The best of all possible worlds

The idea of optimization

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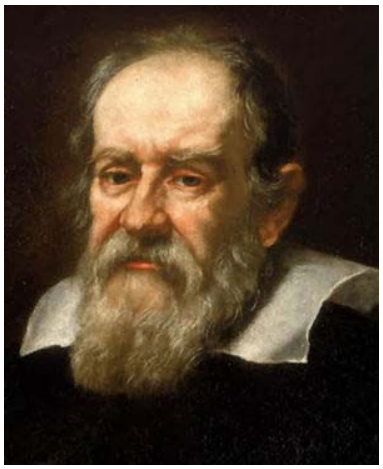
ICIAM 2007

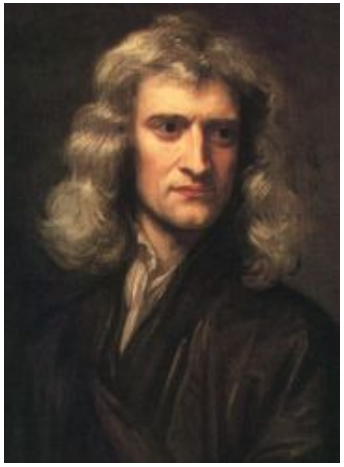
The Renaissance



Galilei 1564 -1642

Descartes 1596 - 1650

Leibniz 1646 - 1716



Newton 1643 - 1727

The two pillars of wisdom

- *And God saw every thing that he had made, and, behold, it was very good*
  - (Genesis, 1-31)
- *[The universe] cannot be read until we have learnt the language and become familiar with the characters in which it is written. It is written in mathematical language, and the letters are triangles, circles and other geometrical figures, without which means it is humanly impossible to understand a single word*
  - (Galileo Galilei, *Il Saggiatore*, 1623)

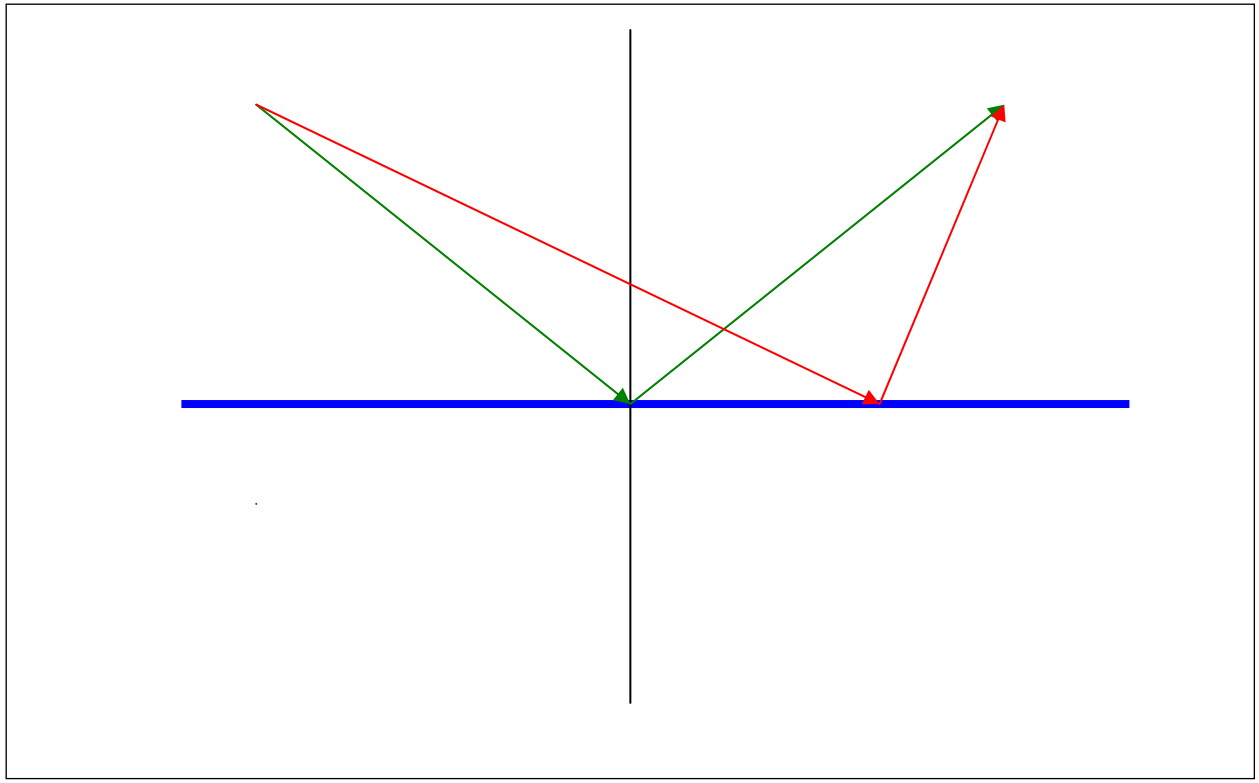
A basic question:

Is the universe optimized ?

A test case:

The theory of light

Reflexion

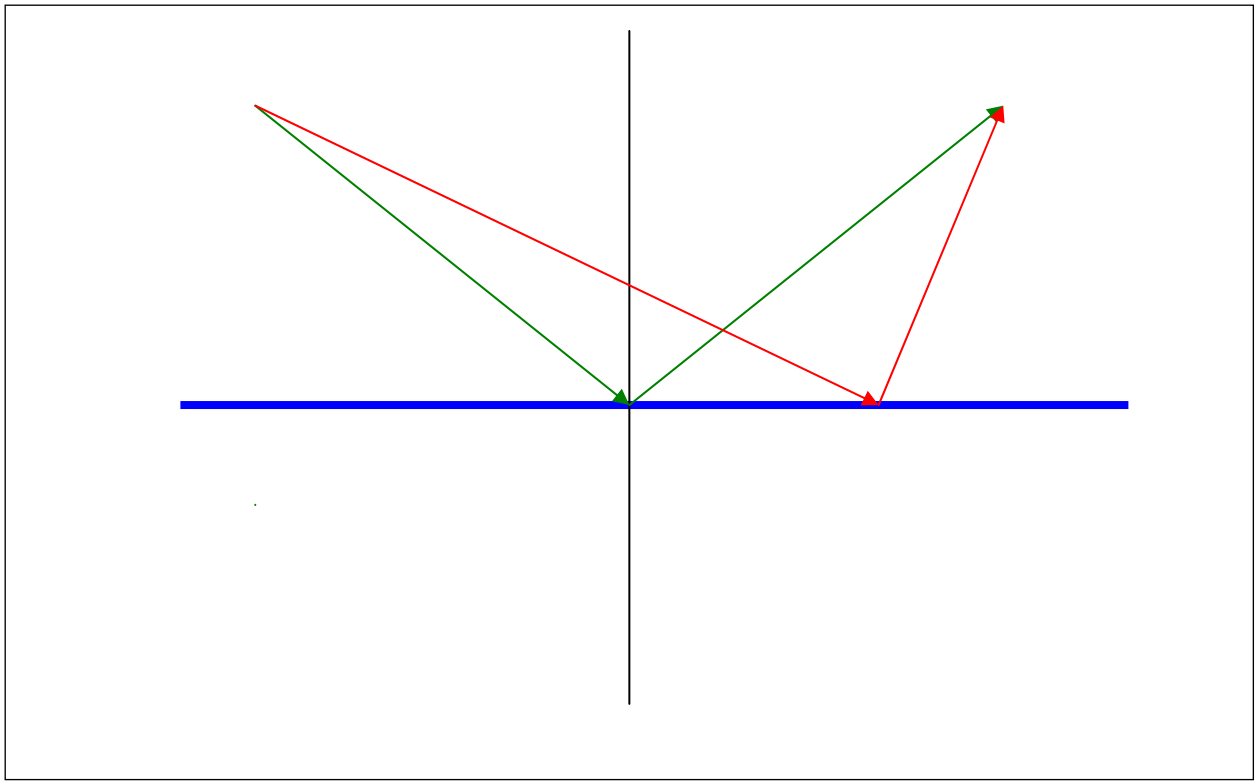


Incoming angle = Outgoing angle.

$i = r$

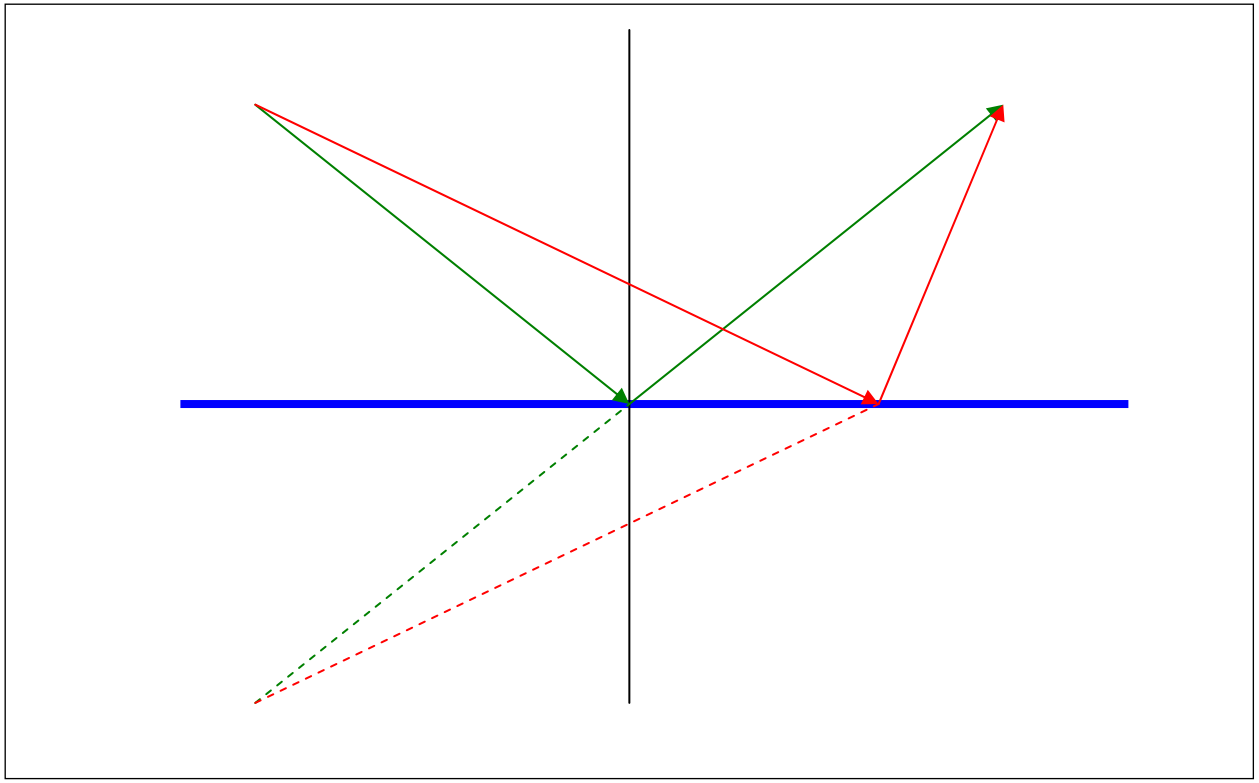
(actual path of light is green – test path is red)

Hero of Alexandria (10-70 AD)



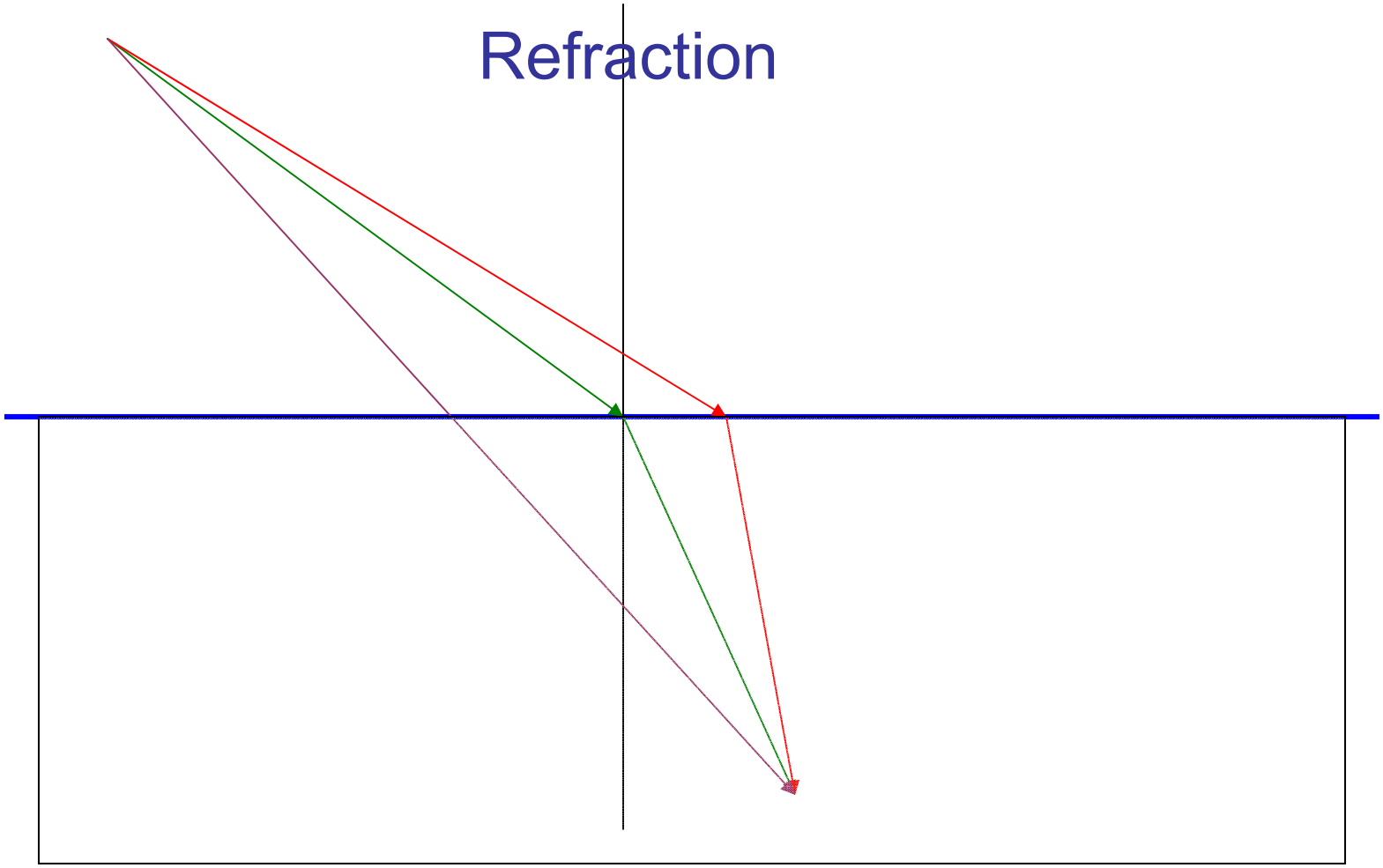
showed that light takes
the shortest path

Hero's proof



Relies on the geometrical fact that the shortest path between two points is a straight line

Refraction



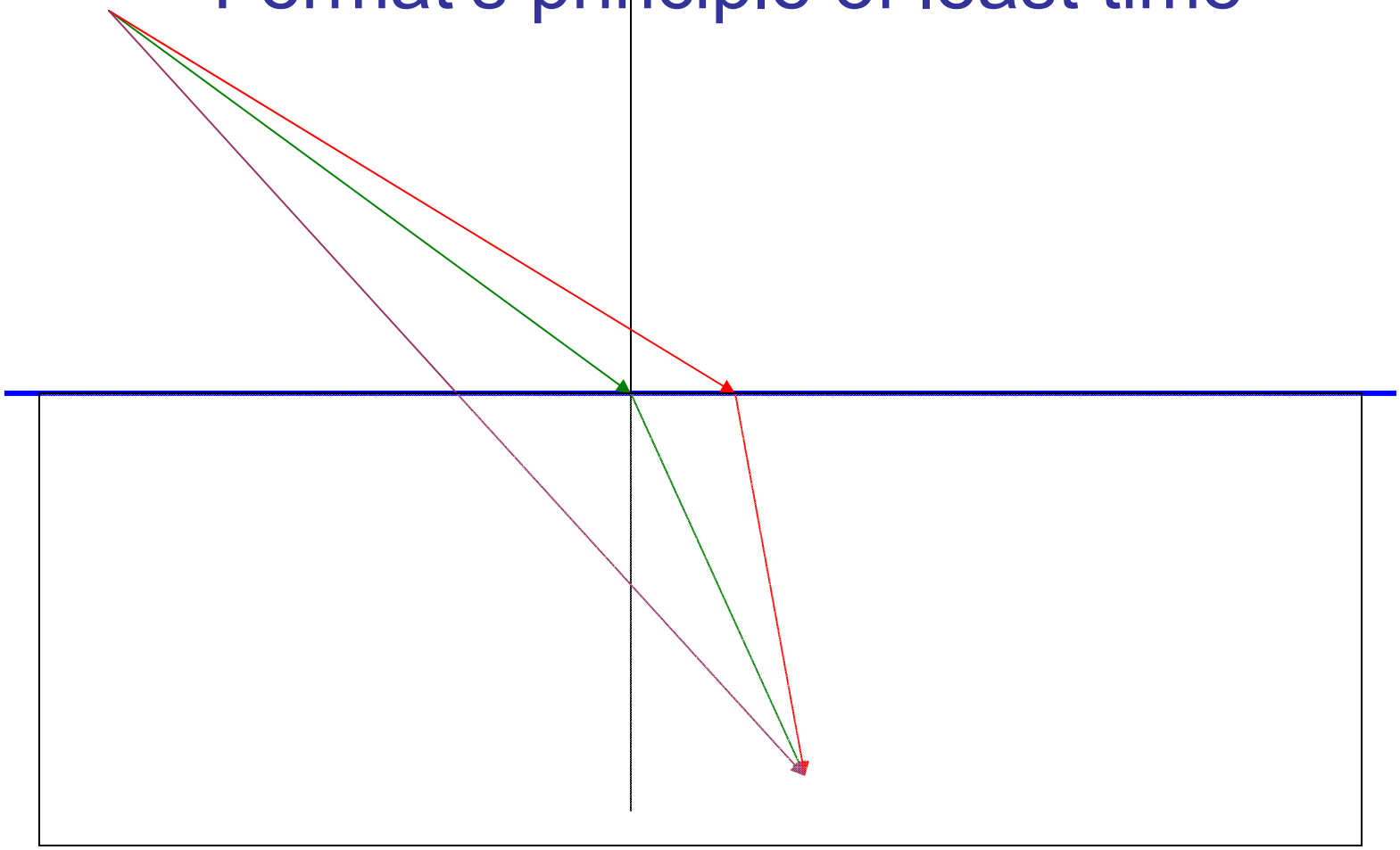
Light no longer takes the shortest path

sin i = 3.2 *sin* r

Fermat (1601-1665)

Light does not take the shortest, but the fastest path

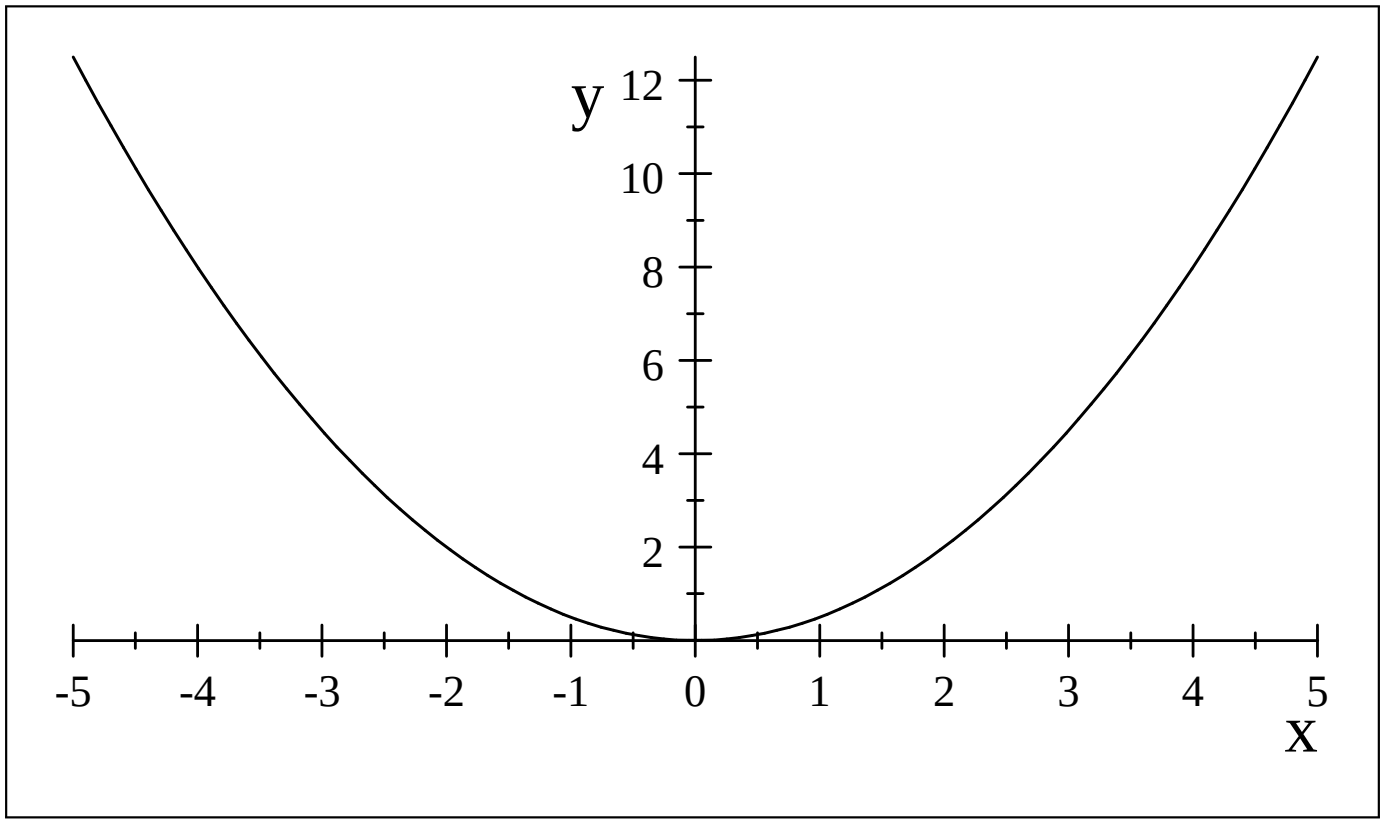
Fermat's principle of least time



Light travels 3.2 times faster in air than in water

The green path is not the shortest, but it is the fastest

Fermat's rule



At the minimum point, the derivative is zero (the tangent is horizontal)

$$f'(x) = 0$$

$$\sin i = 3.2 \sin r$$

Who optimizes ?

The principle upon which you build your proof, namely that nature always acts by the shortest and simplest ways, is but a moral principle, not a physical one, which is not and cannot be the the cause of any effect of nature.

(Clerselier, letter to Fermat, 1662)

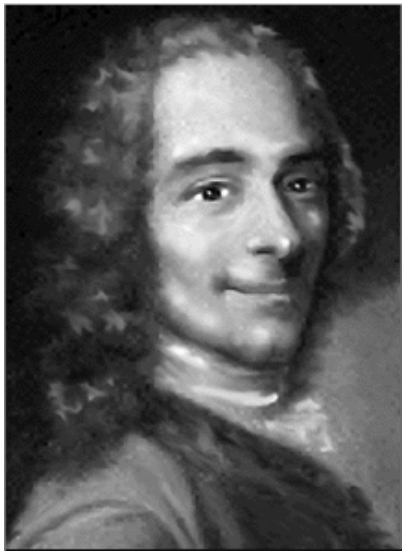
I heartily abandon you my pretended conquest in physics, provided you leave me in possession of my geometrical problem, all pure and in abstracto, by which one can find the path of a moving object which crosses two different mediums, and which tries to end its motion as soon as possible

(Fermat, letter to Clerselier, 1662)

The crossroads

Pierre Louis Moreau
de Maupertuis
1698 - 1759

Leonhard Euler
1707 - 1783



Voltaire
1694 - 1778

The principle of least action

*The general principle … is that the quantity of '**action** required to change anything in nature is the smallest possible*

(Maupertuis, 1745)

*The **action** is proportional to the distance multiplied by the speed along the path*

(Maupertuis, 1744)

Euler shows that the least action principle contains all the known laws of mechanics, plus some new ones: free motion of a rigid body, motion in a constant gravitational field, attraction by a fixed center
The young *Lagrange*, formulates in 1754 the fundamental equations of mechanics:

$$\frac{d}{dt}\frac{\partial L}{\partial x_i'} = \frac{\partial L}{\partial x_i}, \quad 1 \leq i \leq n$$

which are the moral equivalent of *Fermat's rule*

$$f'(x) = 0$$

God the optimizer

Since the constitution of the universe is perfect, and completed by a very wise creator, absolutely nothing happens in this world which could not be explained by some maximizing or minimizing argument

(Euler, 1744)

When it will be realized that all laws of nature rely on the principle of betterment, no one will be able to doubt that they owe their institution to an all-powerful and all-wise being

(Maupertuis, 1752)

Maupertuis suffers two deaths

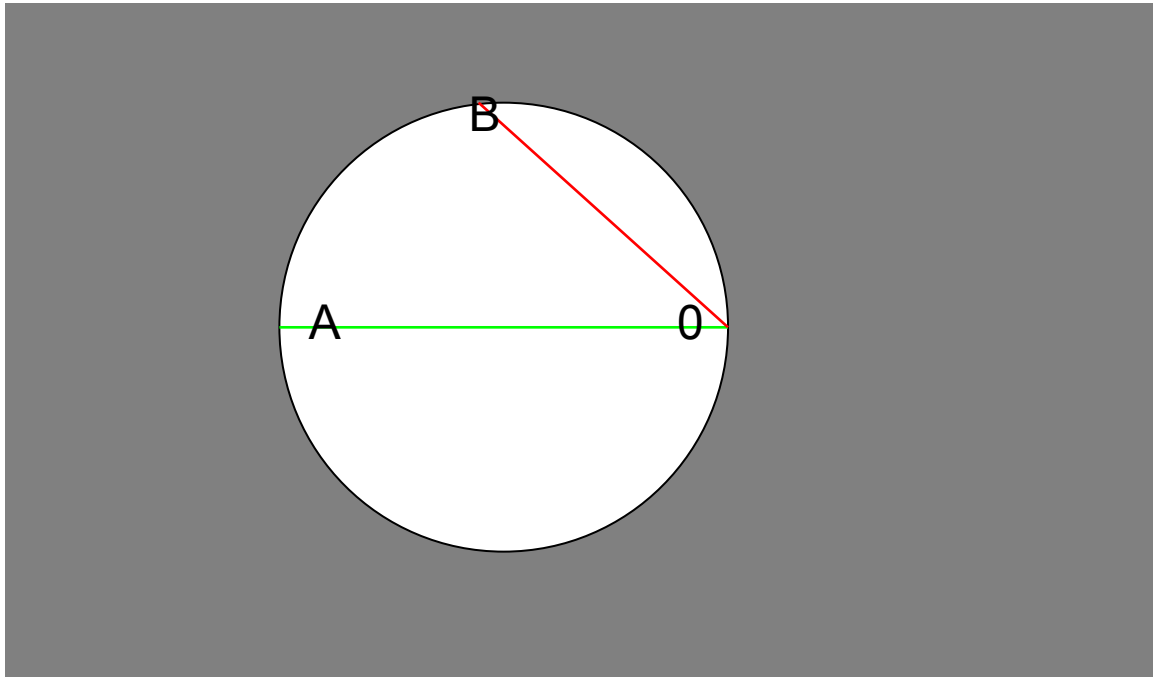
- At the hands of Voltaire
  - Story of Dr. Akakia and the native of Saint-Malo (1753)
  - Candide (1759)
- At the hands of the Chevalier d'Arcy
  - disproves the principle of least action by giving a counter-example (1752)

Voltaire

- Accuses Maupertuis of plagiarizing Leibniz
  - The only witness, Henzi, was beheaded in Bern in 1749
- Ridicules him as Doctor Pangloss:
  - All is well that ends well in the best of all possible worlds
  - "*for, if all this had not happened, we would not be sitting here in the shade eating pistachios*"

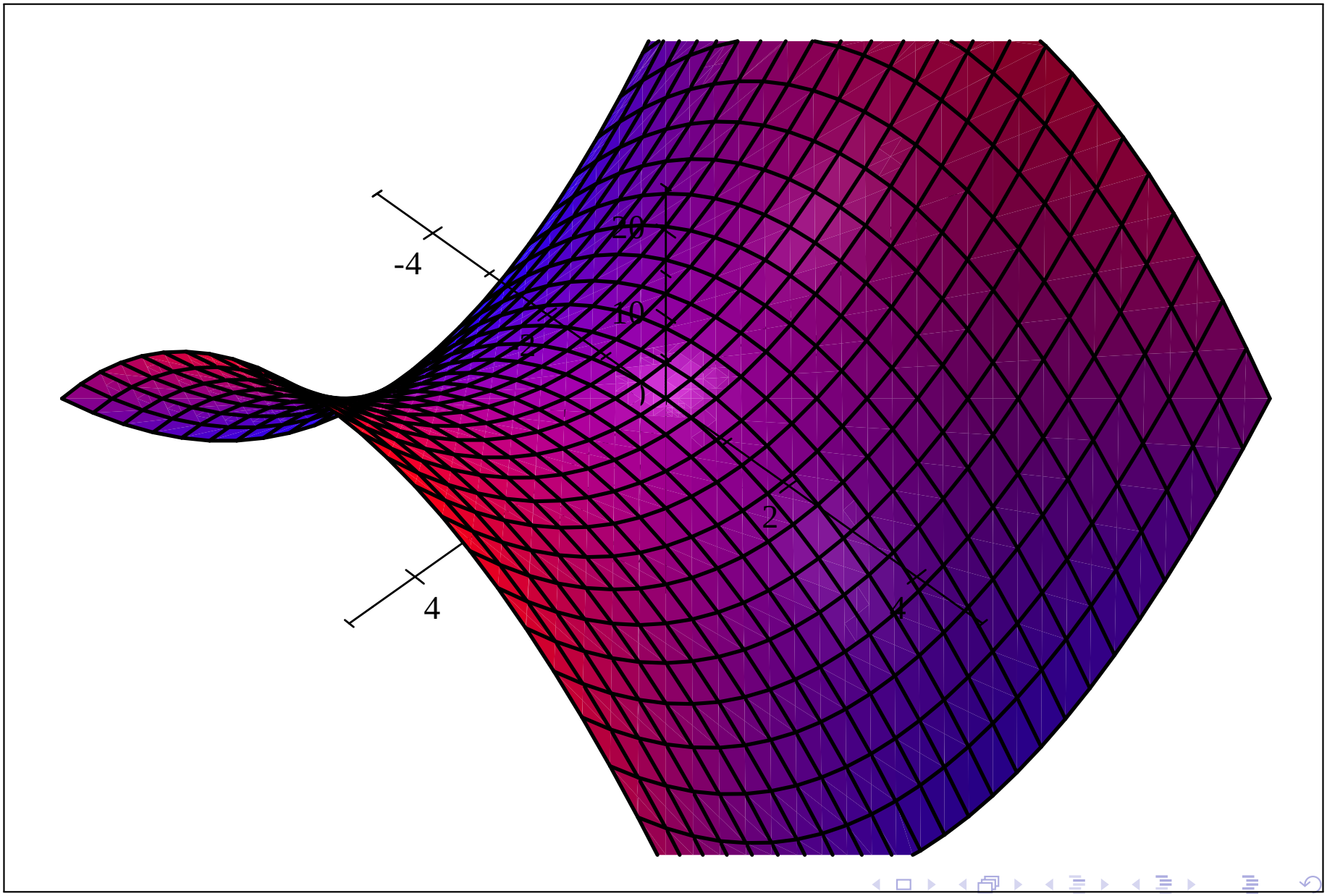
Le chevalier d'Arcy

A spherical mirror, with a source of light at O



The ray OA will reflect back to O, but not the ray OB. Yet OB is shorter and faster than OA

Stationarity versus optimality



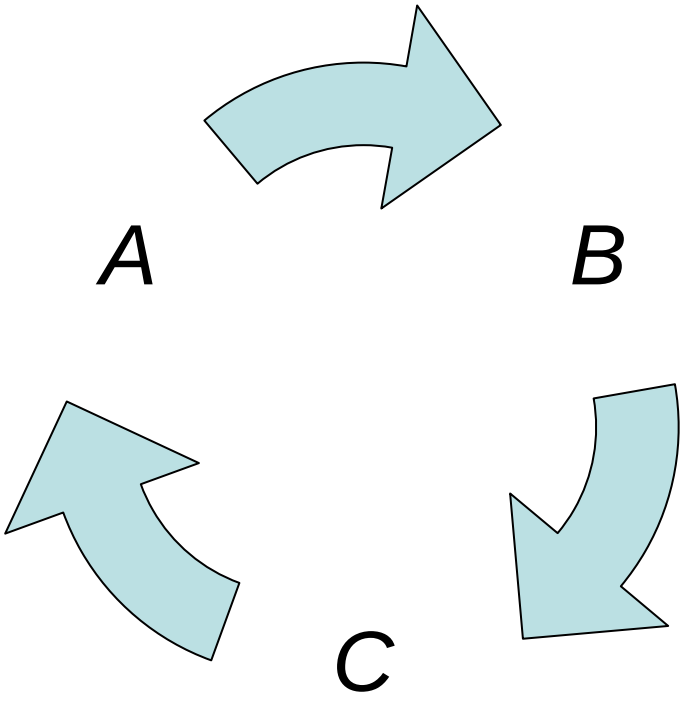
Optimization as a modeling tool

- To optimize, one needs a set of feasible solutions X and a criterion U(x)
- An optimal solution x_{opt} satisfies:

  $U(x_{opt}) \geq U(x)$ for all *x*
- Nature does not optimize:
  - *The criterion was found, but is not maximized*
- Do Humans optimize ?
  - *The economic approach to human behaviour*

What is rationality ?

- The ability to make coherent choices: *if I prefer A to B and B to C, I should prefer A to C*



Modeling rationality

- Given a set of feasible decisions X,
- Assign a utility $U(x)$ to every x in X
- Choose the x with the highest $U(x)$

- This procedure ensures coherence:

if $U(a) > U(b)$ and $U(b) > U(c)$, then $U(a) > U(c)$

Do individuals have utility functions ?

Each consumer is supposed to choose a bundle of goods $\left(x^1, ..., x^N\right)$ by maximizing his utility $u\left(x\right)$ under the budget constraint $\sum_{n=1}^{N} p_n x^n \leq w$, where p_n is the unitary price of good n and w is individual wealth
The resulting consumption of good n is a function $X^n\left(p_1, ..., p_N\right)$ of the price system. It can be observed, and it must satisfy the *Slutsky relations:*

$$\frac{\partial X^i}{\partial p_j} + \sum_k p_k \frac{\partial X^i}{\partial p_k} X^j = \frac{\partial X^j}{\partial p_i} + \sum_k p_k \frac{\partial X^j}{\partial p_k} X^i$$

Using data from the Canadian Family Expenditure survey (7 years from 1974 to 1992) Browning and Chappori (1994) got a fit of 75% for singles

Collective decision-making

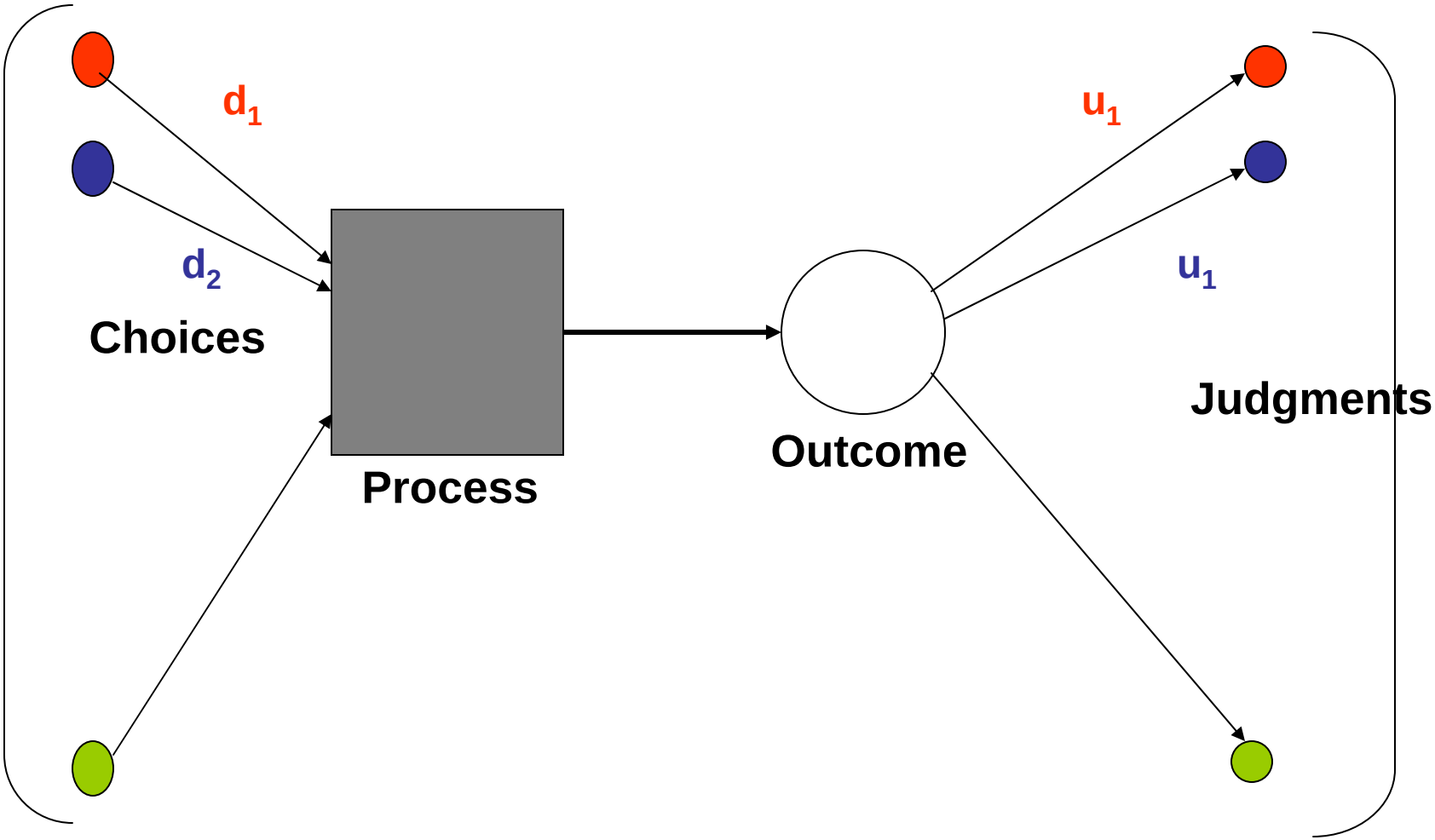
A group consists of individuals with:

- Different endowments (wealth, talents), some of which can be redistributed
- Different utilities
- The ability to misrepresent: they may hide private information, or indulge in hidden actions

who will be affected diversely by the outcome of group decisions.

Decision-making now is a process

Collective decision-making



Collective decision-making

There are two major problems facing any collective decision process:

The incoherence problem

The implementation problem

The incoherence problem

- The benevolent and omniscient dictator
- Majority rule: Condorcet's paradox
  - a simple majority prefers *A* to *B*
  - a simple majority prefers *B* to *C*
  - a simple majority prefers *C* to *A*
- Arrow's theorem: any collective decision depends as much on the procedure as on the individual preferences

The implementation problem

- *1000* individuals in a group *G.*
- Each of them contributes *0* or *100* CHF
- Total gets multiplied by *10* and distributed among the G (non-contributors included)
- Potential gain: *900* CHF for each

- Individual gain:
  - *900* CHF if everyone contributes
  - *999* CHF if I alone shirk
  - *(1000 – n)* CHF if there are *n-1* of us to shirk
- I am always better off shirking, and so is everyone else
- So we all shirk, and end up with gains of *0*, forgoing potential gains of *999* CHF each
- This is known as free-riding

Some lessons from theory

Groups do not optimize:

Personalizing groups is misleading: neither "*the people*" nor "*the international community*" have a will independently from the process by which it is asserted

Small groups acting for special interests

are much more likely to influence public policy than large groups acting for broad interests

The market solution

- Equilibrium prices enable society to reach a collective solution through individual optimisation
  - Each consumer or producer maximizes his own utility under budget constraints
  - The outcome is efficient in a precise sense
  - *"Private Vices by the dextrous Management of a skilful Politician may be turned into Public Benefits"*

    (Mandeville, Fable of the Bees, 1714).

Market failures

- Special assumptions are required for markets to be efficient. If they are not met, there are market failures:
  - Externalities: my activities influences your welfare positively or negatively (pollution)
  - Public goods: I benefit from their availability even if I don't pay for it (clean air)

The Stern review on the economics of climate change (2007)

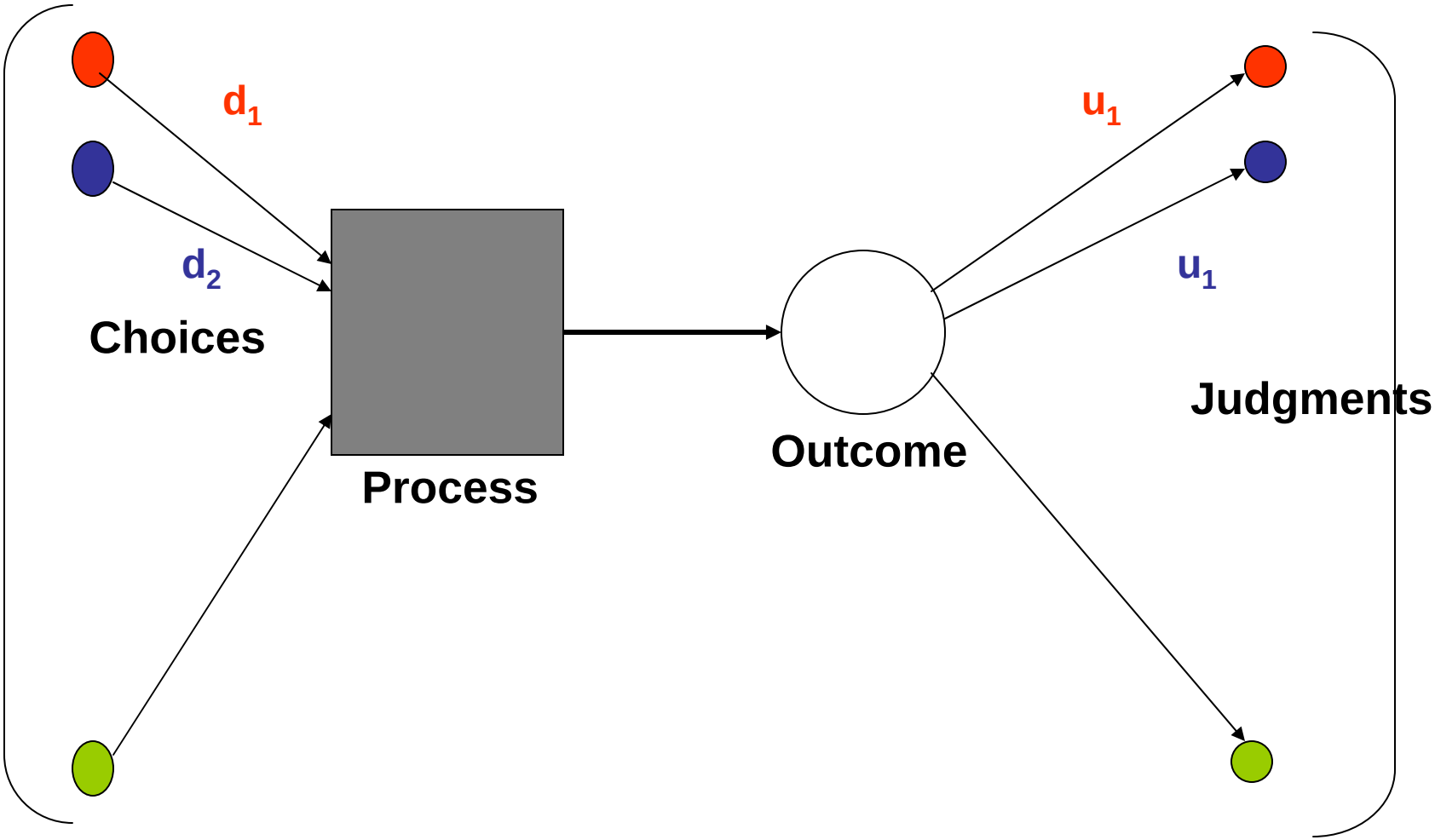
"Climate change is global in its causes and consequences, and international collective action will be critical in driving an effective, efficient and equitable response on the scale required. This response will require deeper international co-operation in many areas [...]

Climate change presents a unique challenge for economics: it is the greatest and widest-ranging market failure ever seen"

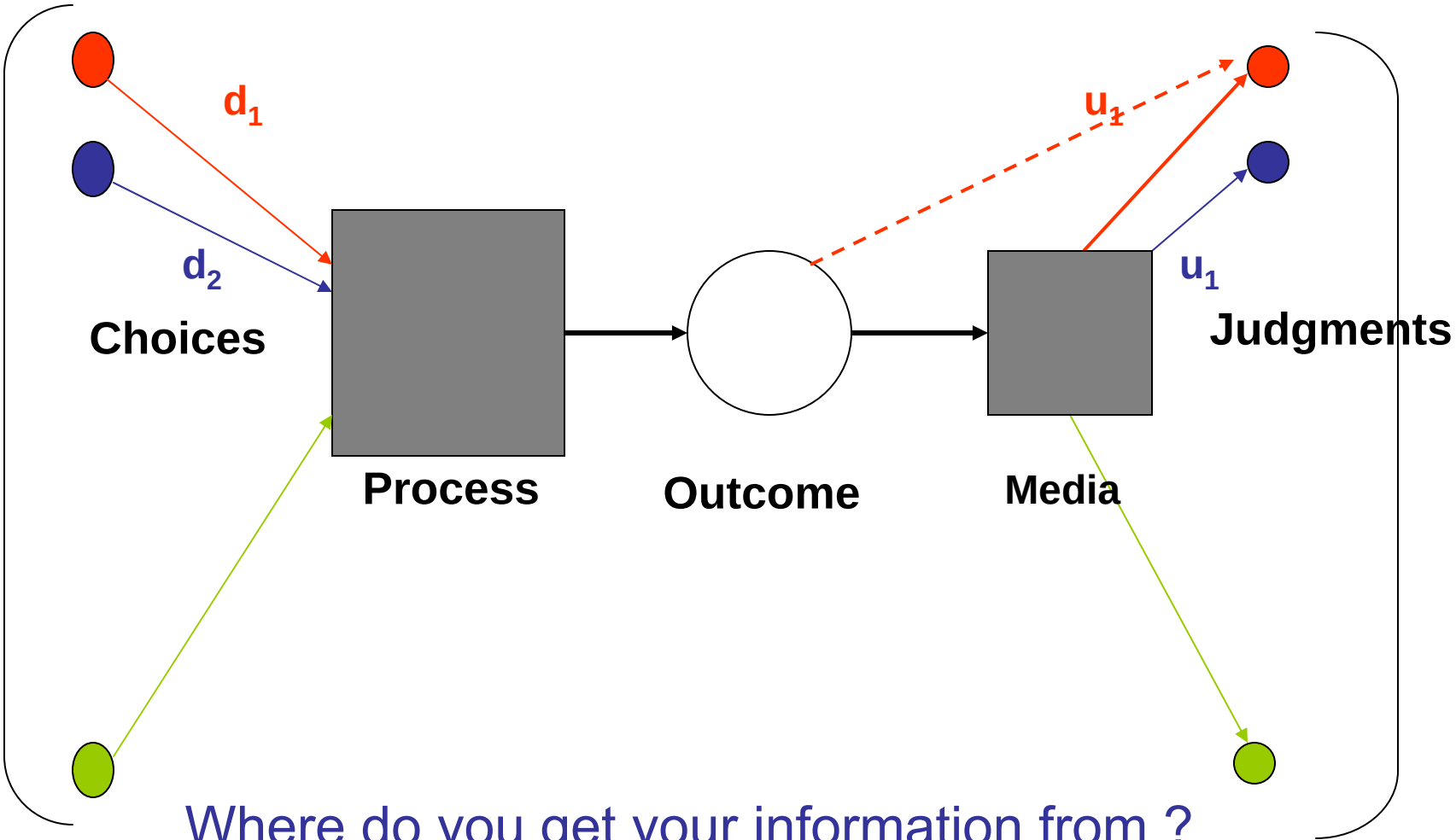
Google it!

Checks and balances:

Angels don't exist, so the devils should be put to watch over one another



Life in the 21st century



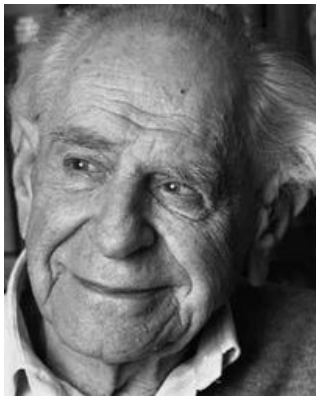
Where do you get your information from ?

Modern views of the world

Leibniz

Darwin

Friedrich von Hayek
1889 1992



Karl Popper
1902 1994