Nash

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December, 2016

John Forbes Nash, Jr., and his wife died in a taxi on May 2015, driving home from the airport after receiving the Abel prize in Oslo. This accident does not conclude his career, for long after a mathematician is dead, he lives on in his work. On the contrary, it puts him firmly in the pantheon of mathematics, along with Nils Henrik Abel and Evariste Galois, whose productive life was cut short by fate. In the case of Nash, fate did not appear in the shape of a duellist, but in the guise of a sickness, schizophrenia, which robbed him of forty years of productive and social life.

His publication list is remarkably small. In 1945, aged seventeen, he published his first paper, [12], jointly with his father. Between 1950 and 1954 he published eight papers in game theory (including his PhD thesis) and one paper on real algebraic geometry. Between 1950 and 1954 he published eight papers in analysis, on the imbedding problem for Riemannian manifolds and on regularity for elliptic and parabolic PDEs, plus one paper published much later (1995). A total of nineteen ! For this (incredibly small) production he was awarded the Nobel prize for economics in 1994, which he shared with John Harsanyi and Reinhard Selten, and the Abel prize for mathematics in 2015, which he shared with Louis Nirenberg. I will now review this extraordinary work

1 Game theory

1.1 Nash's PhD thesis

The twentieth century was the first one where mass killing became an industry: for the first time in history, the war dead were counted in the millions. The whole economy was mobilized towards the war effort: weapons and ammunitions had to be produced and transported thousands of miles away, together with men, oil, and food. This was an unprecedented feat of management, and it was accompanied and sustained by the development of quantitative methods. Linear programming and convex optimization were born in the United States during the WWII, to support the war effort. A new field of mathematics, operations research, was created to accomodate these quantitative methods of management, which would soon incorporate deterministic and stochastic optimal control.

Princeton was the major center of this intellectual effort. There was Albert Tucker, famous nowadays for the Kuhn-Tucker conditions in convex optimization, and for the duality theory in linear programming. There one could also meet John von Neumann, the mathematician, and Oskar Morgenstern, the economist, both refugees from nazi Germany, bringing with them the scientific tradition of Vienna. Vienna was the place were one took seriously the idea that human beings can be thought of as optimizers, that is, that each of us is characterized by a utility function, and strives to maximize it. This is the core of modern economic theory, and indeed the first mathematical proofs of the existence of market equilibrium were done in Vienna, by Abraham Wald, while the whole theory was developed by Abraham Wald, with Friedrich Hayek and Joseph Schumpeter in the background. They also would emigrate, and live on to see the triumph of their ideas: Ronald Reagan and Margaret Thatcher are their intellectual heirs, and we live in the world they envisioned.

However, seeing human beings as optimizing machines is not enough, for one quickly encounters to problem of strategic behaviour: how do I take into account what other people do ? When I run a 100 meter dash against an opponent, for instance, it is clear what I should do: run as fast as possible. But when I play chess ? What is a good move in a given situation ? To answer the question, I have to know what he/she will answer, so I have to put myself in his/her shoes, and there I will find the same problem: he/she cannot find his/her best move without knowing what I would answer, so I am back in my own shoes with the same problem, two steps later and much more complicated !

A game is a situation where the global outcome (a) depends on individual decisions, and (b) affects differently the decision-makers. Chess is a very particular case where there are only two players (two-person game) and one's loss is the other's gain (zero-sum game). Game theory aims to find a "solution" for any given game, that is, to predict the outcome, or to find an agreement that all players will find acceptable. Von Neumann was the first one to provide a solution for two-person zero-sum games (this is the content of his famoux minimax theorem), and together with Morgenstern, in their famous book of 1944, he proposed a solution for general games [23]. Such was the exciting atmosphere in Princeton when Nash was a student. He rubbed shoulders, not only with these masters, but also with other students, no less interested and no less bright: John Milnor, Lloyd Shapley, Gary Becker, Harold Kuhn, David Gale, all of them major contributors to game theory and economics. This was also the topic Nash chose for his thesis: he would give his own solution for n-person games.

1.2 Nash equilibrium

Nash's work is not summarized, but fully contained in his paper [13]. It has two pages, 48 and 49, in fact, only one, because another paper ends on page 48 and another one begins on page 49. The full proof is given, and take rather less than four lines.

This paper, for which Nash was awarded the Nobel prize in economics many years later, defines mathematically the notion of equilibrium and gives conditions under which it exists. A *Nash equilibrium* is a non-cooperative solution for games: unilateral deviations are penalized (but multilateral ones may not be). More precisely, if there are N players, and player n tries to maximize $u_n(x_1, .., x_N)$, a Nash equilibrium is a set of individual actions $(\bar{x}_1, ...\bar{x}_N)$ such that:

$$u_n(\bar{x}_1, ..., \bar{x}_N) \geq u_n(\bar{x}_1, ...\bar{x}_{n-1}, x_n, \bar{x}_{n+1}, ...\bar{x}_N) \text{ for } 1 \leq n \leq N$$

Nash proved that, if the action sets X_n are convex and compact, and if the u_n are upper semi-continuous and concave with respect to x_n, then such an equilibrium exists

Nash's result did not create much excitement at the time, because it was seen as extremely weak. It describes self-fulfilling contracts: once the players have agreed to play $(\bar{x}_1, ..., \bar{x}_N)$, there is no need to worry about enforcing the contract, for each individual finds it in his/her own interest to comply. So, the argument goes, everyone will do his/her part without supervision or police. The trouble with this argument is that it assumes that players do not communicate: what if two of them, n and $n+1$, conspire to play (x_n, x_{n+1}) instead of $(\bar{x}_n, \bar{x}_{n+1})$? It may well be that they would do better than the equilibrium solution ! Isn't it the case in everyday life that people strike deals and cooperate, scratch my back and I will scratch yours ? What then is the use of a definition that assumes collusion away ? Isn't it better to look for cooperative solutions, where players are allowed to form coalitions ?

This was precisely the idea of von Neumann and Morgenstern. Unfortunately the (cooperative) solution they proposed in their book never gained traction or credence, it was just to unwieldy. As the years went by, people lost hope that a cooperative solution would exist, and went back to Nash's noncooperative one. Today, is stands as the central mathematical concept of economic theory. Every social interaction (competitive markets for instance) is understood as some kind of Nash equilibrium. Following Nash, modern economic theory has given up on cooperation. Individuals act within a set of rules, imposed by the state, and within this set of rules they seek a Nash equilibrium.

1.3 The problem of collective action

Nash equilibria can be pretty bad for all players. To give you an example of application, let me explain to you why our civilisation is doomed (I am not joking). As everybody knows by now, we are on track for an increase in mean temperatures $> 5°C$ by the end of the century. As a matter of comparison, this is exactly what separates us from the last ice age, when most of Europe was two or three kilometers of ice. Since we are in Italy, let me mention that temperatures around the Mediterranean are set to increase by $10°C$ in summertime; there is no point to go and seek solace in the Arctic, for temperatures there will increase by $15°C$. It seems clear that something should be done, and recently, at the

COP 21 in Paris, all nations pledged to do something. Will it be done ? No (again, I am serious). Why ?

Let us look at France, for instance. If France participates, benefit for France is NB and cost is c, so the balance is $NB - c$. If France does not participate, benefit for France is $(N-1)B$ but cost is zero. If $(N-1)B > NB - c$, or $c > B$, France will find it to its advantage not to participate. It lets the others do the work, and benefits from the result !

So France will not participate, and think itself smart. The problem is that everyone does the same calculation, everyone tries to free ride on the others, so no one participates. This is exactly what has been going on for twenty years, and which will continue until the end (see my book [4]). In mathematical terms, not participating is a Nash equilibrium, and it is the only one. We are firmly on track for the collapse.

1.4 The Nash bargaining solution

Important though it is, the concept of equilibrium is not the only contribution of Nash to game theory. He also defined a cooperative (!) solution to a very particular game, the bargaining problem.

Let us imagine two individuals engaged in a negociation. We represent the set of possible outcomes as a set $A \subset R_+^2$: individual i seeks to maximize the coordinate x_i. We seek a fair outcome $s(A)$ to the bargaining problem.

(axiom 1) if $(x_1, x_2) \in A$ and there exists $(y_1, y_2) \in A$ such that $y_1 > x_1$ and $y_2 > x_2$, then $(x_1, x_2) \notin s(A)$

(axiom 2) if A is symmetric, then $s(A)$ is the highest point $(x, x) \in S$

(axiom 3) if $A_1 \subset A_2$ and $s(A_2) \subset A_1$, then $s(A_1) = s(A_2)$

In [14], Nash proved that, if A is convex and compact, the only solution point $s(A) \in A$ satisfiying these three axioms for all A is the point where the product $x_1 x_2$ is maximized on A.

Nash's bargaining solution belongs to normative economics: a solution is sought satisfying certain assumptions (fairness), whereas the Nash equilibrium belongs to positive economics (what people actually do)

2 Mathematics

2.1 Real algebraic geometry

We are now entering a field I know very little about. Its difficulty, however, is easy to understand for a non-specialist. Algebraic geometry is the study of the set of zeroes of a family of polynomial equations in $\mathbb{K}^n$. If $\mathbb{K} = \mathbb{C}$, the field of complex numbers, we will benefit from the fact that every polynomial in one variable has a zero, which is no longer the case when $\mathbb{K} = \mathbb{R}$. An algebraic curve, for instance, is defined by a polynomial equation $P(z_1, z_2) = 0$. In $\mathbb{C}^2$, it is a Riemann surface, and its topology is perfectly well understood, but in $\mathbb{R}^2$ the situation is quite different and much more complicated.

Or is it ? The main problem, of course, are the singularities, the points where the equations become linearly dependent or where two sheets cross. Let us be more specific. Consider a set $M \subset \mathbb{R}^n$ defined by a family of equations:

$$M = \left\{ x = (x_1, ..., x_n) \mid f_k(x) = 0, \begin{array}{c} 1 \le n \le N \\ 1 \le k \le K \end{array} \right\}$$

It is a *submanifold* if this system never degenerates:

$$x \in M \Longrightarrow df_1(x) \wedge ... \wedge df_K(x) \neq 0$$

so that the tangent space to M at $x \in M$ is an affine subspace of codimension K:

$$T_x M = \{\xi \mid df_k(x)\, \xi = 0 \ \ \forall k\}$$

It is an *embedded submanifold* if there are no crossing points:

$$\begin{array}{c} x(t) \in M \\ x(t) \to x(0) \text{ when } t \to 0 \end{array} \Longrightarrow \frac{dx}{dt}(0) \in T_{x(0)} M$$

So an embedded submanifold corresponds to our intuitive notion of a smooth hypersurface. Nash, in a famous paper of 1952, raised (and answered) the question: does it make much difference if we want the f_k to be polynomials (M would then be called an algebraic variety) ? His answer was, essentially and suprisingly, no ! More precisely, he proved [17] that if If M is compact, connected and dim$M < \frac{n-1}{2}$, then M can be C^∞ approximated by a nonsingular component of a real algebraic variety.

He also conjectured that any compact embedded submanifold is diffeomorphic to a nonsingular connected component of a algebraic variety, and made a major step towards proving that result. It was established in 1973 by an Italian mathematician, Alberto Tognoli [24], and is known today as the Nash-Tognoli theorem.

Although Nash never again entered the field, and his contribution is limite to this single paper of 1952, it is impossible today to read a paper on real algebraic geometry without encountering his name, not only in his results, but also in the tools he left behind, Nash functions, Nash manifolds, which have been picked up by others and are now ubiquitous.

2.2 Calculus of variations

This is Hilbert's 19th problem (1900). It originates of course with Dirichlet's observation that the Poisson equation $-\Delta u = f$ satisfying $u = 0$ on the boundary is the Euler equation associated with the integral:

$$\int_\Omega \left(\frac{1}{2} |\nabla u|^2 - fu \right) dx$$

so that any minimizer of this integral in the class of functions satisfying the boundary condition must satisfy the Poisson equation $-\Delta u = f$ in the interior.

Riemann took the existence of the minimizer for granted, Weierstrass came up with a counterexample, and the problem was there for Hilbert to state: given a C^∞ map $F : \mathbb{R}^N \times \mathbb{R}^K \times \mathbb{R}^{KN} \to \mathbb{R}$, a domain $\Omega \subset \mathbb{R}^N$ with C^∞ boundary and a map $f : \mathbb{R}^N \to \mathbb{R}^K$, does the problem:

$$\min_u \int_\Omega \left(F\left(x, u, \nabla u\right) - fu\right) dx \tag{1}$$

$$u = 0 \text{ on } \partial\Omega \tag{2}$$

have a C^∞ solution $u\left(x\right)$?

From 1900 onwards, mathematicians benefited from Lebesgue's newly developed theory of the integral. LeonidaTonelli, for instance, put it to full use in his great book *[25]*, which unfortunately deals only with the one-dimensional case, where Ω is an interval and the Euler equation an ODE. Progress on the higher-dimensional case, where $\Omega \subset \mathbb{R}^N$ and the Euler equation is a PDE, came only when the decision was made to separate the problem in two different issues, existence on the one hand and regularity on the other, the pionneer in this direction being Jean Leray, in his thesis on the Navier-Stokes equation. Nowadays, one considers integrals such as (1) as functions over some Sobolev space, typically $u \in W^{k,p}$, and one shows that, under suitable growth and convexity assumptions on F, there is a minimizer, which satisfies the Euler equations in a suitably weak sense, obtained by integrating by parts again C^∞ test functions φ. So the existence problem is solved. The regularity problem consists of showing that the minimizer is not only in $W^{k,p}$ but in C^∞, and the Euler equation will play a crucial role in such a proof.

As an example, consider the problem:

$$\int_\Omega \left(F\left(\nabla u\right) - fu\right) dx$$

where $F \in C^\infty\left(R^{KN}\right)$, $\left|F\left(p\right)\right| \leq c\left|p\right|^2$ and the derivatives $A_k^n\left(p\right) := \partial F / \partial p_n^k$ satisfy the growth and ellipticity conditions:

$$\left|A_k^n\left(p\right)\right| \leq c\left|p\right|, \quad \left|\frac{\partial A_k^n}{\partial p_m^j}\left(p\right)\right| \leq c, \quad \frac{\partial A_k^n}{\partial p_m^j}\left(p\right) \xi_n^k \xi_m^j \geq c\left|\xi\right|^2$$

If u is a minimizer of F, then it satisfies the Euler equation:

$$\sum_{n,k} \int_\Omega A_k^n\left(Du\right) \frac{\partial \varphi^k}{\partial x^n} dx = f\varphi \qquad \forall \varphi \in W_0^{1,2}\left(\Omega; R^K\right)$$

which we rewrite as:

$$-\sum_n \frac{\partial}{\partial x^n}\left[A_k^n\left(Du\right)\right] = f_k$$

Differentiating, we find that any derivative $v_i := \partial u / \partial x^i$ satisfies the elliptic system:

$$-\sum_n \frac{\partial}{\partial x^n}\left[\frac{\partial A_k^n}{\partial p_m^j}\left(Du\right) \frac{\partial v_i^j}{\partial x^m}\right] = \frac{\partial f_k}{\partial x^i}$$

Suppose now $f \in C^\infty$. Is it the case that $u \in C^\infty$? If $u \in C^1$, the coefficients $\frac{\partial A_k^n}{\partial p_m^j}(Du)$ are Lipschitz functions of x, and it follows that $v^j \in C^1$, so that in fact $u \in C^2$. Iterating, we find that eventually $u \in C^\infty$. The problem is to start ! In other words, the difficulty is to prove that, if $u \in W_0^{1,2}\left(\Omega; R^K\right)$ is a weak solution, then u is C^1.

In the scalar case ($K = 1$), the system reduces to a single equation for the derivative $v_n := \partial u / \partial x^n$:

$$\frac{\partial}{\partial x^j}\left(\frac{\partial^2 F}{\partial p_i \partial p_j}(Du)\frac{\partial v}{\partial x^i}\right) = 0$$

If u is a weak solution, the coefficients $\frac{\partial A_k^n}{\partial p_m^j}(Du(x))$ are at best L^∞ (not continuous). So we have an elliptic linear equation with L^∞ coefficients. In [18] and [19], Nash showed that the solution is Hölder continuous, and that is enough to start the bootstrapping argument and go all the way to C^∞. At the same time, Ennio de Giorgi, in Italy, proved the same result by a different method in [6]. Each of them was unaware of the other: it is one of the great coïncidences of mathematics that such an important problem was solved simultaneously by two mathematicians working an ocean away from each other.

So Hilbert's problem was solved in the case of a single equation. The multidimensional case $K > 1$, leading to a system of PDEs, is still open. Nash, as usual, left the field almost immediately. De Giorgi went on to study the multidimensional case. He eventually found an example of a function $F(x, Du)$ with all imaginable blessings, but with a non-smooth solution, namely $u(x) = x/|x|^\gamma$ (see [7]). This was extended by Giaquinta and Giusti in [8] to a function $F(u, Du)$ with minimizer $u(x) = x/|x|$. It seems that in the multidimensional case, even when the integrand is convex (not to mention the nonlinear elasticity problems of continuum mechanics), the existence of singularities (fractures) is the rule, not the exception, although no one has a clear idea of what is going on.

2.3 Can Riemannian geometry be realized ?

2.3.1 The smooth case

In his thesis *Über die Hypothesen, die der Geometrie zugrunde legen* (1854), Bernhard Riemann defined an intrinsic geometry on manifolds by a quadratic form $\sum g_{ij}(x)\xi^i\xi^j$ on the tangent space at x. The question immediately arose: does that bring anything new ? Can every such Riemannian manifold be realized as a submanifold of Euclidian space ?

To restate this problem in a precise way, given a Riemannian manifold M, define an isometry as a one-to-one map $\varphi : M \to \mathbb{R}^N$ of class C^2 which preserves the given quadratic form $g(x)$ on $T_x M$:

$$\sum_{j,k=1}^{K}\left(\frac{\partial\varphi^n}{\partial x^j}(x)\,\xi^j,\frac{\partial\varphi^n}{\partial x^k}(x)\,\xi^k\right)_{R^N} = \sum g_{jk}(x)\,\xi^j\xi^k$$

$$\sum_{n=1}^{N}\frac{\partial\varphi^n}{\partial x^j}(x)\,\frac{\partial\varphi^n}{\partial x^k}(x) = g_{jk}(x)$$

Very early on, Elie Cartan and Maurice Janet solved the local problem: given a point x on a Riemannian manifold, one can alway find some N large enough and a neighbourhood of x small enough to embed isometrically into a neighbourhood of 0 in $\mathbb{R}^N$. Of more interest is the global problem. The particular case when M is a two-dimensional sphere with a Riemannian stucture, which one seeks to embed as a convex hypersurface of $\mathbb{R}^3$, was known as the Weyl problem, and was solved by Hans Lewy in the analytic case, and independently by Louis Nirenberg and Alexei Pogorelov in the C^∞ case.

The general case was solved by Nash, in [21]. He showed any compact Riemannian manifold can be imbedded isometrically into an Euclian space of sufficiently high dimension. Nash's proof goes by showing that the set of Riemannian structures on M (i.e. the set of fields $g(x)$) which can be isometrically embedded is both open and closed. The latter is relatively easy, the former is quite difficult. An isometry φ from M into $\mathbb{R}^N$ induces a map Φ from the Riemannian structures of M to the Riemannian structures of $\mathbb{R}^N$. We want to show that the set of g such that $\Phi(g) = I$ is open. Let us try to apply the inverse function theorem (IVT). Clearly, Φ is of the form $[\Phi(g)](\varphi(x)) = F(x, g(x), Dg(x))$. Differentiating at g_0, we get

$$\Phi'(g_0)\,\gamma = F_x + F_u\gamma + F_p D\gamma$$

So $\Phi'(u)$ maps C^k into C^{k-1}. This derivative is not recovered by inversion: $\Phi'(u)\,\gamma = \omega$ with $\omega \in C^{k-1}$ does not imply that $\omega \in C^k$ except in very special cases. There is a global loss of derivatives, and the usual IVT does not apply. Nash constructed a "hard" IVT to solve the embedding problem, and described his procedure (for the analytic case) in [22]. Simultaneously, Nikolai Kolmogorov in the USSR, with the collaboration of Vladimir Arnol'd constructed such an IVT to solve the resonance problem in celestial mechanics (see [1], [2] and [3]). This is another remarkable case where two great mathematicians, an ocean apart, simultaneously solve a great problem. Their ideas were carried forward by Jürgen Moser, whose name appears both in the Nash-Moser theorem (a generic name for inverse function theorems with loss of regularity) and in the Kolmogorov-Arnol'd-Moser theorem (the existence of invariant tori in near-integrable systems of classical mechanics). These theorems and methods today very active fields of research, with important applications the work of Clément Mouhot and Cedric Villani [11] being the most recent testimony. Together with Eric Séré, I have tried to import new ideas into that field [5].

2.3.2 The non-smooth case

Note that, in 1827, Carl Friedrich Gauss had found an invariant: the curvature is preserved by any isometry. It is the famous *Theorema Egregium*, the theorem which stands out of the flock. As usual, the presence of an invariant leads to obstructions (impossibilities): a sphere of radius R, for instance, cannot be compressed isometrically into a ball of radius $r < R$ (look at the curvature of extreme points).

But what about C^1 embeddings ? If φ is C^1 only, the Riemannian structure (including curvature) no longer makes sense, only the metric is left. Such an embedding will be called isometric if the length of any path $c(t)$, $0 \leq t \leq 1$, on M, coïncides with the length of its image $\varphi(c(t))$ in R^N. This makes excellent sense, and we are rid of the curvature ! Now everything is possible, including sending a sphere of radius R into a ball of radius $r < R$, as the following theorem shows:

Theorem 1 (Nash-Kuiper) *Let $f : M \to R^N$, with $N > \dim M$, be any map which is* 1*-Lipschitz:*

$$\|f(x) - f(y)\| \leq \|x - y\|$$

Then, for any $\varepsilon > 0$, there exists a C^1 embedding $\varphi : M \to R^N$ which is isometric and satisfies

$$\|f(x) - \varphi(x)\| \leq \varepsilon \quad \forall x \in M$$

Nash's original proof gave $N > \dim M + 1$, and the improvement is due to Kuiper. Note that there is still some regularity left: according to the theorem, a piece of paper of format A4 can be put into one's pocket without creating creases or folding !

This result was the first in a line of research with has been extremely active ever since. It was a first step towards Gromov's h-principle (if there are no topological obstructions, there are no holonomy obstructions) and convex integration. Right now, the idea that obstructions that stand in the way of regular solutions disappear for non-regular ones is behind such major advances as the existence of non-energy preserving solutions of the Euler equations in fluid mechanics.

3 The nature of genius

There is no doubt that Nash was a genius. It is not only what he did, but how he did it. His work on game theory is conceptually simple and the proofs are natural. His work on mathematics, on the other hand, is far from being either. Reading one of his proofs is like being led by the hand in a dark forest, following meandering paths until one is utterly lost, and then, at the very last moment, when one has given up all hope of ever leaving the darkness, the path opens into a clearing in front of a beautiful castle, with all the windows lit up. His death seals the fate of a unique personnality. It also leads us to meditate on that fragile boundary between genius and madness. Nash was schizophrenic most of his adult life, when he was not mathematically, productive, but where these subconscious forces already active when he was ?

We do not need to know the answer to render homage to John Nash's greatness. A fitting epitaph would be the one Stéphane Mallarmé composed for Edgar Poe:

Tel qu'en lui-même enfin l'éternité le change
Le poète suscite avec un glaive nu
Son siècle épouvanté de n'avoir pas connu
Que la mort triomphait par cette voix étrange.

Eux, comme un noir sursaut d'hydre oyant jadis l'angre
Donner un sens plus pur aux mots de la tribu
Proclamèrent très haut le sortilège bu
Dans le flot sans honneur de quelque noir mélange..

Du sol et de la nue hostiles, ô grief !
Si notre idée avec ne sculpte un bas-relief
Dont la tombe de Poe éblouissante s'orne,.

Calme bloc, ici-bas chu d'un désastre obscur !
Que ce granite au moins trace à jamais sa borne
Aux noir vol du blasphème épars dans le futur.

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