Equilibrium in quality markets, beyond the transferable case

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Abstract

We consider the market for an indivisible quality good, in the tradition of standard hedonic equilibrium theory but replacing the commonly used quasi-linear assumption on consumers' preferences by a more realistic nonlinear budget constraint. Taking advantage of quasi-linearity on the producer's side, we prove that an optimal transport-like argument can still be used to derive existence of equilibria. We also discuss some simple one-dimensional examples.

Keywords: quality markets, equilibrium, optimal transport, non transferable models.

1 Introduction

In the Arrow-Debreu theory, goods are homogeneous and indefinitely divisible, and equilibrium arises from equating quantity supplied with quantity provided. However, many important goods, such as houses or jobs, do not fit this description: they are heterogeneous (one is not like another one) and equilibrium does not arise from adjusting quantities (after all, one job or one house is usually enough for one individual). The idea of defining a good as a bundle of attributes, originating perhaps with Court [5] and developed by Houthakker [11], Lancaster [12], Becker [1] and Muth [14], provides a systematic framework for the economic analysis of the supply and demand of

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quality, and the proper definition of equilibrium. These are the so-called hedonic models. Typically, in such models, the good comes in separate units, each one characterized by a bundle of qualities, each consumer buys one unit or none, each producer supplies one unit or none. The price of each unit depends on its particular bundle of qualities, consumers maximize utility and producers maximize profit, and a hedonic equilibrium obtains when the market clears. Note that each consumer who enters the market is matched to a producer (who also enters the market).

The first papers showing the existence of a hedonic equilibrium, with separable utilities and in a one-dimensional situation (agents are fully characterized by the value of a single parameter) are due to Rosen [15] and Mussa and Rosen [13]. Ekeland noticed that this problem had the same mathematical structure as the theory of optimal transportation, which was undergoing rapid progress at that time (see [17]). Building on that theory, Ekeland [6], Chiappori, McCann and Nesheim [4] were able to characterize hedonic equilibria as solutions of convex minimization problems and to prove that they exist in higher-dimensional situations, where the agents and the goods have multiple characteristics.

However, all these results come with the price of assuming that utilities are separable: if a consumer buys one unit with quality z and pays p for it, her utility is assumed to be $u(z) - p$. This is mathematically convenient, and leads down the road to very nice properties of the hedonic equilibria, but they are empirically and theoretically unsound: on the one hand, they have no empirical support, on the other, they are cardinal and not ordinal.

This paper will prove the existence of hedonic equilibria in the general situation where utilities are ordinal, and consumers maximize their utility under the budget constraint, that is, given a price system $p(z)$, they seek a bundle of qualities z which maximizes $u(z)$ under the constraint $p(z) \leq w$, where u is the utility function of the consumer and w her wealth.

The model and the definition of equilibrium are presented in section 2, in the simple case where the number of consumers is equal to the number of producers (recall that each consumer buys one unit and each producer sells one unit). One-dimensional examples (including examples of multiplicity or non existence of equilibria) are considered in section 3. Section 4 gives a general existence result. Section 5 extends the model to the case when there is another consumption good on the market, ξ say, which is a quantity good, so that consumers now choose a quantity ξ and a quality z by maximizing $u(\xi, z)$ under the constraint $q \cdot \xi + p(z) \leq w$. If both markets, for the quantity good and for the quality good, clear, then $(q, p(z))$ is an equilibrium price, where q is linear pricing and $p(z)$ is not. Such models have been used in labour economics, where workers have to find a job z paying a salary $p(z)$

and have to buy subsistence goods ξ at price q, thereby striking a compromise between the quality of the job and the salary. See for instance the work of Heckman, [7] and [10].

2 The model

We are interested in a quality good market. The set of all feasible qualities is denoted by Z and assumed to be a compact metric space. The price of one unit with quality z is $p\,(z)$. In the sequel, each consumer buys one unit and each producer sells one unit.

2.1 Consumers and producers

Let us now describe the demand and supply side of the market for the quality good.

Consumers are heterogeneous, more precisely they have types x which have two components $x = (\theta, w)$ where $\theta \in X_0$ (a compact metric space) is a preference parameter and w is a revenue parameter taking its values in the interval $[\underline{w}, \overline{w}] \subset \mathbb{R}_+$. We denote by $X := X_0 \times [\underline{w}, \overline{w}]$ the full type space, the distribution of full types $\mu \in \mathcal{P}(X)$ is known[1] and we write it as

$$\mu(\mathrm{d}\theta, \mathrm{d}w) = \mu(\mathrm{d}\theta|w)\alpha(\mathrm{d}w)$$

so that $\alpha \in \mathcal{P}([\underline{w}, \overline{w}])$ is the distribution of revenues[2] and $\mu(\mathrm{d}\theta|w)$ is the distribution of θ conditional on w. Consumers' preferences are given by a continuous utility function $U \in C(X_0 \times Z)$. Given the price system $z \in Z \mapsto p(z) \in \mathbb{R}_+$, consumers of type x maximize their utility subject to their budget constraint. In other words, they solve:

$$U_p(x) := \max\{U(\theta, z) \ : \ p(z) \leq w\} \tag{2.1}$$

where as usual the maximum is set to $-\infty$ whenever the admissible set is empty.

Producers are heterogeneous and differ in their production cost function for the quality good. More precisely, there is a compact metric space Y and a continuous function $c \in C(Y \times Z)$ such that $c(y, z)$ represents the cost for producers of type y to produce one unit with quality $z \in Z$. The distribution

[1] Throughout the paper, whenever E is a compact metric space, $\mathcal{P}(E)$ denotes the set of Borel probability measures on E, always endowed with the weak star topology.

[2] with no loss of generality, we may assume that $\underline{w}$ is in the support of α which will justify the normalization $\min_Z p = \underline{w}$ in our definition of equilibria.

of y is known and given by a probability measure $\nu \in \mathcal{P}(Y)$. Given a price system $z \in Z \mapsto p(z) \in \mathbb{R}_+$, producers of type y choose to produce a z that minimizes their net cost $c(y,z)$. In other words, setting:

$$p^c(y) := \min_{z \in Z} \{c(y,z) - p(z)\} \tag{2.2}$$

producer of type y choose some quality z such that $p(z) + p^c(y) = c(y,z)$.

Note that we assume throughout that $\mu(X) = \nu(Y)$ (and have normalized this common value to 1) and also that there is no reservation utility (each consumer will end up buying something).

2.2 Equilibrium

Before defining equilibria precisely, let us introduce a few notations. Given two compact metric spaces X_1 and X_2, and $\beta \in \mathcal{P}(X_1, X_2)$ a Borel probability measure, we shall denote by $\pi_{X_1 \#}\beta$ the first marginal of β, which is defined by $\pi_{X_1 \#}\beta(A) := \beta(A \times X_2)$ for every Borel subset A of X_1. The second marginal $\pi_{X_2 \#}\beta$ is defined in the same way. Given $(m_1, m_2) \in \mathcal{P}(X_1) \times \mathcal{P}(X_2)$ we denote by $\Pi(m_1, m_2)$ the set of Borel probability measures on $X_1 \times X_2$ having m_1 and m_2 as marginals:

$$\Pi(m_1, m_2) := \left\{ \beta \in \mathcal{P}(X_1 \times X_2) \ : \ \pi_{X_1 \#}\beta = m_1, \ \pi_{X_2 \#}\beta = m_2 \right\}$$

and recall that it is a nonempty convex and weakly star compact subset of $\mathcal{P}(X_1 \times X_2)$.

Roughly speaking, an equilibrium is a price system which clears the quality market together with the corresponding joint distributions for demand and supply. More precisely:

Definition 2.1. *An equilibrium consists of a price $p \in C(Z, \mathbb{R}_+)$ such that $\min_Z p = \underline{w}$, a quality line $\eta \in \mathcal{P}(Z)$ as well as consumer-quality and producer-quality couplings γ and σ such that*

1. *$\gamma \in \Pi(\mu, \eta)$, $\sigma \in \Pi(\nu, \eta)$,*
2. *for γ-a.e. $(x,z) = (\theta, w, z)$, one has*

$$p(z) \leq w; \quad \text{and } U_p(x) = U(\theta, z) \tag{2.3}$$

3. *for σ-a.e. (y,z), one has*

$$p(z) + p^c(y) = c(y,z). \tag{2.4}$$

We will see in section 4 general regularity assumptions which guarantee existence of equilibria.

3 One-dimensional examples

In this section, we consider the simplest case where $Z = [\underline{z}, \overline{z}] \subset (0, +\infty)$, and $U(\theta, z)$ is increasing in z so that the preference parameter θ becomes irrelevant and we may equivalently take $U(z) = z$ without changing consumers' behavior. On the supply side, we also assume in this section that $Y = [\underline{y}, \overline{y}] \subset (0, +\infty)$, the production cost is $c(y, z) = yz$, the distribution of producers' types ν is atomless and Y is the convex hull of its support.

3.1 Discrete revenues

Let us consider the case where there are only two equiprobable values for the consumers' revenue i.e.

$$\alpha = \frac{1}{2}(\delta_{\underline{w}} + \delta_{\overline{w}})$$

with $\overline{w} > \underline{w} > 0$. Since consumers simply chose the maximal quantity they can afford, if η is an equilibrium quality line associated to a price system p, it is necessarily of the form

$$\eta = \frac{1}{2}(\delta_{z_1} + \delta_{z_2})$$

with

$$z_1 := \max\{z \in [\underline{z}, \overline{z}] \ : \ p(z) \leq \underline{w}\}, \ z_2 := \max\{z \in [\underline{z}, \overline{z}] \ : \ p(z) \leq \overline{w}\}.$$

Since we have imposed in our normalization of equilbria that $p \geq \underline{w}$ we have $p(z_1) = \underline{w}$ and $p(z_2) \leq \overline{w}$ with equality unless if $z_2 = \overline{z}$, hence $z_1 \neq z_2$ unless $z_1 = z_2 = \overline{z}$. But, if $z_1 = z_2 = \overline{z}$, price has to be constant equal to $\underline{w}$ and producers would all chose to produce $\underline{z}$ and all consumers on the contrary would chose to consume $\overline{z}$ a contradiction to the equilibrium requirement. We thus necessarily have $z_1 < z_2$, $p(z_1) = \underline{w}$, $p(z_2) = \overline{w}$. At equilibrium, producers should optimally chose to produce either z_1 or z_2 and by our linear specification of the cost, they should produce z_1 whenever their type y is in the subinterval of Y where $yz_1 - \overline{w} \leq yz_2 - \underline{w}$ (and since ν is atomless the indifference point is negligible) but since half the demand is for good z_1, this subinterval should be $(y^*, \overline{y}]$ where y^* is the median of ν (which exists since ν is atomless and is unique as soon as ν has an increasing cumulative distribution function on Y). One should then have

$$\overline{w} - \underline{w} = y^*(z_2 - z_1) \leq y^*(\overline{z} - \underline{z})$$

so that when $y^*(\overline{z} - \underline{z}) < \overline{w} - \underline{w}$ there is no equilibrium. If $y^*(\overline{z} - \underline{z}) > \overline{w} - \underline{w}$, the situation corresponding to $z_1 = \underline{z}$, $z_2 = \underline{z} + \frac{1}{y^*}(\overline{w} - \underline{w})$ and a concave piecewise linear tariff p with slope y^* on $[\underline{z}, z_2]$ and any slope in $(0, \underline{y}]$ on $[z_2, \overline{z}]$ is an equilibrium. In summary:

- if $y^*(\overline{z} - \underline{z}) < \overline{w} - \underline{w}$ there is no equilibrium,
- If $y^*(\overline{z} - \underline{z}) > \overline{w} - \underline{w}$ there are several equilibria.

By the same arguments, one can see that if ν has several medians then are infinitely many equilibria which may be constructed as previously for each choice of a median y^*.

3.2 Continuous revenues

We consider the same specifications as above but we now assume that revenues and producers' types have continuous distributions, α and ν. To make explicit computations we actually take them uniform i.e. with respective cumulative distribution functions F_α and F_ν given by

$$F_\alpha(w) := \begin{cases} 0 \text{ if } w \leq \underline{w} \\ \frac{w-\underline{w}}{\overline{w}-\underline{w}} \text{ if } w \in [\underline{w}, \overline{w}] \\ 1 \text{ if } w \geq \overline{w} \end{cases} \quad \text{and } F_\nu(y) := \begin{cases} 0 \text{ if } y \leq \underline{y} \\ \frac{y-\underline{y}}{\overline{y}-\underline{y}} \text{ if } y \in [\underline{y}, \overline{y}] \\ 1 \text{ if } y \geq \overline{y}. \end{cases}$$

To find an equilibrium in this setting, it is natural to look for an increasing and concave price system $z \in [\underline{z}, \overline{z}] \mapsto p(z)$ so that $p(\underline{z}) = \underline{w}$. Making this ansatz, quality z is chosen by consumers of revenues $w = p(z)$ and produced by producers with type $y = p'(z)$ at least when $(p(z), p'(z)) \in [\underline{w}, \overline{w}] \times [\underline{y}, \overline{y}]$, so that if we denote by η the quality line distribution, at equilibrium, we should have[3]

$$\alpha = p_{\#}\eta, \ \nu = p'_{\#}\eta.$$

Since p is increasing and concave, by a change of variables argument, and denoting by F_η the cumulative distribution of η this yields

$$F_\eta = F_\alpha \circ p = 1 - F_\nu \circ p'.$$

We therefore have the following Cauchy problem for the tariff:

$$F_\nu(p'(z)) = 1 - F_\alpha(p(z)), \ p(\underline{z}) = \underline{w}. \tag{3.1}$$

[3]here we use the notation $T_{\#}m$ to denote the image measure of a Borel probability measure m by a measurable map T, $T_{\#}m(B) := m(T^{-1}(B))$.

While p remains strictly below $\overline{w}$ (equivalently p' remains strictly above $\underline{y}$), under our uniform specification (and setting $\Delta_w := \overline{w} - \underline{w}$, $\Delta_y := \overline{y} - \underline{y}$), integrating (3.1) directly gives

$$p(z) = \Big(\overline{y}\frac{\Delta_w}{\Delta_y} + \underline{w}\Big) - \overline{y}\frac{\Delta_w}{\Delta_y}\exp\Big(\frac{\Delta_y}{\Delta_w}(\underline{z} - z)\Big). \tag{3.2}$$

Since $\overline{y}\frac{\Delta_w}{\Delta_y} + \underline{w} > \overline{w}$, there is a threshold (which is explicit) $z^* \in [\underline{z}, +\infty)$ for which the concave increasing function of z on the right-hand side of (3.2) remains below $\overline{w}$ on $[\underline{z}, z^*]$. To find an equilibrium by the previous considerations, we thus have to distinguish two cases:

- either $\overline{z} < z^*$, in which case, p is given by (3.2) on $[\underline{z}, \overline{z}]$, and $p(\overline{z}) < \overline{w}$, $p'(\overline{z}) > \underline{y}$, consumers of revenue w less than $p(\overline{z})$ chose the quality $p^{-1}(w)$ and consumers of revenue in $[p(\overline{z}), \overline{w}]$ all chose the maximal quality $\overline{z}$. On the supply side, producers of type $y \in [\underline{y}, p'(\overline{z})]$ all optimally produce $\overline{z}$ and producers of type $y \in [p'(\overline{z}), \overline{y}]$ produce $(p')^{-1}(y)$. Since by construction $\alpha([p(\overline{z}), \overline{w}]) = \nu([\underline{y}, p'(\overline{z})]) = a > 0$, the distribution of supply and demand induced by p coincide and the corresponding equilibrium quality line η has a Dirac mass with weight a at $\overline{z}$.

- or $\overline{z} \geq z^*$, in this case we define p by (3.2) when $z \in [\underline{z}, z^*]$ and $p(z) = p(z^*) + p'(z^*)(z - z^*) = \overline{w} + \underline{y}(z - z^*)$ when $z \in [z^*, \overline{z}]$. Demand and supply distributions match and in this case the corresponding equilibrium quality line is absolutely continuous and supported on $[\underline{z}, z^*]$.

4 An existence result

We have seen in section 3 that, even in an elementary one-dimensional model, if wages are discrete, equilibria may not exist. In this section, we identify structural assumptions, under which existence is ensured. Among our assumptions (see **(H2)**) one rules out the presence of atoms in the distribution of consumers' revenues.

Under the following assumptions

- **(H1)** Y is the closure of an open bounded connected subset of $\mathbb{R}^d$ with a Lebesgue negligible boundary, ν is equivalent to the d-dimensional Lebesgue measure on Y (that is ν has the same negligible sets as the d-dimensional Lebesgue measure), $c \in C(Y \times Z)$, $c(., z)$ is of class C^1 on Y for every $z \in Z$, and its gradient $\nabla_y c\,(y, z)$ is a continuous function.

- **(H2)** $U \in C(X_0 \times Z)$ is continuous and the revenue distribution $\alpha \in \mathcal{P}([\underline{w}, \overline{w}])$ has no atom.

we have the following existence result:

Theorem 4.1. *Under **(H1)** and **(H2)**, there exists at least one equilibrium.*

We are going to prove Theorem 4.1 by a fixed-point argument. The starting point is the observation that $\sigma \in \Pi(\nu, \eta)$ and $p \in C(Z)$ are related by (2.4) if and only if σ solves the mass tranport problem[4]:

$$\inf_{\sigma \in \Pi(\nu,\eta)} \int_{Y \times Z} c(y, z)\sigma(\mathrm{d}y, \mathrm{d}z) \tag{4.1}$$

and for η-a.e. z one has

$$p(z) = q^{\check{c}}(z) := \min_{y \in Y}\{c(y, z) - q(y)\}$$

where q is a Kantorovich potential, that is, a solution of the Kantorovich dual problem of (4.1) with marginals ν and η

$$\sup\Big\{ \int_Y q(y)\nu(\mathrm{d}y) + \int_Z q^{\check{c}}(z)\eta(\mathrm{d}z)\Big\} \tag{4.2}$$

It follows from Prop. 6.1 in [2] that, under **(H1)**, (4.2) admits a unique solution q up to an additive constant. We choose the constant in such a way that

$$\min q^{\check{c}} = \underline{w}$$

and then set

$$p := q^{\check{c}} = F_1(\eta).$$

Then one deduces from the fact that functions of the form $p = q^{\check{c}}$ form an uniformly equicontinuous family[5] and the uniqueness of the optimal q in (4.2) that:

Lemma 4.2. *The map $\eta \mapsto F_1(\eta)$ is continuous from $\mathcal{P}(Z)$ equipped with the weak star topology to $C(Z)$ (with the uniform norm).*

[4]see the textbooks by Villani [17] or Santambrogio [16] for a detailed mathematical presentation of the subject and Galichon [8] for an overview of optimal transport techniques in economics.

[5]This is because c is uniformly continuous on the compact set $Y \times Z$. More precisely, denoting by d_Z the distance on Z, we have $|q^{\check{c}}(z) - q^{\check{c}}(z')| \leq \omega(d_Z(z, z'))$ with $\omega(t) := \max\{|c(y, z) - c(y, z')|,\ y \in Y,\ (z, z') \in Z^2,\ d_Z(z, z') \leq t\} \to 0$ as $t \to 0^+$.

Proof. Let η_n weakly star converge to η, q_n solve (4.2) with marginals μ and η_n and $p_n = F_1(\eta_n)$, by uniqueness, one can assume that $q_n = p_n^{\mathrm{c}}$, then both families p_n and q_n are bounded and uniformly equicontinuous thus by Ascoli's theorem, they admit converging subsequences with respective limits p and q, obviously $\min p = \underline{w}$, $p = q^{\check{c}}$, $q = p^c$ and q solves the Kantorovich problem with marginals ν and η, by uniqueness $p = F_1(\eta)$ which proves the desired result. □

Now given $p \in C(Z)$ such that $\min_Z p = \underline{w}$ define $F_2(p)$ as the set of probability measures γ on $X \times Z$ that satisfy:

- $\pi_{X\#}\gamma = \mu$, i.e. the first marginal of γ is μ,
- $p(z) \leq w$ for γ-a.e. (θ, w, z) (or equivalently on the support of γ since p is continuous) which may also be written as

$$\int_{X\times Z} (p(z) - w)_+ \gamma(\mathrm{d}\theta, \mathrm{d}w, \mathrm{d}z) = 0 \tag{4.3}$$

- defining U_p by (2.1), $U_p(x) = U(\theta, z)$ for γ-a.e. (x, z), which, thanks to (4.3), is equivalent to

$$\int_X U_p(x)\mu(\mathrm{d}x) = \int_{X\times Z} U(\theta, z)\gamma(\mathrm{d}x, \mathrm{d}z). \tag{4.4}$$

It follows from standard measurable selection arguments (see [3]) that $F_2(p)$ is well-defined, nonempty as well as convex and compact. We finally set

$$F(\eta) := \{\pi_{Z\#}\gamma,\ \gamma \in F_2(F_1(\eta))\} \tag{4.5}$$

so that F is the composition of the continuous map F_1, the nonempty-convex-compact valued map F_2 and a linear continuous marginal map, it is thus a nonempty-convex-compact valued self set-valued map of $\mathcal{P}(Z)$. We will deduce Theorem 4.1 from the existence of a fixed-point of F. We thus have to prove that F has a closed graph, obviously setting $A := \{p \in C(Z) \ : \min p = \underline{w}\}$ it is enough to prove that $p \in A \mapsto F_2(p)$ has a closed graph:

Lemma 4.3. *Under **(H2)**, $p \in A \mapsto F_2(p)$ has a closed graph.*

Proof. Let us first study some basic properties of U_p for fixed $p \in A$. First observe that for every $\theta \in X_0$, $w \in [\underline{w}, \overline{w}] \mapsto U_p(\theta, w)$ is nondecreasing, there is therefore an at most countable (left) jump set that we denote by $J_p(\theta)$ on which $U_p(\theta, .)$ is discontinuous:

$$J_p(\theta) := \{w \in (\underline{w}, \overline{w}];:\ U_p(\theta, w) > U_p(\theta, w^-)\}$$

where

$$U_p(\theta; w^-) := \lim_{\delta \to 0^+} U_p(\theta, w - \delta).$$

With respect to the variable θ, U_p is obviously continuous. More precisely denoting by dist the distance on X_0 and

$$\omega(t) := \sup\{|U(\theta, z) - U(\theta', z)|, \ z \in Z, \ (\theta, \theta') \in X_0^2, \ \mathrm{dist}(\theta, \theta') \le t\}$$

then for every $(\theta, \theta', w) \in X_0^2 \times [\underline{w}, \overline{w}]$, one has

$$|U_p(\theta, w) - U_p(\theta', w)| \le \omega(\mathrm{dist}(\theta, \theta')). \tag{4.6}$$

Let D_0 be a countable dense subset of X_0 and define the finite or countable set $J_p := \cup_{\theta \in D_0} J_p(\theta)$, it is then easy to deduce from (4.6) that

$$U_p(\theta, w) = U_p(\theta, w^-), \ \forall \theta \in X_0, \ \forall w \in (\underline{w}, \overline{w}] \setminus J_p. \tag{4.7}$$

Now, let $p_n \in A$ converge to p and $\gamma_n \in F_2(p_n)$ converge to γ, obviously one has $\pi_{X\#}\gamma = \mu$ and (4.3) holds, it remains to deduce from

$$\int_X U_{p_n}(x)\mu(\mathrm{d}x) = \int_{X \times Z} U(\theta, z)\gamma_n(\mathrm{d}x, \mathrm{d}z), \tag{4.8}$$

that (4.4) holds. The right-hand side of (4.8) converges to the right-hand side of (4.4), as for the left-hand side, we proceed as follows. First we observe that

$$\limsup_n U_{p_n} \le U_p$$

and also that for every θ, $w \in (\underline{w}, \overline{w}]$ and $\varepsilon > 0$ such that $w - \varepsilon \ge \underline{w}$ one has

$$\liminf_n U_{p_n}(\theta, w) \ge U_p(\theta, w - \varepsilon)$$

indeed if $U_p(\theta, w - \varepsilon) = U(\theta, z)$ for some $z \in Z$ such that $p(z) \le w - \varepsilon$ then $p_n(z) \le w$ for large enough n so that $U_{p_n}(\theta, w) \ge U_p(\theta, w - \varepsilon)$. We then have, thanks to Fatou's Lemma, for every $\varepsilon > 0$:

$$\begin{aligned} \int_X U_p(x)\mu(\mathrm{d}x) \ge \limsup_n \int_X U_{p_n}(x)\mu(\mathrm{d}x) \ge \liminf_n \int_X U_{p_n}(x)\mu(\mathrm{d}x) \\ \ge \int_X U_p(\theta, w - \varepsilon(w - \underline{w}))\mu(\mathrm{d}\theta, \mathrm{d}w) \end{aligned}$$

thanks to Lebesgue's dominated convergence Theorem, this gives

$$\liminf_n \int_X U_{p_n}(x)\mu(\mathrm{d}x) \ge \int_X U_p(\theta, w^-)\mu(\mathrm{d}\theta, \mathrm{d}w).$$

It follows from (4.7) that $U_p(\theta, w^-) = U_p(\theta, w)$ whenever $w \notin J_p$, but since α has no atoms by **(H2)** and J_p is at most countable $\alpha(J_p) = 0$, hence we have

$$\int_X U_p(\theta, w^-)\mu(\mathrm{d}x, \mathrm{d}w) = \int_X U_p(\theta, w)\mu(\mathrm{d}x, \mathrm{d}w)$$

passing to the limit in (4.8) then gives (4.4) so that $\gamma \in F_2(p)$.

□

This allows us to conclude:

Proof of Theorem 4.1: we deduce from Lemma 4.2 and Lemma 4.3 that the non-empty convex-compact valued map $F : \mathcal{P}(Z) \to 2^{\mathcal{P}(Z)}$ has a closed graph. It thus follows from Glicksberg's fixed-point theorem [9] that there exists $\eta \in \mathcal{P}(Z)$ such that $\eta \in F(\eta)$ i.e. $\eta = \pi_{Z\#}\gamma$ for some $\gamma \in F_2(p)$ where $p = F_1(\eta)$. Letting σ be a solution of the Monge-Kantorovich problem (4.1), it is straightforward to check that the collection p, η, γ, σ is an equilibrium.

5 Extension to the case of an additional quantity good

We now consider an extension of the previous model to the case where consumers do not only consume the quality good but also a divisible good whose price is linear in the quantity and exogenously given.We call this extra good the *quantity* good. The dimension of the quantity good is d, its price vector $q \in (0, +\infty)^d$ is given. As previously, consumers' preferences are hetereogeneous but now they also depend on an extra consumption variable $\xi \in \mathbb{R}^d_+$. Denoting by $\theta \in X_0$ the consumers' preference parameter, the consumers preferences are now given by a utility fonction $V : (\theta, \xi, z) \in X_0 \times \mathbb{R}^d_+ \times Z \to \mathbb{R}$. As in paragraph 2.1, we denote by $x = (\theta, w) \in X := X_0 \times [\underline{w}, \overline{w}]$ the full type of consumers, $\alpha \in \mathcal{P}([\underline{w}, \overline{w}])$ the revenue distribution and $\mu \in \mathcal{P}(X)$ the joint distribution of (θ, w). Given a tariff $z \in [\underline{z}, \overline{z}] \mapsto p(z)$, consumers of type $x = (\theta, w)$ solve

$$\widetilde{V}_p(x) := \max\{V(\theta, \xi, z) \ : \ q \cdot \xi + p(z) \leq w\} \tag{5.1}$$

as we shall again impose that $\min_Z p = \underline{w}$, the budget constraint imposes that ξ a priori remains in the compact set

$$\Delta := \{\xi \in \mathbb{R}^d_+ \ : \ q \cdot \xi \leq \overline{w}\}.$$

The assumptions and notations on the producers of the quality good are exactly the same as in paragraph 2.1. In this setting, equilibria are defined by:

Definition 5.1. *An equilibrium consists of a price* $p \in C(Z, \mathbb{R}_+)$ *such that* $\min_Z p = \underline{w}$, *a quality line* $\eta \in \mathcal{P}(Z)$ *as well as the couplings* $\widetilde{\gamma} \in \mathcal{P}(X \times \Delta \times Z)$ *and* $\sigma \in \mathcal{P}(Y \times Z)$ *such that*

1. $\pi_{X\#}\widetilde{\gamma} = \mu$, $\pi_{Z\#}\widetilde{\gamma} = \eta$, $\sigma \in \Pi(\nu, \eta)$,
2. *for* $\widetilde{\gamma}$*-a.e.* $(x, \xi, z) = (\theta, w, \xi, z)$, *one has*
$$q \cdot \xi + p(z) \le w; \quad \text{and } \widetilde{V}_p(x) = V(\theta, \xi, z) \tag{5.2}$$
3. *for* σ*-a.e.* (y, z), *one has*
$$p(z) + p^c(y) = c(y, z). \tag{5.3}$$

We then have:

Theorem 5.2. *If we assume that* $V \in C(X_0 \times \mathbb{R}^d_+ \times Z)$, *that the revenue distribution* $\alpha \in \mathcal{P}([\underline{w}, \overline{w}])$ *is atomless and* ***(H1)***, *there exists at least one equilibrium in the sense of definition 5.1.*

Proof. The proof is very similar to the one of section 4 and consists in finding a fixed point of the set-valued map $\widetilde{F} : \mathcal{P}(Z) \to 2^{\mathcal{P}(Z)}$ defined by

$$\widetilde{F}(\eta) := \{\pi_{Z\#}\widetilde{\gamma},\ \widetilde{\gamma} \in \widetilde{F}_2(F_1(\eta))\}$$

where F_1 is defined exactly as in section 4 (i.e. by the Kantorovich problem (4.2)) and $\widetilde{F}_2$ is the set-valued map $A := \{p \in C(Z) \ : \ \min_{z \in Z} p = \underline{w}\} \to 2^{\mathcal{P}(X \times \Delta \times Z)}$, which maps $p \in A$ to $\widetilde{F}_2(p)$, the set of probability measures $\widetilde{\gamma}$ on $X \times \Delta \times Z$ that satisfy:

- $\pi_{X\#}\widetilde{\gamma} = \mu$,
- $q \cdot \xi + p(z) \le w$ for $\widetilde{\gamma}$-a.e. (θ, w, ξ, z) (or equivalently on the support of $\widetilde{\gamma}$ since p is continuous)
- $\widetilde{V}_p(x) = V(\theta, \xi, z)$ for $\widetilde{\gamma}$-a.e. (x, ξ, z), (recall that $\widetilde{V}_p$ is defined by (5.1)).

The same proof as in Lemma 4.3 shows that $\widetilde{F}_2$ has a closed graph which exactly as in section 4 gives the existence of a fixed point for $\widetilde{F}$.

□

6 Conclusion

We have achieved our aim, which was to prove the existence of equilibria in hedonic models under the same assumptions on the (ordinal) utility functions that are used in the Arrow-Debreu theory. There is, however, one last extension to make, namely to the case when the number of producers and consumers is different, that is, $\mu(X)$ is different from $\nu(Y)$. An analysis along the lines of [6] is certainly possible, we would have to endow consumers and producers with reservation utilities, and we would reach the same conclusions, namely that, in equilibrium, some agents will be priced out of the market. However, we feel that the mathematical complications would obscure the main achievement which we have presented here

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