Multivariate Risk Measures

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Monovariate risk measures

A *monovariate risk measure* (1-rm) is a function $\rho : L^{\infty}\left(\Omega, \mathcal{F}, P\right) \rightarrow \infty$ such that:

- $\rho\left(0\right) = 0$
- $X \geq Y \Longrightarrow \rho\left(X\right) \leq \rho\left(Y\right)$
- $\rho\left(X + m\right) = \rho\left(X\right) - m$ for $m \in R$

It is

- *convex* if ρ is a convex function
- *coherent* if it is convex and positively homogeneous:

$$\begin{aligned} \rho\left(X+Y\right) &\leq \rho\left(X\right) + \rho\left(Y\right) \\ \rho\left(\lambda X\right) &= \lambda \rho\left(X\right) \end{aligned}$$

Law-invariant 1-r.m.

We shall write $X \sim Y$ to mean that X and Y have the same law. ρ is law-invariant if $X \sim Y$ implies that $\rho(X) = \rho(Y)$

Definition

A 1-rm is strongly coherent if it is convex, law-invariant and:

$$\rho(X) + \rho(Y) = \sup\left\{\rho\left(X + \tilde{Y}\right) \mid Y \sim \tilde{Y}\right\}$$

There are two fundamental examples of s.c. 1-r.m.

1. Let $F \in L^1$ be a probability density, so that $F \geq 0$ and $E[F] = 1$. Define
$$\rho_F := \sup\left\{\mathbb{E}\left[-F\tilde{X}\right] \mid \tilde{X} \sim X\right\}$$
2. Set:
$$\rho_\infty(X) := \operatorname{ess\,sup} -X$$

Both ρ_F and ρ_∞ are strongly coherent.

Kusuoka's theorem

Kusuoka (2001) has shown that all strongly coherent risk measures can be built from ρ_F and ρ_∞

Theorem

ρ is a strongly coherent 1-r.m. if and only if there is a probability density F and a number s with $0 \leq s \leq 1$ such that:

$$\forall X \in L^\infty, \quad \rho(X) = s\rho_\infty(X) + (1-s)\rho_F(X)$$

Multivariate risk measures

A *d-dimensional risk measure* (d-r.m.) is a function $\rho : L^\infty\left(\Omega, \mathcal{F}, P; R^d\right) \to R$ such that:

- $\rho(0) = 0$
- $X \geq Y \Longrightarrow \rho(X) \leq \rho(Y)$
- $\rho(X + me) = \rho(X) - m$ for $m \in R$ and $e = (1, ..., 1)$

It is

- *convex* if ρ is a convex function
- *coherent* if it is convex and positively homogeneous:

$$\begin{aligned} \rho(X+Y) &\leq \rho(X) + \rho(Y) \\ \rho(\lambda X) &= \lambda\rho(X) \end{aligned}$$

Law-invariant d-r.m.

ρ is law-invariant if $X \sim Y$ implies that $\rho(X) = \rho(Y)$

Definition

(Galichon, Henry) A d-rm is *strongly coherent* if it is convex, law-invariant and:

$$\rho(X) + \rho(Y) = \sup\left\{\rho\left(X + \widetilde{Y}\right) \mid Y \sim \widetilde{Y}\right\}$$

Examples

There are two fundamental examples of s.c. 1-r.m.

1. Let $F = (F_1, ..., F_d) \in L^1(R^d)$ satisfy $F_i \geq 0, 1 \leq i \leq d$, and $\sum_i E[F_i] = 1$. Define

$$\rho_F := \sup\left\{\mathbb{E}\left[-F\widetilde{X}\right] \mid \widetilde{X} \sim X\right\} = \sup\left\{\sum_i \mathbb{E}\left[-F_i\widetilde{X}_i\right] \mid \widetilde{X} \sim X\right\}$$

2. Denote by S^d the unit simplex in R^d and let $\xi \in S^d$, so that $\xi_i \geq 0$ and $\sum \xi_i = 1$. Define

$$\rho_\xi(X) := \operatorname{ess\,sup} -X\xi = \operatorname{ess\,sup} -\sum X_i \xi_i$$

3. Let μ be a probability on S^d. Set:

$$\rho_\mu(X) := \int_{S^d} \rho_\xi(X)\, d\mu(\xi)$$

ρ_F and ρ_μ are strongly coherent d-r.m.

Extension of Kusuoka's theorem

Schachermayer and IE have shown that all strongly coherent risk measured-r.m. can be built from ρ_F and ρ_∞

Theorem

ρ is a strongly coherent d-r.m. if and only if there is some $F \in L^1_+\left(R^d\right)$ with $\sum_i E\left[F_i\right] = 1$, a probability μ on S^d and a number s with $0 \leq s \leq 1$ such that:

$$\forall X \in L^\infty, \quad \rho\left(X\right) = s\rho_\mu\left(X\right) + \left(1-s\right)\rho_F\left(X\right)$$

This result builds on an earlier result by Galichon, Henry and IE, which we will explain in due course.

Duality

If $\rho : L^\infty\left(R^d\right) \to R$ is a convex and law-invariant, it is lower semi-continuous wrt $\sigma\left(L^\infty, L^1\right)$ (Jouini, Schachermayer, Touzi, 2006). It follows that ρ is coherent (homogeneous) if and only if there exists a closed, convex, law-invariant subset C of $L^1\left(R^d\right)$ such that.

$$\rho(X) = \sup_{F \in C} \langle F, X \rangle$$

We shall set $\rho = \rho_C$ to remember this fact. Denote by $\mathcal{T}$ the set of measure-preserving maps τ from Ω into itself. Set:

$$K := \{(F, F \circ \tau) \mid F \in C, \; \tau \in \mathcal{T}\}$$

Since C is law-invariant, $K \subset C \times C$. Conversely:

Lemma

ρ is strongly coherent iff $C \times C$ is the closed convex hull of K

Proof

Suppose $\rho = \rho_C$ and $C \times C$ is the closed convex hull of K. Then:

$$\begin{aligned}
\rho_C(X) + \rho_C(Y) &= \sup \{ -\langle F, X \rangle - \langle G, X \rangle \mid (F, G) \in C \times C \} \\
&= \sup \{ -\langle F, X \rangle - \langle F \circ \tau, Y \rangle \mid F \in C, \ \tau \in \mathcal{T} \} \\
&= \sup \{ -\langle F, X + Y \circ \tau^{-1} \rangle \mid F \in C, \ \tau \in \mathcal{T} \} \\
&\leq \sup \left\{ -\langle F, X + \widetilde{Y} \rangle \mid \widetilde{Y} \sim Y \right\} = \rho_C \left(X + \widetilde{Y} \right)
\end{aligned}$$

and the reverse inequality is always true

Geometry of strongly coherent sets

Theorem

Let $C \subset L^1\left(R^d\right)$ *be strongly coherent. Then there exists a number* t*, with* $0 \leq t \leq 1$*, and two closed convex law-invariant subsets* C^r *and* C^s *of* $L^1\left(R^d\right)$ *such that:*

1. $C = (1-t)\, C^r + tC^s$
2. C^r *is* $\sigma\left(L^1, L^\infty\right)$*-compact*
3. $\overline{C^s} \subset \left(L^\infty\right)^*$ *is* $\sigma\left(\left(L^\infty\right)^*, L^\infty\right)$*-compact, and its extreme points are purely singular*

The regular case

This is the case when $C^s = \varnothing$. In other words, $C = C^r$ is weakly compact in $L^1\left(R^d\right)$. For instance, $C \subset L^p\left(R^d\right)$, so that $\rho = \rho_C$ is continuous in the $L^p\left(R^d\right)$ norm. This case was dealt with in an earlier paper by Galichon, Henry and IE.

Since C is weakly compact, it is the closed convex hull of its set of strongly exposed points. Let $F \in C$ be such a point. We claim that $\rho = \rho_F$.

Proof

By definition, there is some $X \in L^{\infty}\left(R^{d}\right)$ such that $\sup_{H \in C}$ $-\langle H, X\rangle = -\langle F, X\rangle$ and if $H_n \in C$ is a maximizing sequence, then $H_n \to F$ strongly in $L^{i}\left(R^{d}\right)$. Since ρ is strongly coherent, we have:

$$\rho(X) + \rho(Y) = \sup\left\{-\langle G, X\rangle - \langle G \circ \tau, Y\rangle \mid G \in C,\ \tau \in \mathcal{T}\right\}$$

Let (G_n, τ_n) be a maximizing sequence. Since we have $=$ instead of $\geq$, we must have $\rho(X) = \sup -\langle G_n, X\rangle$ and $\rho(Y) = \sup -\langle G_n \circ \tau, Y\rangle$ so $G_n \to F$ and:

$$\rho(Y) = \sup_{\tau} -\langle F \circ \tau, Y\rangle = \sup_{\tau} -\langle F, Y \circ \tau^{-1}\rangle = \sup_{Y \sim \widetilde{Y}} -\langle F, \widetilde{Y}\rangle$$

so $\rho = \rho_F$ as desired

The singular case 1

Let $\mathcal{G} = (A_i)$ be a finite partition of Ω. On each A_i we are given a point $x_i \in S^d$ and an element $\beta_i \in (L^\infty(R))^*$. We then build an element $\beta_{\mathcal{G}} \in \left(L^\infty\left(R^d\right)\right)^*$ by the formula

$$\langle \beta_{\mathcal{G}}, X \rangle = \sum \langle \beta_i, (x_i \cdot X) \rangle$$

Denote by δ_i the Dirac mass at x_i. We associate with $\beta_{\mathcal{G}}$ the Borel measure $\mu_{\mathcal{G}}$ on S^d defined by:

$$\mu_{\mathcal{G}} = \sum \delta_i \beta_i (A_i)$$

Lemma

Suppose $\beta_{\mathcal{G}}$ is law-invariant. Then, for $X \in L^\infty\left(R^d\right)$ we have:

$$\sup_{\tau \in \mathcal{T}} -\langle \beta_{\mathcal{G}} \circ \tau, X \rangle = \int_{S^d} \sup - (x \cdot X)\, d\mu(x)$$

Proof

$$\begin{aligned}\int_{S^d} \sup -(x\cdot X)\, d\mu(x) &= \int_{S^d} \sup -\left(x\cdot X\circ\tau^{-1}\right) d\mu(x)\\ &= \sum \sup -\left(x_i\cdot X\circ\tau^{-1}\right)\beta_i(A_i)\\ &\geq -\langle \beta_{\mathcal{G}}\circ\tau, X\rangle\end{aligned}$$

and the reverse inequality follows from the fact that $\beta_{\mathcal{G}}$ is purely singular. Taking points (ξ_i) and disjoint sets (B_i) with $\mathbb{P}(B_i) > 0$ such that

$$\operatorname{ess\,sup} -x_i\cdot X = -x_i\cdot\xi_i \text{ and } B_i \subset \{|X-\xi_i| < \varepsilon\}$$

we can find a set N with $0 < \mathbb{P}(N) < \inf \mathbb{P}(B_i)$ such that $\beta_{\mathcal{G}}\mathbf{1}_N = \beta_{\mathcal{G}}$. We then find a measure-preserving transformation which maps $N\cap A_i$ into B_i and the result follows

The singular case: simple elements

Let $\beta \in \left(L^{\infty}\left(R^{d}\right)\right)^{*} \geq 0$ be a purely singular element. We define a finitely additive measure $|\beta|$ by:

$$|\beta|\,[A] = \langle e\mathbf{1}_{A}, \beta\rangle$$

Let $\mathcal{G} = (A_i)$ be a finite partition of Ω. Set:

$$\begin{aligned} \beta_{\mathcal{G}} &= \sum \xi_i \left(|\beta|\,[G_i]\right) \\ \xi_i &= \frac{\langle e_i \mathbf{1}_{G_i}, \beta\rangle}{\langle e\mathbf{1}_{G_i}, \beta\rangle} \end{aligned}$$

We can approximate β by the $\beta_{\mathcal{G}}$ which have a simpler form. We associate with $\beta_{\mathcal{G}}$ the Borel measure:

$$\mu_{\mathcal{G}} = \sum |\beta|\,[G_i]\,\delta_{\xi_i}$$

and we show that the $\mu_{\mathcal{G}}$ converge to some μ as the $\beta_{\mathcal{G}}$ converge to β.