \neq 0$, such that:

$$\bar{z}(t) \to 0 \quad \text{when} \quad |t| \to \infty. \tag{1.22}$$

Since the equation is autonomous, its Hamiltonian is an integral of the motion:

$$\tfrac{1}{2}(A\bar{z}(t), \bar{z}(t)) + \tilde{R}(\bar{z}(t) = h \ \ \forall t. \tag{1.23}$$

Because of (1.22), the constant h has to be 0. It then follows from (1.18) that $\|\bar{z}(t)\| < \varrho$ for all t, so that (1.15) and (1.21) in fact coincide in a neighbourhood of $\bar{z}(t)$. □

Note that we do not claim multiplicity in the autonomous case. This is because, if $\bar{z}(t)$ is a solution, so is $\bar{z}(t + t_0)$ for any $t_0 \in \mathbb{R}$. So, for any choice of T, condition (1.11) will be satisfied with, for instance, $z_1 = \bar{z}$ and $z_2(t) = \bar{z}\left(t + \frac{T}{2}\right)$.

The paper is organized as follows. In Sect. 2 we set up a variational problem on $W^{1,\beta}(\mathbb{R})$ and we show that its solutions z satisfy (1.4). From the definition of $W^{1,\beta}(\mathbb{R})$ we must have $z(t) \to 0$ when $t \to \infty$, so these solutions are doubly asymptotic to zero. The variational formulation we use is inspired from the dual action principle of Clarke, [3] and requires $R(t, \cdot)$ to be convex. As in [4], we show that the dual action functional has a local minimum at the origin but achieves lower values somewhere else. In the case when one looks for T-periodic solutions, one may then conclude existence by the Ambrosetti-Rabinowitz mountain-pass theorem.

However, since we work in an unbounded domain, there is an inherent lack of compactness, which we overcome by the concentration compactness method of Lions [8, 9]: this is the aim of Sect. 3. Then, in Sect. 4, we prove existence of one homoclinic solution for (1.1). Sections 5 and 6 are devoted to prove Theorem 1. More precisely in Sect. 5 we prove some additional compactness properties which hold for our functional; this is done by introducing a slightly weaker version of the Palais-Smale condition, which is satisfied by our functional if we assume Theorem 1 is wrong, and suffices to prove a deformation lemma. Finally Theorem 1 is proved in Sect. 6, by contradiction.

Independently, P. Rabinowitz has found heteroclinic orbits for a second order system, $x'' + V'(x) = 0$, with a periodic potential V. His method relies on minimizing the corresponding Lagrangian on a suitable subset of path space, [14]. Moreover, an earlier version of our paper has inspired some related work, such as the paper [7] by Hofer and Wysocki, where the existence result (but not the multiplicity) is extended to the non-convex case.

2. Variational formulation

We consider the following problem

$$\begin{cases} x' = JAx + J\nabla_x R(t,x) \\ x(\pm\infty) = 0. \end{cases} \tag{2.1}$$

We assume

(A1) A is a $2N \times 2N$ symmetric matrix;

(A2) All the eigenvalues of $E \equiv JA$ have non-zero real part.

Remark 2.1. (i) Assumption (A1) implies that if α is an eigenvalue of E, then $-\alpha$, $\bar{\alpha}$, $-\bar{\alpha}$ are also eigenvalues of E.

(ii) From (A2) it follows that the flow e^{tE} induced by $x' = Ex$ is hyperbolic. Then $\mathbb{R}^{2N}$ has a direct sum decomposition $\mathbb{R}^{2N} = E_u \oplus E_s$ invariant under E such that the induced flow on E_s is a contraction (i.e. $|e^{tE}x_0| \leqq ke^{-bt}|x_0|$ for some $k, b > 0$ and all $x_0 \in E_s$) and the induced flow on E_u is a expansion (i.e. $|e^{tE}x_0| \geqq ke^{bt}|x_0|$ for some $k, b > 0$ and all $x_0 \in E_s$). Moreover such a decomposition is unique. From part (i) it follows that $\dim E_s = \dim E_u = N$. □

On the nonlinearity we only assume, in this section,

(R1) $R \in C^1(\mathbb{R} \times \mathbb{R}^{2N}; \mathbb{R})$ and $R(t, \cdot)$ is strictly convex;

(R2) $R(t+T, x) = R(t,x)$ for some $T > 0$;

(R3) $R(t,x) \geqq c_1|x|^\alpha$ for some $\alpha > 2$ and $R(t,0) \equiv 0$.

We want to set up a (dual) variational formulation for problem (2.1). We start by studying the linear part of (2.1).

Lemma 2.2. *Let* $\alpha > 2$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. *Suppose* (A1–2) *hold. Then* $\forall u \in L^\beta(\mathbb{R}, \mathbb{R}^{2N})$ *there exists a unique solution* z *of the system*

$$-Jz' - Az = u. \tag{2.2}$$

such that $z \in W^{1,\beta} \cap L^r$ $\forall r \geqq \beta$. *In particular, the equation* $z = Lu$ *defines a self-adjoint, bounded linear operator*

$$L: L^\beta \to L^\alpha.$$

Proof. Take $u \in L^\beta(\mathbb{R}, \mathbb{R}^{2N})$. We want to prove that it exists a unique $z \in L^\alpha(\mathbb{R}, \mathbb{R}^{2N})$ such that

$$-Jz' - Az = u \tag{2.3}$$

or

$$z' - Ez = Ju. \tag{2.4}$$

The general solution of (2.4) is given by

$$z(t)=e^{Et}\xi+\int_0^t e^{E(t-\tau)}Ju(\tau)\,d\tau. \tag{2.5}$$

Let P_s and P_u be the projections onto E_s and E_u [see Remark 2.1 (ii)]. Then

$$z(t)=e^{Et}(P_s\xi+P_u\xi)+\int_0^t e^{E(t-\tau)}P_sJu(\tau)\,d\tau+\int_0^t e^{E(t-\tau)}P_uJu(\tau)\,d\tau. \tag{2.6}$$

We claim that, choosing

$$P_s\xi=\int_{-\infty}^0 e^{-E\tau}P_sJu(\tau)\,d\tau$$

$$P_u\xi=-\int_0^{+\infty} e^{-E\tau}P_uJu(\tau)\,d\tau$$

we have that the corresponding z given by (2.6) belongs to $L^r(\mathbb{R},\mathbb{R}^{2N})$ $\forall r\geqq\beta$. We first observe that x is well defined. Indeed, we know that

$$|e^{-Et}P_sJu(t)|\leqq ke^{bt}|P_sJu(t)|\leqq k_1e^{bt}|u(t)|$$

hence

$$\int_{-\infty}^0 |e^{-E\tau}P_sJu(\tau)|\,d\tau\leqq k_1\int_{-\infty}^0 e^{b\tau}|u(\tau)|\,d\tau$$
$$\leqq k_1\left\{\int_{-\infty}^0 e^{\alpha b\tau}d\tau\right\}^{1/\alpha}\left\{\int_{-\infty}^{+\infty}|u(\tau)|^\beta\,d\tau\right\}^{1/\beta}<+\infty.$$

With this choice of ξ we find the following formula for z

$$z(t)=\int_{-\infty}^t e^{E(t-\tau)}P_sJu(\tau)\,d\tau-\int_t^{+\infty} e^{E(t-\tau)}P_uJu(\tau)\,d\tau$$

or

$$z(t)=\int_{-\infty}^{+\infty} e^{E(t-\tau)}\chi^+(t-\tau)\,P_sJu(\tau)\,d\tau-\int_{-\infty}^{+\infty} e^{E(t-\tau)}\chi^-(t-\tau)\,P_uJu(\tau)\,d\tau, \tag{2.7}$$

where

$$\chi^+(s)=\chi^-(-s)=\begin{cases}1 & \text{if}\quad s\geqq 0\\ 0 & \text{if}\quad s<0.\end{cases}$$

We now prove that

$$z_1(t)=\int_{-\infty}^{+\infty} e^{E(t-\tau)}\chi^+(t-\tau)\,P_sJu(\tau)\,d\tau\in L^\alpha(\mathbb{R},\mathbb{R}^{2N}).$$

We first remark that

$$|e^{E(t-\tau)}\chi^+(t-\tau)\,P_sJu(\tau)|\leqq k_1e^{-b(t-\tau)}\chi^+(t-\tau)\,|u(\tau)|.$$

Set $g(x)=e^{-bx}\chi^+(x)$. Then

$$|z_1(t)|\leqq k_1(g*|u|)(t),$$

where $*$ denotes convolution. Since

$$\int_{-\infty}^{+\infty} |g|^\mu = \int_0^{+\infty} e^{-\mu b x}\, dx = \frac{1}{\mu b}$$

we have that

$$\forall \mu \geqq 1 \quad g \in L^\mu(\mathbb{R}, \mathbb{R}) \quad \text{and} \quad \|g\|_\mu^\mu = \frac{1}{\mu b}.$$

Using the well known convolution inequality

$$\|g * |u|\,\|_r \leqq \|g\|_p \, \|u\|_q$$

which holds for $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$, and $r, p, q \geqq 1$, we find

$$g * |u| \in L^r(\mathbb{R}, \mathbb{R}^{2N}) \quad \forall r \in [\beta, +\infty] \tag{2.8}$$

hence $z_1 \in L^r(\mathbb{R}, \mathbb{R}^{2N})$ for all $r \geqq \beta$. The same arguments also prove that $z_2 \in L^r(\mathbb{R}, \mathbb{R}^{2N})$ so that $z = Lu$ also belongs to $L^r(\mathbb{R}, \mathbb{R}^{2N})$ $\forall r \geqq \beta$, and:

$$\|Lu\|_\alpha \leqq c_3 \|u\|_\beta, \tag{2.9}$$

$$\|Lu\|_\beta \leqq c_4 \|u\|_\beta. \tag{2.10}$$

From the equation $z' - Ez = Ju$ it easily follows that $z \in W^{1,\beta}$. So $z \in W^{1,\beta} \cap L^r$ for all $r \geqq \beta$, as announced. In particular, $z = Lu \in L^\alpha$ (here we use the fact that $\alpha > 2$, so that $\alpha > \beta$).

It remains to show that $L: L^\beta \to L^\alpha$ is self-adjoint. Set $z = Lu$ and $w = Lv$. Then

$$u = -Jz' - Az$$

$$v = -Jw' - Aw$$

$$\begin{aligned}(v, Lu) &= \int_{-\infty}^{+\infty} (v(t), Lu(t))\, dt \\ &= \int_{-\infty}^{+\infty} (-Jw'(t) - Aw(t), z(t))\, dt \\ &= \int_{-\infty}^{+\infty} (-Jw'(t), z(t))\, dt + \int_{-\infty}^{+\infty} (Aw(t), z(t))\, dt\end{aligned}$$

while

$$(Lv, u) = \int_{-\infty}^{+\infty} (-Jz'(t), w(t))\, dt + \int_{-\infty}^{+\infty} (Az(t), w(t))\, dt.$$

Hence, due to the symmetry of A and the skew-symmetry of J,

$$(v, Lu) - (Lv, u) = \int_{-\infty}^{+\infty} \frac{d}{dt}(-Jw(t), z(t))\, dt = 0. \tag{2.11}$$

Remark 2.3. It is not true, in general, that $u \in L^\beta$ implies $Lu \in L^\alpha$ if $\beta > 2$. Take, for example,

$$u(t) = -J\begin{pmatrix} t^{\delta-1} \\ 0 \end{pmatrix} - A\begin{pmatrix} t^\delta \\ 0 \end{pmatrix}$$

for t large and positive. Then $u \in L^\beta$ provided $\delta < -\frac{1}{\beta}$. Since in such a case $Lu(t) = t^\delta$ (for t large and positive), $Lu \in L^\alpha$ if and only if $\delta < -\frac{1}{\alpha}$. On the other hand we have that $-\frac{1}{\alpha} < -\frac{1}{\beta}$. □

We now introduce the Legendre transform of R. We set

$$G(t,y) = \max\{(x,y) - R(t,x) \mid x \in \mathbb{R}^{2N}\}. \tag{2.12}$$

It follows from (R1), (R2), (R3) that

(G1) $G \in C^1(\mathbb{R} \times \mathbb{R}^{2N}; \mathbb{R})$, and $G(t,\cdot)$ is strictly convex;

(G2) $G(t+T,y) = G(t,y)$;

(G3) $0 \leqq G(t,y) \leqq c_5 |y|^\beta$;

(G4) $|\nabla_y G(t,y)| \leqq c_6 |y|^{\beta-1}$.

We can now state our dual variational principle

Lemma 2.4. *Suppose* (A1–2), (R1–3) *hold. Then the functional* $f: L^\beta(\mathbb{R}, \mathbb{R}^{2N}) \to \mathbb{R}$ *defined by*

$$f(u) = \int_{\mathbb{R}} G(t,u) - \tfrac{1}{2} \int_{\mathbb{R}} (u, Lu)$$

is well defined and of class C^1. *If* $u \in L^\beta(\mathbb{R}, \mathbb{R}^{2N})$ *is a critical point for* f, *then* $v(t) = \nabla_y G(t, u(t))$ *is a (classical) solution of* (2.1).

Proof. From (G3) it follows that

$$\int_{\mathbb{R}} |G(t,u)| \leqq c_5 \int_{\mathbb{R}} |u|^\beta$$

which implies that the first term of f is well defined. From Lemma 2.2 it follows that $Lu \in L^\alpha$ $\forall u \in L^\beta$, so that the quadratic term also is well defined. The fact that f is of class C^1 follows from (G1) and (G4).

Suppose now that u is a critical point for f on L^β. Then

$$\nabla_y G(t, u(t)) - Lu(t) = 0 \quad \text{for a.e.} \quad t \in \mathbb{R}.$$

Set $v(t) = \nabla_y G(t, u(t))$. Then

$$\begin{cases} v(t) = \nabla_y G(t, u(t)) \\ v(t) = Lu(t). \end{cases}$$

Using the Legendre reciprocity formula

$$\nabla_y G(t,y) = x \Leftrightarrow \nabla_x R(t,x) = y$$

together with Lemma 2.2 we get

$$\begin{cases} u(t) = \nabla_x R(t, u(t)) \\ u(t) = -Jv'(t) - Av(t) \end{cases}$$

i.e.

$$-Jv'(t) = Av(t) + \nabla_x R(t, v(t)).$$

Standard bootstrap arguments then show that v is actually a classical solution. □

3. First compactness properties of f

To prove existence of critical points of f we need to show that f has some compactness properties. We remark that our functional does not satisfy the Palais-Smale (PS) condition. Indeed, suppose that $v(t)$ is a critical point of f. Then, for every $k \in \mathbf{Z}$, $v_k(t) \equiv v(t+kT)$ is also a critical point for f and $f(v) = f(v_k)$. In spite of that there is no subsequence of (v_k) which converges.

This phenomenon is a consequence of the fact that f is invariant through the action of the non-compact group $\mathbf{Z}$ and is reflected in the non-compactness of the linear operator L (this in contrast to the periodic solution problem, where the setting is very similar to ours but the corresponding linear operator is compact).

A general theory to deal with this kind of non-compactness has been described by Lions [8, 9] under the name of "compactness by concentration", and we shall make use of it. We start by studying the compactness properties of L.

Lemma 3.1. *Let $(v_n) \in L^\beta(\mathbb{R}, \mathbb{R}^{2N})$ be such that*
(i) $\int_{\mathbb{R}} |v_n(t)|^\beta = \lambda$;
(ii) $\forall \varepsilon > 0\ \exists R > 0$ *such that*

$$\lambda - \varepsilon \leqq \int_{-R}^{R} |v_n(t)|^\beta \leqq \lambda \ \forall n. \tag{3.1}$$

Then there exists a subsequence v_{n_k} such that $Lv_{n_k} \to w$ in L^α.

Proof. For $h \in \mathbb{R}$, set $\tau_h w(t) = w(t+h)$.

We know that a sequence $(Lv_n) \in L^\alpha$ is precompact if

$$\begin{cases} \forall \varepsilon > 0\ \forall \omega \Subset \mathbb{R}\ \exists \delta > 0 \quad \text{such that} \quad |h| < \delta \\ \Rightarrow \|\tau_h L v_n - L v_n\|_{L^\alpha(\omega)} < \varepsilon\ \forall n; \end{cases} \tag{3.2}$$

and

$$\forall \varepsilon > 0\ \exists \omega \Subset \mathbb{R} \quad \text{such that} \quad \|Lv_n\|_{L^\alpha(\mathbb{R}\setminus\omega)} < \varepsilon\ \forall n. \tag{3.3}$$

To prove (3.2), we remark that from (2.7) it follows that

$$Lv_n = \mathscr{L} * v_n, \tag{3.4}$$

where

$$\mathscr{L}(t) = e^{Et}(\chi^+(t) P_s J - \chi^-(t) P_u J) = \mathscr{L}_s(t) - \mathscr{L}_u(t) \tag{3.5}$$

so that

$$\tau_h L v_n - L v_n = \tau_h(\mathscr{L} * v_n) - \mathscr{L} * v_n = (\tau_h \mathscr{L} - \mathscr{L}) * v_n .$$

Hence, $\forall \omega \Subset \mathbb{R}$

$$\begin{aligned} \|\tau_h L v_n - L v_n\|_{L^\alpha(\omega)} &\leqq \|(\tau_h \mathscr{L} - \mathscr{L}) * v_n\|_{L^\alpha(\mathbb{R})} \\ &\leqq \|(\tau_h \mathscr{L}_s - \mathscr{L}_s) * v_n\|_{L^\alpha(\mathbb{R})} + \|(\tau_h \mathscr{L}_u - \mathscr{L}_u) * v_n\|_{L^\alpha(\mathbb{R})} \\ &\leqq c_7 \lambda (\|\tau_h \mathscr{L}_s - \mathscr{L}_s\|_{L^{\alpha/2}(\mathbb{R})} + \|\tau_h \mathscr{L}_u - \mathscr{L}_u\|_{L^{\alpha/2}(\mathbb{R})}) \end{aligned}$$

(we have used here the convolution inequalities described in Lemma 2.2). Since $\mathscr{L}_s$ and $\mathscr{L}_u$ belong to $L^{\alpha/2}(\mathbb{R})$, we get $\|\tau_h \mathscr{L}_s - \mathscr{L}_s\|_{L^{\alpha/2}(\mathbb{R})} \to 0$ as $h \to 0$ and (3.2) follows.

To prove (3.3) we remark that for $f, u \geqq 0$ we have

$$\left\{\int_\Omega dx\left(\int_{-\infty}^{+\infty} f(x-t)\,u(t)\,dt\right)^\alpha\right\}^{1/\alpha}$$

$$\leqq \left\{\int_\Omega dx\left[\left(\int_{-\infty}^{+\infty} f(x-t)^{\beta/2}\,u(t)^\beta\,dt\right)^{1/\beta}\left(\int_{-\infty}^{+\infty} f(x-t)^{\alpha/2}\,dt\right)^{1/\alpha}\right]^\alpha\right\}^{1/\alpha}$$

$$= \|f\|^{1/2}_{L^{\alpha/2}(\mathbb{R})}\left\{\int_\Omega dx\left(\int_{-\infty}^{+\infty} f(x-t)^{\beta/2}\,u(t)^\beta\,dt\right)^{\alpha/\beta}\right\}^{(\beta/\alpha)(1/\beta)}.$$

By a standard inequality

$$\left\{\int_\Omega dx\left(\int_{-\infty}^{+\infty} g(x,y)\,dy\right)^k\right\}^{1/k} \leqq \int_{-\infty}^{+\infty} dy\left(\int_\Omega g(x,y)^k\,dx\right)^{1/k}$$

hence

$$\left\{\int_\Omega dx\left(\int_{-\infty}^{+\infty} f(x-t)\,u(t)\,dt\right)^\alpha\right\}^{1/\alpha}$$

$$\leqq \|f\|^{1/2}_{L^{\alpha/2}(\mathbb{R})}\left\{\int_{-\infty}^{+\infty} dt|u(t)|^\beta\left(\int_\Omega f(x-t)^{\alpha/2}\,dx\right)^{\beta/\alpha}\right\}^{1/\beta}. \tag{3.6}$$

Let us now take $\Omega \Subset \mathbb{R}$. Denote $\Omega = \mathbb{R}\backslash\omega$. We have

$$\|Lv_n\|_{L^\alpha(\Omega)} \leqq \|\mathscr{L}_s * v_n\|_{L^\alpha(\Omega)} + \|\mathscr{L}_u * v_n\|_{L^\alpha(\Omega)}$$

and

$$\|\mathscr{L}_s * v_n\|_{L^\alpha(\Omega)} = \left\{\int_\Omega dt\left|\int_{-\infty}^{+\infty} e^{E(t-\tau)}\chi^+(t-\tau)\,P_sJv_n(\tau)\,d\tau\right|^\alpha\right\}^{1/\alpha}$$

$$\leqq \left\{\int_\Omega dt\left|\int_{-\infty}^{+\infty} ke^{-b(t-\tau)}\chi^+(t-\tau)\,P_sJv_n(\tau)\,d\tau\right|^\alpha\right\}^{1/\alpha}$$

and using (3.6) we find

$$\|\mathscr{L}_s * v_n\|_{L^\alpha(\Omega)} \leqq k_1\|e^{-bt}\chi^+(t)\|^{1/2}_{L^{\alpha/2}(\mathbb{R})}$$
$$\times\left\{\int_{-\infty}^{+\infty} d\tau|v_n(\tau)|^\beta\left(\int_\Omega e^{-\frac{\alpha b(t-\tau)}{2}}\chi^+(t-\tau)\,dt\right)^{\beta/\alpha}\right\}^{1/\beta},$$

i.e.

$$\|\mathscr{L}_s * v_n\|_{L^\alpha(\Omega)} \leqq c_8\left\{\int_{-\infty}^{+\infty} d\tau|v_n(\tau)|^\beta\left(\int_\Omega e^{-\frac{\alpha b(t-\tau)}{2}}\chi^+(t-\tau)\,dt\right)^{\beta/\alpha}\right\}^{1/\beta}.$$

Let us now estimate

$$f(\tau) = \int_\Omega e^{-\frac{\alpha b(t-\tau)}{2}}\chi^+(t-\tau)\,dt.$$

Take $\Omega =]-\infty, -A] \cup [A, +\infty[$, $A > 0$. Then

$$\int_{-\infty}^{-A} e^{-\frac{\alpha b(t-\tau)}{2}}\chi^+(t-\tau)\,dt = \begin{cases} 0 & \text{if } \tau \geqq -A \\ \dfrac{2}{\alpha b}\left(1-e^{\frac{\alpha b(A+\tau)}{2}}\right) & \text{if } \tau \leqq -A \end{cases}$$

and

$$\int_A^{+\infty} e^{-\frac{\alpha b(t-\tau)}{2}} \chi^+(t-\tau)\,dt = \begin{cases} \dfrac{2}{\alpha b} e^{-\frac{\alpha b(A-\tau)}{2}} & \text{if } \tau \leqq A \\ \dfrac{2}{\alpha b} & \text{if } \tau \geqq A \end{cases}$$

so that

$$f(\tau) = \begin{cases} \dfrac{2}{\alpha b}\left(1 - e^{\frac{\alpha b(A+\tau)}{2}} + e^{-\frac{\alpha b(A-\tau)}{2}}\right) & \text{if } \tau \leqq -A \\ \dfrac{2}{\alpha b} e^{-\frac{\alpha b(A-\tau)}{2}} & \text{if } -A \leqq \tau \leqq A \\ \dfrac{2}{\alpha b} & \text{if } \tau \leqq A. \end{cases}$$

In particular $f(\tau) \leqq \dfrac{4}{\alpha b}$. Take $R \in]1, A[$. Then

$$f(t) \leqq \frac{2}{\alpha b} e^{-\frac{\alpha b(A-R)}{2}} \quad \forall t \in]-R, R[.$$

We deduce, taking $\Omega =]-\infty, -A] \cup [A, +\infty[$ and $R \in]1, A[$, that

$$\|\mathscr{L}_s * v_n\|_{L^\alpha(\Omega)} \leqq c_8 \left\{ \int_{-\infty}^{+\infty} |v_n(\tau)|^\beta f(\tau)^{\beta/\alpha}\,d\tau \right\}^{1/\beta}$$

$$\leqq c_8 \left\{ \int_{-\infty}^{-R} |v_n(\tau)|^\beta f(\tau)^{\beta/\alpha}\,d\tau + \int_R^{+\infty} |v_n(\tau)|^\beta f(\tau)^{\beta/\alpha}\,d\tau + \int_{-R}^{+R} |v_n(\tau)|^\beta f(\tau)^{\beta/\alpha}\,d\tau \right\}^{1/\beta}$$

and hence

$$\|\mathscr{L}_s * v_n\|^\beta_{L^\alpha(\Omega)} \leqq c_9 \left\{ \int_{-\infty}^{-R} |v_n(\tau)|^\beta\,d\tau + \int_R^{+\infty} |v_n(\tau)|^\beta\,d\tau \right\} + c_{10} e^{-\frac{\beta b(A-R)}{2}} \int_{-R}^{R} |v_n(t)|^\beta\,d\tau.$$

Fix $\varepsilon > 0$ and choose $R > 0$ such that

$$c_9 \left\{ \int_{-\infty}^{-R} |v_n(t)|^\beta\,d\tau + \int_R^{+\infty} |v_n(t)|^\beta\,d\tau \right\} \leqq \frac{1}{2}\left(\frac{\varepsilon}{2}\right)^\beta.$$

Then choose $A > R$ such that

$$c_{10} e^{-\frac{\beta b(A-R)}{2}} \lambda^\beta < \frac{1}{2}\left(\frac{\varepsilon}{2}\right)^\beta.$$

It follows that

$$\|\mathscr{L}_s * v_n\|^\beta_{L^\alpha(\Omega)} \leqq \frac{1}{2}\left(\frac{\varepsilon}{2}\right)^\beta + \frac{1}{2}\left(\frac{\varepsilon}{2}\right)^\beta,$$

i.e.

$$\|\mathscr{L}_s * v_n\|_{L^\alpha(\Omega)} \leqq \frac{\varepsilon}{2}.$$

Since we can prove with the same argument that also $\|\mathscr{L}_u * v_n\|_{L^\alpha(\Omega)} \leqq \frac{\varepsilon}{2}$, we see that (3.3) holds and hence that $(Lv_n) \in L^\alpha(\mathbb{R})$ is precompact. □

We are now in position to study the compactness properties of f. We need to make some more assumptions on the non-linear term R (essentially the usual superquadraticity assumption introduced by Ambrosetti and Rabinowitz [1]).

(R4) $R(t,x) \leqq \frac{1}{\alpha}(\nabla_x R(t,x), x)$;

(R5) $R(t,x) \leqq k_1 |x|^\alpha$.

(R4) and (R5) imply

(G5) $0 \leqq \frac{1}{\beta}(\nabla_y G(t,y), y) \leqq G(t,y)$;

(G6) $G(t,y) \geqq k_2 |y|^\beta$.

Lemma 3.2. *Suppose that* (A1–2) *and* (R1–5) *hold. Then every sequence* (u_n) *in* L^β *such that* $f(u_n) \to c$ *and* $f'(u_n) \to 0$ *is bounded in* L^β. *Moreover* $c \geqq 0$ *and* $\inf\{\|u\| \mid u \neq 0$ *and* $f'(u) = 0\} > 0$.

Proof. Let $(u_n) \in L^\beta(\mathbb{R}, \mathbb{R}^{2N})$ be such that $f(u_n) \to c$ and $f'(u_n) \to 0$. Then $c_{11} \leqq f(u_n) \leqq c_{12}$, so that

$$c_{11} \leqq \int_{-\infty}^{+\infty} G(t,u_n) - \frac{1}{2}\int_{-\infty}^{+\infty} (u_n, Lu_n) \leqq c_{12}$$

$$\int_{-\infty}^{+\infty} (\nabla_y G(t,u_n), u_n) - \int_{-\infty}^{+\infty} (u_n, Lu_n) = (f'(u_n), u_n).$$

Eliminating (u_n, Lu_n), we get

$$f(u_n) = \int_{-\infty}^{+\infty} G(t,u_n) - \frac{1}{2}\int_{-\infty}^{+\infty} (\nabla_y G(t,u_n), u_n) + \frac{1}{2}(f'(u_n), u_n) \leqq c_{12}. \tag{3.7}$$

Using (G5) we then deduce

$$\left(1 - \frac{\beta}{2}\right)\int_{-\infty}^{+\infty} G(t,u_n) \leqq c_{12} + \frac{1}{2}\|f'(u_n)\|_{L^\alpha}\|u_n\|_{L^\beta},$$

and using (G6) we deduce

$$c_{13}\int_{-\infty}^{+\infty} |u_n|^\beta \leqq c_{12} + \frac{1}{2}\|f'(u_n)\|_{L^\alpha}\|u_n\|_{L^\beta}$$

and since $\|f'(u_n)\|_{L^\alpha} \to 0$ we deduce that $\|u_n\|_{L^\beta}$ is bounded. Using this information and (G5), it follows from (3.7) that

$$f(u_n) \geqq -\tfrac{1}{2}\|f'(u_n)\|_{L^\alpha}\|u_n\|_{L^\beta} \to 0.$$

Suppose now that $u \in L^\beta$ is such that $f'(u) = 0$. Then

$$0 = \int_{-\infty}^{+\infty} (\nabla_y G(t,u), u) - \int_{-\infty}^{+\infty} (u, Lu)$$

or

$$\int_{-\infty}^{+\infty} (\nabla_y G(t,u), u) = \int_{-\infty}^{+\infty} (u, Lu).$$

Since $G(t,\cdot)$ attains its minimum at the origin, we have $G(t,u) \leqq (\nabla_u G(t,u), u)$. Using condition (G5) and the boundedness of the operator L, we get:

$$k_2 \int_{-\infty}^{+\infty} |u|^\beta \leqq c_3 \|u\|_\beta^2$$

from which we deduce

$$\frac{k_2}{c_3} \leqq \|u\|_\beta^{2-\beta}. \quad \square$$

We can now state the main lemma concerning the compactness behavior of the "Palais-Smale" sequences, i.e. of the sequences $(u_n) \in L^\beta$ such that $f(u_n) \to c$ and $f'(u_n) \to 0$.

Lemma 3.3. *Suppose* (A1–2) *and* (R1–5) *hold. Suppose that* $(u_n) \in L^\beta$ *is such that* $f(u_n) \to c > 0$ *and* $f'(u_n) \to 0$. *Then there exist* $1 \leqq m < +\infty$ *critical points* $(u^{(k)})_{1 \leqq k \leqq m}$ *of* f *(not necessarily distinct) and a subsequence* $(u_{n_p})_{p \geqq 0}$ *such that*

$$\left\| u_{n_p} - \sum_{k=1}^{m} u^{(k)}(\cdot + y_p^{(k)}) \right\|_{L^\beta} \to 0, \tag{3.8}$$

where $|y_p^{(k)} - y_p^{(k')}| \to +\infty$ *as* $p \to +\infty$ *if* $k \neq k'$. *We also have*

$$f(u_n) \to \sum_{k=1}^{m} f(u^{(k)}) \tag{3.9}$$

and the $y_p^{(k)}$ *can be taken to be integer multiples of* T.

Proof. The proof is based on a lemma by Lions contained, for example, in [8] which we recall here in the form best suited for our application.

Lemma ("Concentration compactness"). *Suppose* $\varrho_n \in L^1(\mathbb{R}, \mathbb{R})$, $\varrho_n \geqq 0$, $\int_{-\infty}^{+\infty} \varrho_n = 1$. *Then there exists a subsequence, which we still denote by* ϱ_n, *for which one of the three following possibilities happens*

(i) *vanishing:*

$$\sup_{y \in \mathbb{R}} \int_{y-R}^{y+R} \varrho_n \to 0 \quad as \quad n \to +\infty \ \forall R > 0;$$

(ii) *concentration:*

$$\exists y_n \in \mathbb{R}: \ \forall \varepsilon > 0 \ \exists R > 0: \int_{y_n - R}^{y_n + R} \varrho_n \geqq 1 - \varepsilon \ \forall n;$$

(iii) *dichotomy:*
$\exists y_n \in \mathbb{R}$, $\exists \lambda \in]0,1[$, $\exists R_n^1, R_n^2 \in \mathbb{R}$ *such that*

(a) $$R_n^1, R_n^2 \to +\infty, \frac{R_n^1}{R_n^2} \to 0;$$

(b) $$\int_{y_n - R_n^1}^{y_n + R_n^1} \varrho_n \to \lambda ;$$

(c) $$\forall \varepsilon > 0 \ \exists R > 0 : \int_{y_n - R}^{y_n + R} \varrho_n \geqq \lambda - \varepsilon \ \forall n ;$$

(d) $$\int_{y_n - R_n^2}^{y_n + R_n^2} \varrho_n \to \lambda \quad as \quad n \to +\infty . \quad \square$$

We will apply this lemma to $\varrho_n(t) = \dfrac{|u_n(t)|^\beta}{\|u_n\|_\beta^\beta}$.

Step 0. There is some $\delta > 0$ such that $\|u_n\|_\beta \geqq \delta$.

Suppose, by contradiction, that $\|u_n\|_\beta \to 0$ (up to subsequences). Then from (G3) it follows

$$0 \leqq \int_{-\infty}^{+\infty} G(t, u_n) \leqq c_5 \|u_n\|_\beta^\beta \to 0$$

and

$$\left| \int_{-\infty}^{+\infty} (u_n, Lu_n) \right| \leqq c_3 \|u_n\|_\beta^2 \to 0$$

so that

$$f(u_n) \to 0 ,$$

which contradicts the assumption that $\lim f(u_n) > 0$. It then follows that ϱ_n satisfies the assumptions of the concentration compactness lemma. The proof will now be divided into three steps.

Step 1. Vanishing cannot occur.

Suppose vanishing occurs. This implies, in particular, that there is a sequence $\varepsilon_n \to 0$ such that

$$\int_{s-1}^{s+1} |u_n(t)|^\beta \, dt : \leqq \varepsilon_n \|u_n\|_\beta^\beta \ \forall s \in \mathbb{R} .$$

Then, using the same notations as in (3.5), we write

$$|Lu_n(t)| \leqq |\mathscr{L}_s * u_n(t)| + |\mathscr{L}_u * u_n(t)|$$

with

$$|\mathscr{L}_s * u_n(t)| = \left| \int_{-\infty}^{+\infty} e^{E(t-\tau)} \chi^+(t-\tau) \, P_s J u_n(\tau) \, d\tau \right|$$

$$\leqq k_1 e^{-bt} \int_{-\infty}^{t} e^{b\tau} |u_n(\tau)| \, d\tau$$

$$\leqq k_1 e^{-bt} \sum_{k=0}^{+\infty} \int_{t-k-1}^{t-k} e^{b\tau} |u_n(\tau)| \, d\tau$$

$$\leqq k_1 e^{-bt} \sum_{k=0}^{+\infty} \left\{ \int_{t-k-1}^{t-k} e^{\alpha b \tau} \, d\tau \right\}^{1/\alpha} \left\{ \int_{t-k-1}^{t-k} |u_n(\tau)|^\beta \, d\tau \right\}^{1/\beta}$$

$$\leqq k_1 \varepsilon_n^{1/\beta} \|u_n\|_\beta e^{-bt} \left(\frac{1 - e^{-\alpha b}}{\alpha b} \right)^{1/\alpha} e^{bt} \sum_{k=0}^{+\infty} e^{-bk}$$

and finally

$$|\mathscr{L}_s * u_n(t)| \leqq k_1 \varepsilon_n^{1/\beta} \|u_n\|_\beta \left(\frac{1-e^{-\alpha b}}{\alpha b}\right)^{1/\alpha} \frac{1}{1-e^{-b}}$$

so that

$$\|Lu_n\|_\infty \to 0 \quad \text{as} \quad n \to +\infty \quad \text{if vanishing occurs}. \tag{3.10}$$

Since

$$\|Lu_n\|_\alpha^\alpha = \int_{-\infty}^{+\infty} |Lu_n|^\alpha \leqq \|Lu_n\|_\infty^{\alpha-\beta} \int_{-\infty}^{+\infty} |Lu_n|^\beta$$

and since $\|Lu_n\|_\beta \leqq c_4 \|u_n\|_\beta$ by Lemma 2.2 we find that

$$\|Lu_n\|_\alpha \to 0$$

and hence

$$|(u_n, Lu_n)| \leqq \|u_n\|_\beta \, \|Lu_n\|_\alpha \to 0 .$$

Using now the fact that $f'(u_n) \to 0$ we deduce

$$\nabla_y G(t, u_n) = Lu_n + f'(u_n) \to 0 \quad \text{in} \quad L^\alpha$$

and using condition (G6) again

$$\int_{-\infty}^{+\infty} |u_n|^\beta \leqq \frac{1}{k_2} \int_{-\infty}^{+\infty} (\nabla_y G(t, u_n), u_n) \leqq \frac{1}{k_2} \|\nabla_y G(t, u_n)\|_\alpha \, \|u_n\|_\beta \to 0 ,$$

i.e. $\|u_n\|_\beta \to 0$, which contradicts Step 0. So vanishing cannot occur.

Step 2. Concentration implies compactness.
If concentration occurs, we set

$$w_n(t) = u_n(t - y_n), \qquad v_n(t) = \frac{w_n(t)}{\|w_n\|_\beta} .$$

Then

$$\int_{-\infty}^{+\infty} |v_n|^\beta = 1$$

and for every $\varepsilon > 0$ there is some $R > 0$ such that

$$1 - \varepsilon \leqq \int_{-R}^{+R} |v_n|^\beta \leqq 1 .$$

Using Lemma 3.1 we deduce that it exists a subsequence (which we still denote by v_n) and some $\bar{z}$ such that

$$Lv_n \to \bar{z} \quad \text{in} \quad L^\alpha .$$

By Lemma 3.2 and Step 0, the sequence $\|u_n\|_\beta$ is bounded away from 0 and infinity. We can therefore find some z in L^α such that

$$Lw_n \to z \quad \text{in} \quad L^\alpha .$$

We can always assume, without loss of generality, that $y_k = n_k T$ is an integer multiple of T (if this is not the case it is enough to take $w_n(t) = u_n\left(t + \left[\frac{y_n}{T}\right] T\right)$ where $[\sigma]$ is the integer part of σ). It follows that $f(u_n) = f(w_n)$ and that $f'(w_n)(t) = f'(u_n)(\cdot + y_k)$, so that $f(w_n) \to c$, $f'(w_n) \to 0$. But then

$$z_n(t) \equiv \nabla_y G(t, w_n(t)) = f'(w_n)(t) + (Lw_n)(t) \to z(t) \quad \text{in} \quad L^\alpha .$$

Using the Legendre reciprocity formula we then deduce

$$w_n(t) = \nabla_x R(t, z_n(t)) \to \nabla_x R(t, z(t)) = w(t) \quad \text{in} \quad L^\beta$$

since $z(t) \to \nabla_x R(t, z(t))$ is continuous from L^α into L^β. From

$$w_n \to w \quad \text{in} \quad L^\beta$$

we deduce

$$Lw_n \to Lw \quad \text{in} \quad L^\alpha$$

and

$$\nabla_y G(t, w_n) \to \nabla_y G(t, w) \quad \text{in} \quad L^\alpha ,$$

so that

$$\nabla_y G(t, w) - Lw = \lim_{n \to \infty} (\nabla_y G(t, w_n) - Lw_n) = \lim_{n \to \infty} f'(w_n) = 0 ,$$

i.e., w is a critical point for f. It is clear that in such a situation Lemma 3.3 holds with $m = 1$.

Step 3. Dichotomy.

If dichotomy occurs, we set

$$w_n(t) = u_n(t + y_n) ,$$

where y_n is an integer multiple of T and

$$w_n^{(1)}(t) = w_n(t)\, \chi_{B(R_n^1)}(t) ,$$
$$w_n^{(2)}(t) = w_n(t)\, (1 - \chi_{B(R_n^2)}(t)) ,$$
$$w_n^{(3)}(t) = w_n(t) - w_n^{(1)}(t) - w_n^{(2)}(t) .$$

Set also $v_n^{(1)}(t) = \dfrac{w_n^{(1)}(t)}{\|w_n^{(1)}\|_\beta}$. The sequence $v_n^{(1)}$ satisfies the assumptions of Lemma 3.1. As in Step 2 we deduce

$$w_n^{(1)} \to z \quad \text{in} \quad L^\alpha . \tag{3.11}$$

We first prove that $f(u_n)$ splits into $f(w_n^{(1)}) + f(w_n^{(2)})$ up to a small reminder. Consider

$$f(w_n) = f(w_n^{(1)}) + f(w_n^{(2)}) + f(w_n^{(3)}) - \int_{-\infty}^{+\infty} (w_n^{(1)}, Lw_n^{(2)}) - \int_{-\infty}^{+\infty} (w_n^{(3)}, L(w_n^{(1)} + w_n^{(2)})) .$$

Since

$$\int_{-\infty}^{+\infty} |w_n^{(3)}|^\beta = \int_{-\infty}^{+\infty} |w_n|^\beta - \int_{-\infty}^{+\infty} |w_n^{(1)}|^\beta - \int_{-\infty}^{+\infty} |w_n^{(2)}|^\beta \to 0$$

we have that $f(w_n^{(3)}) \to 0$ as well as $\int_{-\infty}^{+\infty} (w_n^{(3)}, L(w_n^{(1)} + w_n^{(2)})) \to 0$.

Next we consider the term

$$\int_{-\infty}^{+\infty} (w_n^{(1)}, L_s w_n^{(2)}) = \int_{B(R_n^1)} dt\, e^{-Et} \Big(w_n(t), \int_{B(R_n^2)^c} e^{E\tau} \chi^+(t-\tau) P_s J w_n(\tau) \Big) d\tau .$$

We know that

$$|e^{E(t-\tau)} \chi^+(t-\tau) P_s| \leqq \begin{cases} 0 & \text{if } t \geqq \tau \\ k_1 e^{-b(t-\tau)} & \text{if } t < \tau . \end{cases}$$

In particular

$$|e^{E(t-\tau)} \chi^+(t-\tau) P_s| \leqq k_1 e^{-b|t-\tau|} \quad \forall t, \tau .$$

It follows that

$$\left| \int_{-\infty}^{+\infty} (w_n^{(1)}, L_s w_n^{(2)}) \right| \leqq k_1 \int_{B(R_n^1)} dt \int_{B(R_n^2)^c} d\tau\, e^{-b|t-\tau|} |w_n(t)|\, |w_n(\tau)|$$

$$\leqq k_1 e^{-\frac{b}{2}|R_n^1 - R_n^2|} \int_{B(R_n^1)} dt \int_{B(R_n^2)^c} d\tau\, e^{-\frac{b}{2}|t-\tau|} |w_n(t)|\, |w_n(\tau)| .$$

It is easy to show that the double integral is bounded independently from n. Since $|R_n^1 - R_n^2| \to +\infty$, we have that

$$\int_{-\infty}^{+\infty} (w_n^{(1)}, L_s w_n^{(2)}) \to 0 \quad \text{as} \quad n \to +\infty .$$

This proves the splitting formula

$$f(w_n) = f(w_n^{(1)}) + f(w_n^{(2)}) + o(n), \tag{3.12}$$

where $o(n) \to 0$ as $n \to +\infty$.

We now claim that $f'(w_n^{(1)}) \to 0$. Let us remark that, for every n

$$L^\beta(\mathbb{R}, \mathbb{R}^{2N}) = L^\beta(B(R_n^1), \mathbb{R}^{2N}) \oplus L^\beta(B(R_n^2)^c, \mathbb{R}^{2N}) \oplus L^\beta(\mathbb{R} \setminus (B(R_n^1) \cup B(R_n^2)^c), \mathbb{R}^{2N}) .$$

We write accordingly $u = u^{(1)} + u^{(2)} + u^{(3)}$ for every u in L^β.

We now prove

$$(f'(w_n), u^{(1)}) = (f'(w_n^{(1)}), u^{(1)}) + o_1(n)\, \|u^{(1)}\|_\beta \tag{3.13}$$

$$(f'(w_n), u^{(2)}) = (f'(w_n^{(2)}), u^{(2)}) + o_2(n)\, \|u^{(1)}\|_\beta \tag{3.14}$$

where $o_1(n)$, $o_2(n) \to 0$ as $n \to +\infty$. In fact

$$(f'(w_n), u^{(1)}) = \int_{-\infty}^{+\infty} (\nabla_y G(t, w_n), u^{(1)}) - \int_{-\infty}^{+\infty} (L w_n, u^{(1)})$$

$$= (f'(w_n^{(1)}), u^{(1)}) - \int_{-\infty}^{+\infty} (L w_n^{(2)}, u^{(1)}) - \int_{-\infty}^{+\infty} (L w_n^{(3)}, u^{(1)}) .$$

We have already seen that

$$\left|\int_{-\infty}^{+\infty}(Lw_n^{(2)},u^{(1)})\right| \leqq o_1(n)\,\|u^{(1)}\|_\beta \qquad (\operatorname{dist}(\operatorname{Supp} w_n^{(2)}, \operatorname{Supp} u^{(1)})\to\infty)$$

and that

$$\left|\int_{-\infty}^{+\infty}(Lw_n^{(3)},u^{(1)})\right| \leqq o_2(n)\,\|u^{(1)}\|_\beta \qquad (w_n^{(3)}\to 0 \quad \text{in} \quad L^\beta).$$

From this (3.13) and (3.14) follow. Then, for every $u\in L^\beta(\mathbb{R},\mathbb{R}^{2N})$

$$\begin{aligned}(f'(w_n^{(1)}),u) &= (f'(w_n^{(1)}),u^{(1)})+(f'(w_n^{(1)}),u^{(2)})+(f'(w_n^{(1)}),u^{(3)})\\ &=(f'(w_n),u^{(1)}+u^{(2)})+(f'(w_n^{(1)}),u^{(3)})-(o_1(n)+o_2(n))\,\|u^{(1)}\|_\beta . \qquad (3.15)\end{aligned}$$

The first and last term on the right clearly go to zero. We investigate the second one. Note that:

$$\begin{aligned}(f'(w_n^{(1)}),u^{(3)}) &= \int_{-\infty}^{+\infty}(\nabla_y G(t,w_n^{(1)}),u^{(3)})-\int_{-\infty}^{+\infty}(Lw_n^{(1)},u^{(3)})\\ &=-\int_{-\infty}^{+\infty}(Lw_n^{(1)},u^{(3)}).\end{aligned}$$

Moreover,

$$\left|\int_{-\infty}^{+\infty}(L_s w_n^{(1)},u^{(3)})\right| \leqq k_1 \int_{[-R_n^2,-R_n^1)]\cup[R_n^1,R_n^2]} dt\,|u(t)| \int_{-R_n^1}^{R_n^1} e^{-b|t-\tau|}|w(\tau)|\,d\tau .$$

Taking now R such that $\int_{-R}^{R}|w(\tau)|^\beta\,d\tau \geqq \lambda-\varepsilon$ we have that

$$\int_{-R_n^1}^{R_n^1} e^{-b|t-\tau|}|w(\tau)|\,d\tau = g(t)$$

with $g(t)=g_1(t)+g_2(t)$, where

$$g_1(t)=\int_{-R}^{R} e^{-b|t-\tau|}|w(\tau)|\,d\tau \leqq c_{14}e^{-b|t|}$$

while

$$g_2(t)=\int_{[-R_n^1,-R)]\cup[R,R_n^1]} e^{-b|t-\tau|}|w(\tau)|\,d\tau \leqq c_{15}\varepsilon e^{-b|t-R_n^1|}.$$

Finally, we deduce

$$\begin{aligned}\left|\int_{-\infty}^{+\infty}(L_s w_n^{(1)},u^{(3)})\right| &\leqq k_1 \int_{[-R_n^2,-R_n^1]\cup[R_n^1,R_n^2]} dt\{|u(t)|\,(c_{14}e^{-b|t|}+c_{15}\varepsilon e^{-b|t-R_n^1|})\}\\ &\leqq c_{16}(e^{-b|R_n^1|}+\varepsilon)\,\|u^{(3)}\|_\beta .\end{aligned}$$

Writing this into (3.15), we find that $f'(w_n^{(1)})\to 0$. We know that $w_n^{(1)}$ converges to a limit z in L^α by (3.11). So z has to be a critical point of f, that is, $f'(z)=0$. Now consider the sequence

$$\tilde{w}_n(t)=w_n(t)-z(t).$$

We have, by the splitting (3.12):

$$\int_{-\infty}^{+\infty} |\tilde{w}_n|^\beta \to 1-\lambda > 0$$

$$f(\tilde{w}_n) \to c - f(z)$$

$$f'(\tilde{w}_n) \to 0 .$$

We can argue for $\tilde{w}_n$ as for u_n; vanishing cannot occur since $\|\tilde{w}_n\|_\beta \to 1-\lambda$, hence either concentration of dichotomy applies.

Iterating this procedure K times, we get $z^1, \ldots, z^K$ such that $f'(z^k)=0$ and $f(w_n) \to f(z^1)+ \ldots f(z^K)$. As we have seen in Lemma 3.2, we must have $\|z^k\|_\beta^{2-\beta} \geqq \frac{k_2}{c_3} > 0$ for every k, while $\|w_n\|_\beta \to \|z^1\|_\beta + \ldots \|z^K\|_\beta$. So K has to be bounded. This is the content of Lemma 3.3.

4. Existence of a critical point for f

To prove existence of a critical point for f we will apply the Mountain Pass lemma of Ambrosetti and Rabinowitz [1] together with Lemma 3.3. First of all we check that the geometrical conditions of the MP lemma are verified.

Lemma 4.1. *Suppose* (A1–2) *and* (R1–5) *hold. Then there exist* $r>0$, $\delta>0$, *and* $v \in L^\beta(\mathbb{R}, \mathbb{R}^{2N})$ *such that*

(a) $f(u) \geqq \delta$ *for* $\|u\|_\beta = r$, $f(u) \geqq 0$ *for* $\|u\|_\beta \leqq r$;

(b) $\|v\|_\beta \geqq r$ *and* $f(v) < 0$.

Proof. (a) follows from

$$\begin{aligned} f(u) &= \int_{-\infty}^{+\infty} G(t,u) - \frac{1}{2} \int_{-\infty}^{+\infty} (u, Lu) \\ &\geqq k_2 \int_{-\infty}^{+\infty} |u|^\beta - \frac{1}{2} c_3 \|u\|_\beta^2 \\ &= k_2 \|u\|_\beta^\beta - \frac{1}{2} c_3 \|u\|_\beta^2 . \end{aligned}$$

Since $\beta < 2$, (a) follows if $\|u\|_\beta$ is small enough.

(b) We claim that there exists $v_0 \in L^\beta(\mathbb{R}, \mathbb{R}^{2N})$ such that

$$\int_{-\infty}^{+\infty} (v_0, Lv_0) > 0 .$$

Then, considering

$$\begin{aligned} f(\lambda v_0) &= \int_{-\infty}^{+\infty} G(t, \lambda v_0) - \frac{\lambda^2}{2} \int_{-\infty}^{+\infty} (v_0, Lv_0) \\ &\leqq c_5 \lambda^\beta \int_{-\infty}^{+\infty} |v_0|^\beta - \frac{\lambda^2}{2} \int_{-\infty}^{+\infty} (v_0, Lv_0) \end{aligned}$$

we have that (b) follows for λ large enough. To prove the claim, let

$$v_0(t) = -Jz'(t) + Az(t),$$

where $z(t) \in C^\infty$ and

$$z(t) = \begin{cases} \sin kt \\ 0 \\ \cos kt \\ 0 \end{cases} \quad \text{for} \quad t \in [-a, a];$$

$$z(t) = 0 \quad \text{for} \quad |t| \geqq a;$$

$$|z(t)| \leqq 1 \quad \forall t;$$

$$|z'(t)| \leqq k \quad \forall t,$$

where $k \in \mathbb{N}$ and $a \in \mathbb{R}$ will be determined presently. Since $Lv_0 = z$, we find

$$\int_{-\infty}^{+\infty} (v_0, Lv_0) = \int_{-\infty}^{+\infty} (-Jz' + Az, z)$$

since $-(z(t), Jz'(t)) = k[\sin^2 kt + \cos^2 kt] = k$ for $|t| \leqq a$, we have that

$$\begin{aligned} \int_{-\infty}^{+\infty} (v_0, Lv_0) &= 2ka - \int_{a \leqq |t| \leqq a+2} (z, -Jz') + \int_{a \leqq |t| \leqq a+2} (z, Az) \\ &\geqq 2ka - 4k - 2\|A\|(a+2) \\ &= 2k(a-2) - 2\|A\|(a+2), \end{aligned}$$

which is positive provided $a > 2$, $k > \|A\| \dfrac{a+2}{a-2}$. □

From Lemma 3.3 and Lemma 4.1 follows.

Theorem 4.2. *Suppose* (A1–2) *and* (R1–5) *hold. Then problem* (2.1) *has at least one solution* $u \neq 0$.

Proof. It is well known that, using Ekeland's variational principle (see [2, Chap. 5]) applied to the functional

$$I(\gamma) = \sup_\gamma f,$$

where $\gamma \in \Gamma \equiv \{\gamma \in C([0, 1]; L^\beta(\mathbb{R}, \mathbb{R}^{2N}))$ such that $\gamma(0) = 0$, $f(\gamma(1)) < 0\}$, one can deduce the existence of a sequence $u_n \in L^\beta$ such that

$$f(u_n) \to a \equiv \inf_\Gamma \sup_\gamma f$$

$$f'(u_n) \to 0.$$

From Lemma 3.3 immediately follows the existence of a critical point for f. □

Remark 4.3. Even if the critical point u found via Theorem 4.2 arises from a sequence u_n such that $f(u_n) \to a \equiv \inf_\Gamma \sup_\gamma f$, we only get in the limit $f(u) \leqq a$. Note however that $f(u) > 0$. □

5. Further compactness properties of f

We saw in Sect. 3 that f does not satisfy the Palais-Smale condition, because of its invariance for the non-compact group $\mathbb{Z}$. In this paragraph we will use the fact that $\mathbb{Z}$ is discrete to introduce a slightly weaker condition, which does not seem to have been noted before. We denote it by $(\overline{PS})$.

Definition 5.1. We will say that a C^1 function f on a Banach space E satisfies condition $(\overline{PS})$ if, whenever u_n is a sequence satisfying:

$$f(u_n) \to c;$$

$$f'(u_n) \to 0;$$

$$\|u_{n+1} - u_n\| \to 0,$$

then (u_n) is convergent. □

This condition is sufficient for the deformation lemma to hold:

Lemma 5.2. *Let E be a Banach space, $f: E \to \mathbb{R}$ a C^1 functional which satisfies $(\overline{PS})$. Suppose $\exists \varepsilon > 0$ such that*

$$f^{-1}([a-\varepsilon, b+\varepsilon]) \cup \{u \in E \mid f'(u) = 0\} = \emptyset.$$

Then $\exists \eta \in C([0,1] \times E; E)$ such that
(i) $\eta(0,x) = x \quad \forall x \in E$;
(ii) $f(x) \leqq a - \varepsilon$ *or* $f(x) \geqq b + \varepsilon \Rightarrow \eta(s,x) = x \quad \forall s \in [0,1]$;
(iii) $f(x) \leqq b \Rightarrow f(\eta(1,x)) \leqq a$.

Suppose, moreover, that f is invariant for the isometric representation T of a group G, such that any orbit $G_x = \{T(g)x \mid g \in G\}$ with $f(x) \in [a-\varepsilon, b+\varepsilon]$ is such that

$$\varrho_x = \inf\{\|x - T(g)x\|_E \mid g \in G - \{0\}\} > 0.$$

Then we may choose η such that
(iv) $\forall (g,t,x) \in G \times [0,1] \times E,\ T(g)\eta(t,x) = \eta(t, T(g)x)$. □

Proof. Consider the projection $p: E \to E/T,\ x \mapsto \bar{x} = G_x$. Define $\delta(\bar{x}, \bar{y}) = \inf\{\|x - T(g)y\|_E \mid g \in G\}$. δ is well defined on $(E/T)^2$, symmetric and satisfies the triangular inequality. Moreover, if $\|x-y\|_E \leqq \frac{1}{2}\min(\varrho_x, \varrho_y)$, then $\delta(\bar{x}, \bar{y}) = \|x-y\|_E$. Hence $\delta(\bar{x}, \bar{y}) = 0$ implies $\bar{x} = \bar{y}$, and δ is a metric.

Now, to each $x \in Y = \{y \in E; f'(y) \neq 0,\ f(y) \in [a-\varepsilon,\ b+\varepsilon]\}$ we may associate a vector $V_x \in E$ and a radius $r_x < \frac{\varrho_x}{2}$ such that, for any $y \in B_E(x, r_x)$:

$$\begin{cases} \|f'(y)\|_{E^*} \|V_x\|_E \leqq 2(b-a) \\ \langle f'(y), V_x \rangle \leqq a - b. \end{cases}$$

We may also impose, for any $(x,g) \in Y \times G$:

$$\begin{cases} r_x = r_{T(g)x} = r_{\bar{x}} \\ V_{T(g)x} = T(g) V_x. \end{cases}$$

$\mathscr{U} = \{B_\delta(\bar{x}, r_{\bar{x}}) \mid \bar{x} \in Y/T\}$ is then an open covering of the metric, thus precompact space $(Y/T, \delta)$. $\mathscr{U}$ has therefore a locally finite refinement $\{W_i \mid i \in I\}$. Now to any i we may associate x_i such that $W_i \subset B_\delta(\bar{x}_i, r_{x_i})$. The family

$$\{\mathcal{O}_{i,g} = p^{-1}(W_i) \cap B_E(T(g)x_i, r_{T(g)x_i}) \mid (i,g) \in I \times G\}$$

is then a locally finite covering of Y.

Choose $\varphi : \mathbb{R} \to \mathbb{R}^+$, smooth, with $\varphi(x) = 1$ for all $x \in [a,b]$ and $\varphi(x) = 0$ for all $x \in]-\infty, a-\varepsilon] \cup [b+\varepsilon, +\infty[$.

Define $V(x)$ by

$$V(x) = \varphi(f(x)) \frac{\left(\sum_{(i,g) \in I \times G} d(x, Y \backslash \mathcal{O}_{i,g}) V_{T(g)x_i}\right)}{\left(\sum_{(i,g) \in I \times G} d(x, Y \backslash \mathcal{O}_{i,g})\right)}.$$

$V(\cdot)$ is a locally Lipschitz vector field such that

(a) $\forall x \in E,\ \|f'(x)\|_{E^*} \|V(x)\|_E \leqq 2(b-a)$;

(b) $f(x) \notin [a-\varepsilon, b+\varepsilon] \Rightarrow V(x) = 0$;

(c) $\forall x \in E,\ \langle f'(x), V(x) \rangle \leqq 0$;

(d) $f(x) \in [a,b] \Rightarrow \langle f'(x), V(x) \rangle \leqq a-b$;

(e) $\forall g \in G,\ V \circ T(g) = T(g) \circ V$.

Define η by

$$\begin{cases} \eta(0,x) = x & x \in E \\ \dfrac{\partial}{\partial s} \eta(s,x) = V \circ \eta(s,x) & s \geqq 0. \end{cases}$$

Let $[0, L_x[$ be the maximal interval of definition of $\eta(\cdot, x)$.

Suppose that for some $x \in E$, $L_x < +\infty$. This implies

$$\begin{cases} \int_0^{L_x} \|V \circ \eta(t,x)\|\, dt = +\infty, \\ \forall t \in [0, L_x[,\ f \circ \eta(t,x) \in [a-\varepsilon, b+\varepsilon]. \end{cases}$$

Consider the sequence $(t_n)_{n \geqq 0}$ such that

$$\begin{cases} t_0 = 0,\ t_n \leqq t_{n+1} < L_x \\ \int_{t_n}^{t_{n+1}} \|V \circ \eta(t,x)\|\, dt = \sqrt{L_x - t_n}. \end{cases}$$

We have

(α) $$\inf_{[t_n, t_{n+1}]} \|f' \circ \eta(t,x)\| \leqq 2(b-a) \Big[\sup_{[t_n, t_{n+1}]} \|V \circ \eta(t,x)\| \Big]^{-1} \leqq 2(b-a) \sqrt{L_x - t_n}$$

(β) $$\sup_{(u,v) \in [t_n, t_{n+1}]^2} \|\eta(u,x) - \eta(v,x)\| \leqq \sqrt{L_x - t_n}$$

(γ) $$\int_0^{l_x} \|V \circ \eta(t,x)\|\, dt = \sum_{n=0}^{+\infty} \sqrt{L_x - t_n}, \quad \text{where} \quad l_x = \lim_{n \to \infty} t_n \leqq L_x.$$

If $l_x < L_x$, the left term of (γ) is finite, and the right one infinite. So we always have $l_x = L_x$, and $\sqrt{L_x - t_n} \to 0$.

Choosing $u_n = \eta(s_n, x)$, with $s_n \in [t_n, t_{n+1}]$ and $\|V \circ \eta(s_n, x)\|_E \leqq 4(b-a)\sqrt{L_x - t_n}$, and applying $\overline{PS}$ to (u_n), we obtain a contradiction with $f(u_n)$ decreasing to $c \in [a-\varepsilon, b+\varepsilon]$ [see (b)–(c)].

We have thus proved that η is defined on $\mathbb{R}^+ \times E$. Then (b) gives (ii), (c) and (d) give (iii) and (e) gives (iv). □

Lemma 5.3. *Suppose* (A1–2) *and* (R1–5) *hold. Suppose, moreover, that f has only critical points of the form $u(\cdot + kT)$, u given by Theorem 4.2. Then* ($\overline{PS}$) *holds for f.*

Proof. Consider a sequence $(u_n) \subset L^\beta$ such that $f(u_n) \to c$, $f'(u_n) \to 0$ and $\|u_n - u_{n-1}\| \to 0$.

We know, by Lemma 3.3, that $c = mf(u)$ for some integer $m > 0$.

Suppose that for some $A > 0$ and $\varepsilon > 0$ we could extract $(u_{n_p})_{p \geqq 0}$ such that

$$\forall p \;\; u_{n_p} \notin \bigcup_{(y_1, \ldots, y_m) \in D} B(u(\cdot + y_1) + \ldots + u(\cdot + y_m), \varepsilon),$$

where $D = \{(y_1, \ldots, y_m) \in \mathbf{Z}^m T \,|\, i \neq j \Rightarrow |y_i - y_j| > A\}$, $D = \mathbf{Z}$ for $m = 1$.

The existence of (u_{n_p}) contradicts Lemma 3.3. So there is no such subsequence and we have

$$\left\| u_n - \sum_{k=1}^{m} u(\cdot + y_n^{(k)}) \right\|_{L^\beta} \to 0, \tag{5.2}$$

where $|y_n^{(k)} - y_n^{(k')}| \to +\infty$ if $k \neq k'$ and where the y_n are integer multiples of T.

Let us remark that

$$\inf_{k \geqq 1} \|u(\cdot) - u(\cdot + kT)\|_{L^\beta} = \delta > 0. \tag{5.3}$$

This follows easily noticing that

$$\lim_{k \to +\infty} \|u(\cdot) - u(\cdot + kT)\|_{L^\beta} = 2\|u\|_{L^\beta} > 0$$

and that

$$u(t) = u(t + kT)$$

implies $u(t) = 0$ for every t.

If $m = 1$ in (5.2) we have that

$$\begin{aligned} \|u_{n+1} - u_n\| &= \|u_{n+1} - u(\cdot + y_{n+1}) + u(\cdot + y_{n+1}) - u(\cdot + y_n) + u(\cdot + y_n) - u_n\| \\ &\geqq \|u(\cdot + y_{n+1}) - u(\cdot + y_n)\| - \|u_{n+1} - u(\cdot + y_{n+1})\| - \|u(\cdot + y_n) - u_n\| \end{aligned}$$

which implies

$$\begin{aligned} \|u(\cdot + y_{n+1}) - u(\cdot + y_n)\| &\leqq \|u_{n+1} - u_n\| \\ &\quad + \|u_{n+1} - u(\cdot + y_{n+1})\| + \|u(\cdot + y_n) - u_n\| < \delta; \end{aligned}$$

hence

$$y_n = \xi \quad \text{for} \quad n \quad \text{sufficiently large}.$$

It follows that

$$u_n \to u(\cdot + \xi)$$

and ($\overline{PS}$) holds.

For $m>1$ we have, proceeding analogously:

$$\left\|\sum_{k=1}^{m}(u(\cdot+y_{n+1}^{(k)})-u(\cdot+y_n^{(k)}))\right\|$$
$$\leqq \|u_{n+1}-u_n\|+\left\|u_{n+1}-\sum_{k=1}^{m}u(\cdot+y_{n+1}^{(k)})\right\|+\left\|\sum_{k=1}^{m}u(\cdot+y_n^{(k)})-u_n\right\|.$$

On the other hand, for n large,

$$\left\|\sum_{k=1}^{m}(u(\cdot+y_{n+1}^{(k)})-u(\cdot+y_n^{(k)}))\right\|\geqq \sum_{k=1}^{m}\|u(\cdot+y_{n+1}^{(k)})-u(\cdot+y_n^{(k)})\|-\varepsilon.$$

Taking n large and recalling (5.3) we deduce from the above inequalities that

$$y_{n+1}^{(k)}=y_n^{(k)} \quad \text{for} \quad n \quad \text{sufficiently large},$$

contradiction which proves that $m=1$ and that $(\overline{PS})$ holds. □

6. Existence of a second critical point

In this section we will prove the existence of a second solution for system 2.1. It will be found by a suitable inf-max procedure.

From now on, we set

$$s*u(t)=u(t+s).$$

The map $u\mapsto s*u$ is clearly an isometry of L^β into itself. Because of the periodicity of R, we have:

$$f(nT*u)=f(u) \ \forall u\in L^\beta, \ \forall n\in\mathbf{Z}.$$

We now prove

Theorem 6.1. *Assume* (A1–2) *and* (R1–5) *hold. Then problem* (2.1) *has at least two solutions* v *and* w *such that* $kT*v\neq w\neq 0 \ \forall k\in\mathbf{Z}$.

Proof. We know by Theorem 4.2 that f has a critical point $v\neq 0$. We will assume that all critical points of f are of the form $kT*v$ for some $k\in\mathbf{Z}$, and we will derive a contradiction.

Set:

$$\|v\|_\beta=\mu_1$$
$$\inf_{k\neq 0}\|v-kT*v\|=\mu_2.$$

We know that $\mu_1>0$, $\mu_2>0$, and $\mu_2\leqq 2\mu_1$. Now take $\mu>0$ so small that

$$\mu<\tfrac{1}{3}\mu_2 \quad \text{and} \quad \|u\|\leqq\mu\Rightarrow f(u)<c=f(v).$$

Next consider the sets:

$$\Gamma^s=\{\gamma\in C^\infty([0,1];L^\beta)\,|\,\gamma(0)=0, f(s*\gamma(1))<0 \ \ \forall s\}$$

and, for every $\gamma\in\Gamma^s$,

$$\Sigma_\gamma^s=\left\{\gamma\in C^\infty([0,1]^2;L^\beta)\,\middle|\, \begin{matrix}\sigma(0,s)=0 & \sigma(1,s)=sT*\gamma(1)\\ \sigma(t,0)=\gamma(t) & \sigma(t,1)=T*\gamma(t)\end{matrix}\right\}.$$

Consider also the related sets Γ^c and Σ_γ^c, which satisfy the same conditions

$$\Gamma^c=\{\gamma\in C^0([0,1];L^\beta)\,|\,\gamma(0)=0, f(s*\gamma(1))<0\ \ \forall s\}$$

and, for every $\gamma\in\Gamma^c$,

$$\Sigma_\gamma^c=\left\{\sigma\in C^0([0,1]^2;L^\beta)\,\middle|\,\begin{matrix}\sigma(0,s)=0 & \sigma(1,s)=sT*\gamma(1)\\ \sigma(t,0)=\gamma(t) & \sigma(t,1)=T*\gamma(t)\end{matrix}\right\},$$

where $\gamma\in\Gamma^c$.

Since C^∞ is dense in C^0, we must have:

$$\inf_{\gamma\in\Gamma^s}\max f\circ\gamma=\inf_{\gamma\in\Gamma^c}\max f\circ\gamma\equiv c_1$$

$$\inf_{\gamma\in\Gamma^s}\inf_{\sigma\in\Sigma^s}\max f\circ\gamma=\inf_{\gamma\in\Gamma^c}\inf_{\sigma\in\Sigma^c}\max f\circ\gamma\equiv c_2\,.$$

Taking $\sigma(t,s)=sT*\gamma(t)$, we find $c_1\leqq c_2$. By Lemma 4.1, we have $c_2\geqq c_1\geqq\delta>0$. Since we have assumed that f has only critical points of the form $kT*v$, condition $(\overline{PS})$ holds, and hence the deformation Lemma 5.2 with $G=\mathbb{Z}$, $T(n)=nT*$. Using the corresponding deformation, we find that

$$c_1=c=c_2\,.$$

So, for every $\varepsilon>0$, there exists $\gamma_\varepsilon\in\Gamma^s$ and $\sigma_\varepsilon\in\Sigma^s_{\gamma_\varepsilon}$ such that

$$\sup f\circ\sigma_\varepsilon\leqq c+\varepsilon\,.$$

The family $(\varphi_k)_{k\geqq 0}$, $\varphi_k:[0,1]^2\to\mathbb{R}$ defined by

$$\varphi_k(s,s')=\|\gamma_\varepsilon(s)-kT*\gamma(s')\|-\|\gamma_\varepsilon(s)\|-\|\gamma_\varepsilon(s')\|$$

is equicontinuous.

Indeed

$$|\varphi_k(s+h,s'+h')-\varphi_k(s,s')|\leqq 2\|\gamma_\varepsilon(s+h)-\gamma_\varepsilon(s)\|+2\|\gamma_\varepsilon(s'+h')-\gamma_\varepsilon(s')\|\,.$$

Moreover, $\varphi_k(0,0)=0\ \ \forall k\geqq 0$. By the Ascoli-Arzelá theorem (φ_k) is thus precompact for the uniform topology.

But, for all (s,s'),

$$\|\gamma_\varepsilon(s)-kT*\gamma_\varepsilon(s')\|\xrightarrow[k\to+\infty]{}\|\gamma_\varepsilon(s)\|+\|\gamma_\varepsilon(s')\|\,.$$

So, if a subsequence (φ_{k_p}) converges uniformly the limit must be zero. Finally (φ_k) converges uniformly to zero.

Hence, setting

$$\mu_\varepsilon(k)=\inf\{\|\gamma_\varepsilon(t)-kT*\gamma_\varepsilon(t')\|\,|\,\|\gamma_\varepsilon(t)\|\geqq\mu,\ \|\gamma_\varepsilon(t')\|\geqq\mu\}$$

we see that there exists $K_\varepsilon>0$ such that, whenever $k\geqq K_\varepsilon$, we have

$$\mu_\varepsilon(k)\geqq\frac{3\mu}{2}\,.$$

Now define a map $\tilde{\sigma}_\varepsilon:[0,1]\times[0,K_\varepsilon]\to L^\beta$ by:

$$\tilde{\sigma}_\varepsilon(t,s)=kT*\sigma_\varepsilon(t,s-kT),$$

where k is the integer part of s/T.

Note that the maps $d_k:[0,1]\times[0,K_\varepsilon]\to\mathbb{R}$ defined by

$$d_k(t,s)=\|\tilde{\sigma}_\varepsilon(t,s)-kT*v\|$$

are C^∞, so that, by Sard's theorem, almost every $\mu\in\mathbb{R}$ is a regular value. We may therefore assume that the subsets $d_k^{-1}\left(\frac{\mu}{2}\right)$ are either empty or closed submanifolds of dimension 1, that is, either loops contained in the interior part of the rectangle or arcs starting and ending on the boundary.

These curves have no self-intersections since they are submanifolds. By the definition of μ_2, they are pairwise disjoint. We claim that there is no arc connecting the side $s=0$ with the side $s=K_\varepsilon$.

Indeed, suppose there is such an arc, starting at $(t_0,0)$ and ending at (t_1,K_ε). Then there is some k_0 such that

$$\|\sigma_\varepsilon(t_0,0)-k_0T*v\|=\frac{\mu}{2}$$

$$\|\sigma_\varepsilon(t_1,K_\varepsilon)-k_0T*v\|=\frac{\mu}{2}$$

and hence

$$\|\sigma_\varepsilon(t_0,0)-\sigma_\varepsilon(t_1,K_\varepsilon)\|\leqq\mu.$$

Now $\sigma_\varepsilon(t_0,0)=\gamma_\varepsilon(t_0)$ and $\sigma_\varepsilon(t_1,K_\varepsilon)=K_\varepsilon T*\gamma_\varepsilon(t_1)$. We have thus contradicted the definition of K_ε, unless

$$\|\sigma_\varepsilon(t_0,0)\|<\mu$$

or

$$\|\sigma_\varepsilon(t_1,K_\varepsilon)\|=\|\sigma_\varepsilon(t_1,0)\|<\mu.$$

Writing this in the preceding inequalities, we get $\|k_0T*v\|=\|v\|<\frac{3\mu}{2}<\mu_1$, which contradicts the definition of μ_1. The claim is proved.

Since there is no arc connecting $s=0$ with $s=1$, we can construct a C^∞ path

$$\xi:[0,1]\to[0,1]\times[0,K_\varepsilon]$$

such that $\xi(0)$ lies on $t=0$, $\xi(1)$ lies on $t=1$ and ξ does not intersect any of the $d_k^{-1}\left(\frac{\mu}{2}\right)$:

$$\forall h,\ \forall k\in\mathbf{Z},\qquad \|\sigma_\varepsilon(\xi(h))-kT*v\|\geqq\frac{\mu}{2},$$

Set

$$\tilde{\gamma}_\varepsilon(h)=\sigma_\varepsilon(\xi(h)),\qquad 0\leqq h\leqq 1.$$

The $\tilde{\gamma}_\varepsilon$ belong to Γ for every $\varepsilon > 0$, and

$$\max f \circ \tilde{\gamma}_\varepsilon \to \inf_{\gamma \in \Gamma} \max f \circ \gamma = c.$$

Extract a sequence $\tilde{\gamma}_n$, $n \in \mathbb{N}$, such that

$$\begin{cases} \forall h, \ \forall k \in \mathbf{Z}, \quad \|\tilde{\gamma}_n(h) - kT * v\| \geqq \dfrac{\mu}{2} \\ \max f \circ \tilde{\gamma}_n \leqq \inf\limits_{\gamma \in \Gamma} \max f \circ \gamma + \dfrac{1}{n}. \end{cases}$$

By Ekeland's variational principle there will be for each n some $\hat{\gamma}_n \in \Gamma$ such that

$$\begin{cases} \max f \circ \hat{\gamma}_n \leqq \inf\limits_{\gamma \in \Gamma} \max f \circ \gamma + \dfrac{1}{n} \\ \|\hat{\gamma}_n(h) - \tilde{\gamma}_n(h)\| \leqq \dfrac{1}{\sqrt{n}} \quad \forall h \in [0,1] \\ \forall \gamma \in \Gamma, \max g \circ \gamma \geqq \max f \circ \hat{\gamma}_n - \dfrac{1}{\sqrt{n}} \max\limits_h \|\hat{\gamma}(h) - \tilde{\gamma}_n(h)\| \end{cases}$$

and the latter condition translates into (see [1])

$$\begin{cases} \exists h_n : f \circ \hat{\gamma}_n(h_n) = \max\limits_h f \circ \hat{\gamma}_n(h) \\ \|f'(\hat{\gamma}_n(h_n))\| \leqq \dfrac{1}{\sqrt{n}}. \end{cases}$$

Setting $\hat{\gamma}_n(h_n) = u_n$, we have found a sequence such that

$$f(u_n) \to c, \qquad f'(u_n) \to 0$$

$$\forall k \in \mathbf{Z}, \qquad \|u_n - kT * v\| \geqq \frac{\mu}{2}.$$

This contradicts the concentration-compactness lemma and ends the proof. □

Acknowledgement. The first author thanks I. Ekeland and all the members of CEREMADE for their kind hospitality.

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