1 Trading.

1.1 What is a trade ?

An agreement between two parties, one of which (the buyer) pays the price, and the second one (the seller) commits to deliver. Also called a contract. Every trade is characterized by:

- the identities of the parties (buyer and seller)
- the price. There are two kind of prices: over the counter prices (OTC) and market prices. The differences are huge. OTC prices are not known beforehand; they are the result of a negociation process. If the negociation breaks down, there is no price, and if the parties do not communicate the result, the price will not be known. Market prices are publicly posted, so they can be quoted even without a transaction, they are anonymous (everyone gets the same price) and liquid (any quantity can be had for that price). The structure of the market determines whether prices are competitive, oligopolistic or monopolistic.
- the delivery:
  - What ? The asset: precise definition of quantity and quality.
  - When ? Immediate delivery (spot contract) vs. deferred delivery (forward contract)
  - Where ?
  - On most organized markets, there are standardized forward contract, called futures. See http://www.cmegroup.com/trading/commodities/

Further distinctions of importance:

- financial assets (flows of money) vs. physical assets (commodities, energy). Stock exchanges vs mercantile exchanges. Physical delivery vs cash delivery
- contingent assets: the delivery (what ? when ?) depends on some well-defined event, which may or may not occur in the future. Example: insurance

1.2 Basic financial assets

- future contracts with cash delivery
- stocks
- bonds. Zero-coupon bonds, coupon bonds and interest rates. The yield curve

The prices change around the clock.

1.3 Financial risk

Risk = commitment to an (partially) unknown financial flow

- prices go up or down (market risk). Example: interest rate risk and mortgages
- one may not be able to sell/buy when required (liquidity risk)
- counterparty may default (credit risk). Example: Madoff
- defaults may spread mechanically (systemic risk).

Ideally, an investor should bear only the market risk. This is why organised markets exist: they are tightly regulated in order to eliminate all but the market risk:

- standardized contracts
- clearing house
- margin calls

There is a further question: should the market risk be the same for all, ie should everyone have the same information ? In most stock exchanges, the answer is yes (no insider trading)

1.4 Basic attitudes towards risk

- arbitrage (cash and carry, contango)
- hedging (use of futures contracts)
- investing (running the risk for an expected reward)

2 Risk measures

Prices change around the clock. There are two possibilities to model this process. Either you understand how prices are formed, and you have a deterministic model, or you don't. In that case, the only know way of modeling uncertainty is through probability, and we get a stochastic model. From now on, we will consider financial positions as random variables. There are two approaches:

2.1 The scenario approach

There is a set of events Ω, one and only one of which will occur, thereby determining the value of all financial assets. We turn it inot a probability space $(\Omega, \mathcal{A}, P)$. A position is then a random variable $X : \Omega \to R$. It is usually restricted to be in L^∞.

2.1.1 Definitions

A *monetary risk measure* is a map $\rho : L^\infty \longrightarrow R$ which is

- monotonic: $X \leq Y \Longrightarrow \rho(X) \geq \rho(Y)$
- cash invariant: $\rho(X+m) = \rho(X) - m$ (so that $\rho(X + \rho(X)) = 0$ and $\rho(1) = -1$)
- normalization: $\rho(0) = 0$

It is *convex* if ρ is a convex function. A *coherent* risk measure is a monetary risk measure which is

- positively homogeneous: $\rho(\lambda X) = \lambda \rho(X)$ for $\lambda > 0$
- subadditive $\rho(X+Y) \leq \rho(X) + \rho(Y)$

Note that a coherent risk measure is convex. The converse is not true (e.g. the entropic risk measure)

Subadditivity is a desirable property, because it means that the task of managing risk can be decentralized. Unfortunately, the $V@R$, which is the most common risk measure does not have it.

2.1.2 Examples

1: The worst-case risk measure

$$\rho_{\max}(X) := -\inf X$$

This is a coherent measure of risk. Note that it is finite on L^∞ only.

2: The V@R

$$V@R_\lambda(X) := \inf\{m \mid P[m + X < 0] \leq \lambda\}$$

This is a monetary measure of risk, it is positively homogeneous but it is not subadditive (and hence not convex). It is the one used in practice (Basel 2 accounting regulations). Having a $V@R_\lambda$ of m means that the chance of losing m or more is less than λ. Altenatively, m is the least amount of equity capital which has to be added to the position so that the likelihood of a negative outcome is less than λ.

3: The entropic risk measure

$$\rho_e(X) := \frac{1}{\beta} \log E\left[e^{-\beta X}\right]$$

This is a convex risk measure, but it is not coherent (neither positively homogeneous nor subadditive). Note that it is finite on L^∞ only.

2.1.3 The acceptance set

Given a risk measure ρ, we define the associated acceptance set $\mathcal{A}_\rho$ as follows:

$$\mathcal{A}_\rho = (X \mid \rho(X) \leq 0)$$

Note that ρ can be recovered from $\mathcal{A}_\rho$:

$$\rho(X) = \inf \{m \mid m + X \in \mathcal{A}_\rho\}$$

2.1.4 Convex risk measures: the penalty function

From now on, we assume that ρ is a convex risk measure (note that this excludes the $V@R$) which is lower semi-continuous (lsc) with respect to the L^p topology, for some $p \in]1, \infty[$. Note that this is always the case when Ω is finite. It then extends to a convex lsc function $\rho : L^p \to R \cup \{+\infty\}$, and we can use standard Fenchel duality.. We have:

$$\rho^*(X^*) = \sup_X \{< X, X^* > -\rho(X)\}$$

$\rho^*(X^*)$ is usually called the *penalty function* associated with the risk measure $\rho(X)$. By Fenchel duality:

$$\rho(X) = \sup_{X^*} \{< X, X^* > -\rho^*(X^*)\}$$

Proposition 1 *If ρ is a monetary risk measure, and $\rho^*(X^*)$ is finite, then $-X^*$ is a probability density*

$$\rho^*(X^*) \leq +\infty \Longrightarrow \{\ X^* \leq 0 \text{ and } E[X^*] = 1\}$$

Proof. By cash-invariance, we have:

$$\begin{aligned}
\rho^*(X^*) &= \sup_X \{< X, X^* > -\rho(X)\} \\
&= \sup_X \{< X, X^* > -\rho(X + m) + m\} \\
&= \sup_X \{< X + m, X^* > -\rho(X + m) - < m, X^* > +m\} \\
&= \rho^*(X^*) - < m, X^* > +m
\end{aligned}$$

which implies that $m = < m, X^* >$, or $1 = < 1, X^* >$ so that X^* integrates to 1.

Suppose $X^* > 0$ for an event $A \subset \Omega$ set with $P[A] > 0$. Consider the characteristic function 1_A of A, and the random variable $\lambda 1_A$, for some constant $\lambda > 0$. We have:

$$\begin{aligned}
\rho^*(X^*) &= \sup_X \{< X, X^* > -\rho(X)\} \\
&\geq \sup_{\lambda > 0} < \lambda 1_A, X^* > -\rho(\lambda 1_A) \\
&= \sup_{\lambda > 0} \{\lambda < 1_A, X^* > -\rho(\lambda 1_A)\} \\
&\geq \sup_{\lambda > 0} \lambda < 1_A, X^* >= +\infty
\end{aligned}$$

■

The entropic risk measure The penalty function is:

$$\rho^* (X^*) = \sup_X \left\{ < X^*, X > -\frac{1}{\beta} \log E \left[e^{-\beta X} \right] \right\} \tag{1}$$

The function $X \to < X^*, X > -\frac{1}{\beta} \log E \left[e^{-\beta X} \right]$ is concave We seek the point $\bar{X}$ where it attains its maximum. It is given by:

$$X^* - \frac{1}{\beta} \frac{F'(X)}{F(X)} = 0$$

where $F(X) := E \left[e^{-\beta X} \right]$. Computing the derivative, we find:

$$\begin{aligned}
< F'(X), Y > &= \lim_{h \to 0} \frac{F(X + hY) - F(X)}{h} = \lim_{h \to 0} \int \frac{e^{-\beta(X+hY)} - e^{-\beta X}}{h} dP \\
&= \lim_{h \to 0} \int \frac{e^{-\beta Y h} - 1}{h} e^{-\beta X} dP = \int -\beta e^{-\beta X} Y dP \\
&= < -\beta e^{-\beta X}, Y >
\end{aligned}$$

so that $F'(X) = -\beta e^{-\beta X}$. Substituting, we get:

$$X^* = -\frac{e^{-\beta X}}{E \left[e^{-\beta X} \right]}$$

We see that $-X^*$ is a probability density, as announced by the general theory. Substituting into formula (1), and setting $E \left[e^{-\beta X} \right] = c$, we get:

$$\begin{aligned}
\rho^* (X^*) &= \sup_c \left\{ -\frac{1}{\beta} < X^*, \log c + \log(-X^*) > -\frac{1}{\beta} \log E \left[e^{\log c + \log(-X^*)} \right] \right\} \\
&= \frac{1}{\beta} < -X^*, \log(-X^*) > \qquad (2)
\end{aligned}$$

$-X^*$ is the density of a certain probability Q with respect to P. The relative entropy of Q with respect to P is defined to be the number:

$$H(Q \mid P) := E \left[-X^* \log(-X^*) \right]$$

so that formula (2) is usually written:

$$\rho^* (X^*) = \frac{1}{\beta} H(Q \mid P)$$

2.1.5 Coherent risk measures

Recall that a coherent risk measure is convex.

Proposition 2 *If ρ is coherent, then $\rho^* = \delta(-X^* \mid C)$ for some closed convex set $C \subset X^*$ of probability densities:*

$$\begin{aligned}\rho^*(X^*) &= \begin{cases} 0 \text{ if } X^* \in -C \\ +\infty \text{ otherwise}\end{cases} \\ \rho(X) &= \sup_{X^* \in -C} \{< X^*, X >\}\end{aligned}$$

This is usually written as follows:

$$\rho(X) = \sup E_{Q \in \mathcal{Q}}[-X]$$

where $\mathcal{Q}$ is the set of probability measures Q with density in $-C$.

The worst-case risk measure. Denoting, as usual, by Q the probability which has density $-X^*$ with respect to P, so that $dQ = -X^* dP$

$$\begin{aligned}\rho^*_{\max}(X^*) &= \sup_X \{< X^*, X > + \inf X\} \\ &= \sup_X \left\{ -\int X dQ \right\} + \inf X \\ &= 0 \text{ for all probabilities } Q\end{aligned}$$

so $\mathcal{Q}$ is the set of all probability densities in L^q

Proposition 2 has a converse, which will allow us to build systematically coherent risk measures:

Proposition 3 *Given any closed convex subset $-C$ of probability densities, and denoting by $\mathcal{Q}$ the set of probability measures Q with density in $-C$, the formula*

$$\begin{aligned}\rho(X) &= < X^*, X > = \sup_{X^* \in C} \int_\Omega (-X)(-X^* dP) \\ &= \sup_{Q \in \mathcal{Q}} E_Q[-X]\end{aligned}$$

defines a coherent risk measure.

Quantiles From now on, we will suppose that the law of X is continous. The λ-*quantile* of X is the only number $q_X(\lambda)$ such that

$$P[X \leq q_X(\lambda)] = \lambda \tag{3}$$

Alternatively, $P[X \geq q_X(\lambda)] = 1 - \lambda$. We have:

$$q_X(\lambda) = \sup\{q \mid P[X \leq q] \leq \lambda\} = \inf\{q \mid P[X \leq q] \geq \lambda\}$$

We can also express q_X in terms of the distribution function F_X, defined by

$$F_X(x) := P[X \leq x]. \tag{4}$$

Comparing (3) and (4), we find $\lambda = F(q_X(\lambda))$ and hence:

$$q_X(\lambda) = F^{-1}(\lambda)$$

Lemma 4 *Note that the $V@R$ can be expressed in terms of quantiles:*

$$V@R_\lambda(X) = -q_X(\lambda) = q_{-X}(1-\lambda) \tag{5}$$

The average value at risk AV@R. We take:

$$C := \left\{ X^* \mid |X^*(\omega)| \leq \frac{1}{\lambda} \ \text{ a.e} \right\}$$

and we define the *average value at risk* at level λ :

$$AV@R_\lambda(X) := \sup_{Q \in \mathcal{Q}} E_Q[-X]$$

It is a coherent measure of risk.

Proposition 5 *Suppose the law of X is continuous. Then*

$$AV@R_\lambda(X) = -\frac{1}{\lambda} \int_{X \leq q} X dP$$

where $q = q_{-X}(\lambda)$ is the λ-quantile of X

Proof. We compute:

$$\sup E_Q[-X] = \sup_{Z^*} \left\{ \int Z^*(-X) dP \mid \int Z^* = 1, \ \frac{1}{\lambda} \geq Z^* \geq 0 \right\} \tag{6}$$

We have to find the $\bar{Z}^*$ where the maximum is attained. To do this, we rewrite the problem as a convex optimisation problem:

$$\begin{gathered} \sup_{Z^*} \int \left(Z^* X \ + \delta \left(Z^* | \left[0, \ \frac{1}{\lambda} \right] \right) \right) dP \\ \int Z^* = 1 \end{gathered}$$

From convex optimization theory, there exists a Lagrange multiplier μ such that $\bar{Z}^*$ solves the unconstrained problem:

$$\sup_{Z^*} \left\{ \int \left(Z^* X \ + \delta \left(Z^* | \left[0, \ \frac{1}{\lambda} \right] \right) \right) dP \ + \mu \int Z^* \right\} = \sup_{Z^*} \left\{ \int \left(Z^*(\omega) X(\omega) \ + \delta \left(Z^*(\omega) | \left[0, \ \frac{1}{\lambda} \right] \right) + \mu \right) dP \right.$$

The integral of course can be maximized pointwise, leading to the solution:

$$\bar{Z}(\omega)^* = \begin{cases} 0 \text{ if } -X(\omega) < a \\ \frac{1}{\lambda} \text{ if } -X(\omega) > a \end{cases}$$

The level a is determined by the fact that Z^* integrates to 1, so that:$P[-X > a] = \lambda$. In other words, $-a = q_X(\lambda)$ is the λ-quantile of X. ■

We are now able the explain why it is called the "average" value at risk

Corollary 6 *Suppose the law of X is continous. Then:*

$$AV@R_\lambda(X) = \frac{1}{\lambda}\int_0^\lambda V@R_\gamma(X)\,d\gamma$$

Proof. By formula (5), we have:

$$\begin{aligned} \int_0^\lambda V@R_\gamma(X)\,d\gamma &= -\int_0^\lambda q_X(\gamma)\,d\gamma = -\int_0^\lambda F_X^{-1}(\gamma)\,d\gamma \\ -\int_{X \le q_X(\lambda)} X dP &= -\int_{-\infty}^{q_X(\lambda)} x dF(x) = -q_X(\lambda)\,F(q_X(\lambda)) + \int_{-\infty}^{q_X(\lambda)} F_X(\gamma)\,d\gamma \end{aligned}$$

and a glance at the graph is enough to convince us that both quantities are equal:

$$q\lambda = \int_{-\infty}^q F(\gamma)\,d\gamma + \int_0^\lambda F^{-1}(\gamma)\,d\gamma$$

■

2.2 The law approach

In this approach, one know only the law of the random variable X. In other words, one concentrates on law-invariant risk measures:

Given two random variables X and Y, on the same probability space $(\Omega, \mathcal{A}, P)$, we shall write $X\tilde{}Y$ to express that X and Y have the same law:

$$E_P[f(X)] = E_P[f(Y)] \text{ for any non-negative Borel function } f : R \to R$$

Definition 7 *A risk measure is law-invariant if $\rho(X) = \rho(Y)$ whenever $X\tilde{}Y$*

AV@R is law invariant. In fact, it serves as a building stone for all law-invariant coherent risk measures. The basic result is the following:

Theorem 8 *A lower semi-continuous risk measure ρ is coherent if and only it:*

$$\rho(X) = \sup_{\pi \in \mathcal{P}} \int AV@R_\lambda(X)\,d\pi(\lambda)$$

where $\mathcal{P}$ is a set of probabilities on the real line.

3 Modelling asset prices by diffusions

The basic model is the following:

$$
\begin{array}{c}
dX^n = \mu^n (t, X) \, dt + \sum_{k=1}^{K} \sigma^{nj} (X) \, dW^k, \quad 1 \leq n \leq N \\
S^i = f^i \left(X^1, ..., X^N \right), ...1 \leq i \leq I
\end{array} \tag{7}
$$

where the S_i are the asset prices, which are determined by outside factors X_n, the evolution of which is partially deterministic and partly stochastic, the stochastic part reflecting the uncertainty.

3.1 Mathematical issues:

See Oksendal, chapters 2,3,4, or Bjork, chapters 2,3,4,5, or Shreve chapters 1,2,3,4

3.1.1 Characterization of 1-d Brownian motion (BM)

3.1.2 Information

Filtration associated with BM. Adapted processes.

3.1.3 The stochastic integral

Definition for adapted step processes. Extension to general processes. Proof in a particular case:

$$
\sum \left(W \left(t_{k+1} \right) - W \left(t_k \right) \right) W \left(t_k \right) \rightarrow \frac{1}{2} W (T)^2 - \frac{T}{2} \quad \text{in } L^2
$$

3.1.4 n-d Brownian motion

Definition, filtration, and stochastic integrals of the form:

$$
\int f \left(W_1 \right) dW_2
$$

3.1.5 The 1-d Ito formula

$$
dW dW = dt
$$

$$\int W dW$$
$$\int W dt$$
$$\int W^2 dt$$
$$\begin{aligned} dX &= \mu_t dt + \sigma_t dW \\ \frac{dX}{X} &= \mu_t dt + \sigma_t dW \\ dX &= \mu (a - X) + \sigma dW \end{aligned}$$

3.1.6 The n-d Ito formula

$$dW_i dW_j = \delta^i_j dt$$

3.2 The reality:

Are financial markets correctly modelled by diffusions?

- Theoretical issues: the noise as extrinsic to the market.
- Practical issues: Levy processes
- Statistical issues: the drift is impossible to estimate in any practical sense.

4 Portfolio management

4.1 Marketed assets and portfolios

Using the Ito formula on the basic model (7), we find that the asset prices are Ito processes. In the following, we will bypass the factors and write the model directly in terms of the S_i. This leads to the modified model;

$$\begin{matrix} dS^n = \mu^n_t dt + \sum_{k=1}^K \sigma^{nk}_t dW^k_t, \quad 1 \leq n \leq N \\ dB = r_t B_t dt \end{matrix} \tag{8}$$

where the W^k, $1 \leq k \leq K$ are independent Brownian motions, μ^i_t, σ^{nj}_t,r_t are stochastic processes, adapted to the filtration $F^{W_1,\ldots,W_K}$.

The interpretation is as follows:

- we are modelling a financial market where $(N+1)$ assets are traded (typically, bonds and/or stocks): S^i is the unitary price of asset i
- the assets can be held in negative quantity: this is called *shorting* the asset (basically, this means you owe it to a counterparty)

- the S^i are market prices in the sense we described at the beginning of this course; for instance, they are *liquid*, meaning that you can buy or sell any quantity without affecting the price

- there are no *transaction costs*: asset i trades at the price S^i; in real life, market-makers typically sell the asset at a higher price (the ask price) then they buy it (the bid price), and they make a profit on the difference

- B is the *riskless asset*. It represent the accumulated deposit of a current account, where interest is earned at the spot (overnight) rate. It is normalized by $B_0 = 1$, so B_t is the amount accrued at time t for one \$ invested at time 0. Note that:

$$B_t = \exp \int_0^t r_s ds$$

  where the integral in taken along each trajectory (so that it is not a stochastic integral).

In the sequel, we shall denote by F_t^W the filtration generated by $\left(W^1, ..., W^K\right)$ and by H the class of all stochastic processes h on (Ω, P) such that:

- h_t is F_t^W-adapted
- $E_P\left[\int_0^t h(t,\omega)^2 ds\right] < \infty$

Definition 9 *A* portfolio *is a family* $\left(h_t^0, h_t^1, ..., h_t^N\right)$ *of adapted processes such that* $h^n \in H$ *for* $0 \leq n \leq N$

The value *of the portfolio is given by:*

$$X_t := h_t^0 B_t + \sum_{n=1}^N h_t^n S_t^n$$

The portfolio is self-financing *if*

$$\begin{aligned} dX_t &= h_t^0 dB_t + \sum_{n=1}^N h_t^n dS_t^n \\ &= \left(h_t^0 r_t B_t + \sum_{n=1}^N h_t^n \mu_t^n\right) dt + \sum_{n=1}^N \sum_{k=1}^K h_t^n \sigma_t^{nk} dW_t^k \end{aligned}$$

Introduce the *relative portfolio weights*:

$$\xi^n = \frac{h^n S^n}{X}$$

The the above formula can be rewritten as:

$$\frac{dX_t}{X_t} = r_t^0 \xi_t^0 + \sum_{n=1}^N \xi_t^n \frac{dS_t^n}{S_t^n}$$

4.2 Discounted values

By definition, the discounted value of B_t is 1 (constant dollars), and the discounted value of a risky asset n is:

$$\tilde{S}_t^n := \frac{S^n}{B_t} = S^n \exp\left(-\int_0^t r_s ds\right)$$

We have:

$$d\tilde{S}^n = \left(\mu_t^n dt + \sum_{k=1}^K \sigma_t^{nk} dW_t^k\right) \exp\left(-\int_0^t r_s ds\right) - r_t \tilde{S}^n dt, \quad 1 \leq n \leq N$$

The discounted value of portfolio $\left(h_t^0, h_t^1, ..., h_t^N\right)$ is:

$$\begin{aligned} \tilde{X}_t &:= h_t^0 + \sum_{n=1}^N h_t^n \tilde{S}_t^n \\ \frac{d\tilde{X}_t}{\tilde{X}_t} &= \sum_{n=1}^N \xi_t^n \frac{dS_t^n}{S_t^n} \end{aligned}$$

and $\left(h_t^0, h_t^1, ..., h_t^N\right)$ is self-financing if:

$$d\tilde{X}_t = \sum_{n=1}^N h_t^n d\tilde{S}_t^n = \sum_{n=1}^N \xi_t^n \frac{d\tilde{S}_t^n}{\tilde{S}_t^n} \tag{9}$$

where $\left(\xi_t^0, \xi_t^1, ..., \xi_t^N\right)$ are the relative portfolio weights.

Lemma 10 *Any adapted process* $\left(h_t^1, ..., h_t^N\right)$ *with* $h^n \in H$ *for* $1 \leq n \leq N$ *is the risky part of a self-financing portfolio. The initial value* x *can be chosen arbitrarily.*

Proof. If the $\left(h_t^1, ..., h_t^N\right)$ are given, we derive $\tilde{X}_t$ from (9) and the initial value x. Then $X_t = \tilde{X}_t B_t$, and we get h_t^0 from the equation:

$$h_t^0 B_t = X_t - \sum_{n=1}^N h_t^n S_t^n$$

■

4.3 Utility function and portfolio management

Preference for money. The Saint-Peterburg paradox. The utility function $u : R \longrightarrow \{-\infty\} \cup R$ and the von Neumann-Morgenstern model. Concavity and risk aversion (the Jensen inequality).

Preference for the present. Psychological discount rate.

The terminal-wealth problem:

$$\begin{array}{c}\max e^{-\rho t} E_P \left[u\left(\tilde{X}_t B_t\right)\right] \\ h^n \in H,\ 1 \leq n \leq N \\ d\tilde{X}_t = \sum_{n=1}^N h_t^n d\tilde{S}_t^n \\ X_0 = x \end{array} \tag{10}$$

Note that there are no condition on the h^n beyond the fact that they should belong to H. This is the import of Lemma 10

Note that the $e^{-\rho t}$ factor drops out of the integral: this is because we are not giving the investor the option to stop earlier, so preference for the present plays no role. It will not be the case in more general porfolio management problems.

4.4 Admissible portfolios and AOA

Let $T > 0$ (the investment horizon) be given.

Definition 11 *An admissible portfolio is an* arbitrage opportunity *if its value X_t satisfies the following:*

$$\begin{array}{c} X_0 = 0 \\ X_T \geq 0 \quad a.s. \\ P\left[X_T > 0\right] > 0 \quad a.s \end{array} \tag{11}$$

Note that if $\left(h_t^0, h_t^1, ..., h_t^N\right)$ is an arbitrage opportunity, so is $\lambda\left(h_t^0, h_t^1, ..., h_t^N\right)$ for every $\lambda > 0$. If there exists an arbitrage opportunity, then the maximum expected value in problem (10) is $+\infty$. In other words, there are investment strategies which require no initial investment, which in the worst case will cost nothing in the future, and will bring in money with positive probability. It will be assumed that such opportunities no longer exist in the market:

Definition 12 *We shall say that the market is* arbitrage free *if there are no arbitrage opportunities*

We shall now draw the consequences of AOA.

Denote by V the set of all terminal discounted values which can be reached by an admissible self-financing portfolio with 0 initial value:

$$V = \left\{ \tilde{X}_T \mid h^n \in H,\ \ d\tilde{X}_t = \sum_{n=1}^N h_t^n d\tilde{S}_t^n,\ \ \tilde{X}_0 = 0 \right\}$$

.

It is clearly a linear subspace of $L^2\left(\Omega, F_T^W, P\right)$. Now denote by C the set of all contingent claims which are non-negative at time T and positive with positive probability:

$$C := \left\{ Y \in L^2\left(\Omega, F_T^W, P\right) \mid Y(\omega) \geq 0 \text{ a.s. and } P\left[Y > 0\right] > 0 \right\}$$

It is a convex subset.of $L^2\left(\Omega, F_T^W, P\right)$ If there is AOA, then:

$$K \cap C = \varnothing$$

Separating K from C, we get some $Z \in L^2\left(\Omega, F_T^W, P\right)$ with $Z \neq 0$ such that:

$$\begin{aligned} E_P[ZY] &\geq 0 \text{ for all } Y \in C \\ E_P[ZX] &\leq 0 \text{ for all } X \in V \end{aligned}$$

Since V is a vector space, we have in fact:

$$E_P[ZX] = 0 \text{ for all } X \in V$$

If $Y^1 \in C$ so does $Y^1 + \lambda Y^2$ for all $Y^2 \geq 0$. Plugging this into the preceding equation, we find that:

$$E_P\left[ZY^1\right] + \lambda E_P\left[ZY^2\right] \geq 0 \text{ for all } \lambda \geq 0$$

and hence:

$$E_P\left[ZY^2\right] \geq 0 \text{ for all } Y^2 \geq 0$$

so that:

$$Z \geq 0$$

Setting $Z_0 = Z/E_P[Z]$, we find that Z_0 is a probability density. Define a new probability Q on $\left(\Omega, F_T^W\right)$ by:

$$E_Q[X] := E_P[ZX] \quad \forall X \in L^2\left(\Omega, F_T^W, P\right)$$

Proposition 13 *The probability Q is absolutely continuous with respect to P and the discounted value of any self-financing portfolio is a Q-martingale*

Proof. Let X be a self-financing portfolio. Take any t between 0 and T, and any event A in F_t^W. Consider a portfolio Y consisting of holding X until time t, and then:

- if A occurs, cashing in the portfolio and putting the money in a current account until time T
- if A does not occur, keeping the portfolio $X_s, s \geq t$ until time T

■

Proof. We have, for $s \geq t$:

$$Y_s = \mathbf{1}_A \frac{X_t}{B_t} B_s + (1 - \mathbf{1}_A) X_s$$

Discounting, this becomes:

$$\frac{Y_s}{B_s} = \mathbf{1}_A \frac{X_t}{B_t} + (1 - \mathbf{1}_A) \frac{X_s}{B_s}$$

so that:

$$\begin{aligned} E_Q\left[\frac{Y_T}{B_T}\right] &= E_Q\left[\frac{X_T}{B_T}\right] \\ E_Q\left[\mathbf{1}_A\frac{X_t}{B_t}+(1-\mathbf{1}_A)\frac{X_T}{B_T}\right] &= E_Q\left[\frac{X_T}{B_T}\right] \\ E_Q\left[\mathbf{1}_A\frac{X_t}{B_t}\right] &= E_Q\left[\mathbf{1}_A\frac{X_T}{B_T}\right] \end{aligned}$$

But the latter equality means precisely that:

$$\frac{X_t}{B_t} = E_Q\left[\frac{X_T}{B_T} \mid F_t^W\right] \tag{12}$$

■

Definition 14 *Any probability Q equivalent to P and satisfying (12) is called a* martingale measure

Theorem 15 *The market is arbitrage-free if and only if there is a martingale measure.*

Proof. We have proved that there is a probability Q, absolutely continuous with respect to P, satisfying (12). With more effort, we can prove that $Z > 0$ almost everywhere, so that Q is in fact equivalent to P, and this would prove the "only if" part. Now suppose there is a martingale measure Q and an arbitrage opportunity, ie a process X_t satisfying (11). Then:

$$X_0 = E_Q\left[\frac{X_T}{B_T}\right] > 0$$

contradicting the fact that $X_0 = 0$ ■

4.5 Solving the terminal-wealth problem

Problem (10) is meaningful only if the market is arbitrage-free. In that case, it can be rewritten as follows, with Z denoting the Radon-Nikodym derivative of Q with respect to P :

$$\begin{gathered} \sup E_P\left[u\left(\tilde{X}_T B_T\right)\right] \\ E_P\left[Z\tilde{X}_T\right] = x \\ \tilde{X}_T \in L^2\left(\Omega, \mathcal{F}_T, P\right) \end{gathered}$$

which we can rewrite in a more geometric way, involving the interior product in $L^2\left(\Omega, \mathcal{F}_T, P\right)$:

$$\begin{gathered} \sup \int_\Omega u\left(\tilde{X}_T B_T\right) dP \\ \int_\Omega Z\tilde{X} dP = (Z, \tilde{X})_{L^2} = x \end{gathered}$$

We are consists maximizing a concave function on a closed linear subspace of L^2. Assume there is a maximizer $\hat{X}$.. If the usual theory of Lagrange multipliers applies, there will be some $\lambda \in \mathbb{R}$ such that $\hat{X}$ actually minimizes the functional:

$$\int_\Omega \left[u\left(\tilde{X}_T B_T\right) - \lambda Z \tilde{X}_T \right] dP$$

over all of L^2. Maximizing pointwise under the integral, and bearing in mind that u is concave, we are led to the equation:

$$B_T u'\left(\tilde{X}_T B_T\right) = \lambda Z\left(\omega\right) \quad P\text{-a.e.}$$

which fully characterizes the solution $\hat{X}$.

So, if the maximizer exists, it must be given by the formula:

$$\hat{X} = \frac{1}{B_T}\left[u'\right]^{-1}\left(\lambda Z\right) \tag{13}$$

Proposition 16 *Suppose there is some $\lambda > 0$ such that $\hat{X}_T$ given by formula (13) is square-integrable and satisfies the constraint $(Z, \tilde{X}_T)_{L^2} = x$. Then $\hat{X}_T$ solves the terminal-wealth problem*

Proof. Take another $\tilde{Y} \in L^2$ such that $(Z, \tilde{Y})_{L^2} = x$. Since u is concave, we have:

$$u\left(\tilde{Y}\left(\omega\right) B_T\left(\omega\right)\right) \leq u\left(\tilde{X}_T\left(\omega\right) B_T\left(\omega\right)\right) + B_T\left(\omega\right)(\tilde{Y}\left(\omega\right) - \hat{X}\left(\omega\right)) u'\left(\tilde{X}_T\left(\omega\right) B_T\left(\omega\right)\right) \quad P\text{-a.e.}$$

By definition, $B_T u'\left(\tilde{X}_T B_T\right) = \lambda Z$. Substituting into the inequality and integrating, we get:

$$\int_\Omega u\left(\tilde{Y} B_T\right) dP \leq \int_\Omega u\left(\tilde{X}_T B_T\right) dP + \lambda \int_\Omega (\tilde{Y} - \hat{X}_T) Z dP$$

and the last term vanishes because it is just $\lambda\left(x - x\right)$. So $\hat{X}$ is indeed a minimizer, and the result follows.

The optimal wealth process is then given by:

$$\tilde{X}_t = B_t E_Q\left[\tilde{X}_T \mid t\right] = B_t E_P\left[\tilde{X}_T Z \mid t\right]$$

■

4.5.1 Example 1: $u\left(x\right) = \ln x$

Equation (13) becomes:

$$\hat{X} = \frac{1}{\lambda Z}.$$

and we adjust λ to satisfy the constraint:

$$x = \left(Z, \hat{X}\right) = \frac{1}{\lambda}$$

Hence the optimal wealth process and the optimal value:

$$\begin{aligned}
\hat{X}_T &= \frac{x}{Z} \\
\hat{X}_t &= xE_Q\left[\frac{1}{Z} \mid t\right] \qquad (14)\\
E_P\left[u\left(\hat{X}_T\right)\right] &= E_P\left[\ln\left(\frac{x}{Z}B_T\right)\right] = \ln x + E_P\left[\ln\left(\frac{B_T}{Z}\right)\right]
\end{aligned}$$

Note that $E_Q\left[\tilde{X}_T\right] = E_P\left[Z\tilde{X}_T\right] = x$, as it should be. For $0 < t < T$, we use the following result, which is an elaborated form of *Bayes' rule:*

Lemma 17 *Consider a probability space $(\Omega, \mathcal{F}, P)$, and let Q be another probability on (Ω, F), absolutely continuous wrt P, so that $dQ = ZdP$, $Z \in L^1(\Omega,)$. Assume $\mathcal{G} \subset \mathcal{F}$ is another, smaller, σ-algebra. Then:*

1. *For any random variable X on $(\Omega, \mathcal{F}, P)$, we have:*
2. *Denote by $P_{\mathcal{G}}$ and $Q_{\mathcal{G}}$ the restrictions of P and Q to $\mathcal{G}$. Then:*

$$E_Q[X \mid \mathcal{G}] = \frac{E_P[ZX \mid \mathcal{G}]}{E_P[Z \mid \mathcal{G}]} \qquad (15)$$

$$dQ_{\mathcal{G}} = E_P[Z \mid \mathcal{G}]\, dP \qquad (16)$$

Proof. Let us prove that $E_Q[X \mid \mathcal{G}]\, E_P[Z \mid \mathcal{G}] = E_P[ZX \mid \mathcal{G}]$. It is suffient to prove that, for any $G \in \mathcal{G}$, the integrals of both sides over G are equal:

$$\begin{aligned}
\int_G E_Q[X \mid \mathcal{G}]\, E_P[Z \mid \mathcal{G}]\, dP &= \int_G E_P[ZE_Q[X \mid \mathcal{G}] \mid \mathcal{G}]\, dP \\
&= \int_G ZE_Q[X \mid \mathcal{G}]\ dP \\
&= \int_G E_Q[X \mid \mathcal{G}]\ dQ = \int_G XdQ
\end{aligned}$$

■

As a consequence, formula (14) becomes:

$$\hat{X}_t = x\frac{E_P\left[\frac{1}{Z}Z \mid t\right]}{E_P[Z \mid t]} = x\frac{1}{E_P[Z \mid t]}$$

5 Arbitrage pricing

5.1 Finding the martingale measure(s)

Let W be a one-dimensional BM on (Ω, F_T, P). Consider an Ito process:

$$dS_t = \mu_t dt + \sigma_t dW_t$$

Let Q be a martingale measure. What does it look like ? Can there be several of them ?

Suppose we have found one. We have:

$$dQ = ZdP \text{ on } F_T$$

Definining $Z_t = E\left[Z \mid t\right]$, Bayes' rule gives us:

$$dQ = Z_t dP \text{ on } F_t$$

The process Z_t constitutes a martingale on the filtration F_t, $0 \leq t \leq T$, so it is natural to try to write it as a stochastic integral. In addition it is positive, so we may well write it in the form:

$$\begin{aligned} \frac{dZ_t}{Z_t} &= \lambda_t dW_t \\ Z_0 &= 1 \end{aligned}$$

Applying Ito's formula, we can express Z_t in terms of the process λ_t :

$$Z_t = \exp\left[\int_0^t \lambda_s dW_s - \frac{1}{2}\int_0^t \lambda_s^2 ds\right] \tag{17}$$

We will now adjust λ_t so that S_t is a martingale under Q. We have:

$$\begin{aligned} E_P\left[dX_t \mid t\right] &= \mu_t dt \\ E_P\left[\left(dX_t\right)^2 \mid t\right] &= \sigma_t^2 dt \end{aligned}$$

and by repeated application of Bayes' rule (formula (15))

$$\begin{aligned} E_Q\left[dX_t \mid t\right] &= \frac{E_P\left[Z_{t+dt} dX_t \mid t\right]}{Z_t} \\ &= \frac{E_P\left[\left(Z_t + dZ_t\right) dX_t \mid t\right]}{Z_t} \\ &= \frac{E_P\left[Z_t dX_t \mid t\right]}{Z_t} + \frac{E_P\left[dZ_t dX_t \mid t\right]}{Z_t} \\ &= Z_t \frac{E_P\left[dX_t \mid t\right]}{Z_t} + \frac{E_P\left[dZ_t dX_t \mid t\right]}{Z_t} \\ &= \mu_t dt + \lambda_t \sigma dt \end{aligned}$$

$$\begin{aligned} E_Q\left[\left(dX_t\right)^2 \mid t\right] &= \frac{E_Q\left[Z_{t+dt}\left(dX_t\right)^2 \mid t\right]}{E_P\left[Z_{t+dt} \mid t\right]} \\ &= \frac{E_P\left[Z_t\left(X_{t+dt} - X_t\right)^2 \mid t\right]}{Z_t} + \frac{E_P\left[dZ_t\left(X_{t+dt} - X_t\right)^2 \mid t\right]}{Z_t} \\ &= Z_t \frac{E_P\left[\left(X_{t+dt} - X_t\right)^2 \mid t\right]}{Z_t} \\ &= \sigma_t^2 dt \end{aligned}$$

So X_t is an Ito process under Q:

$$dS_t = \mu_t' dt + \sigma_t' d\tilde{W}$$

with $\mu_t' = \mu_t + \lambda_t \sigma_t$ and $\sigma_t' = \sigma_t$. so that S_t will be a martingale under Q if and only if:

$$\lambda_t = -\frac{\mu_t}{\sigma_t}$$

The whole procedure can be made rigorous. This is the content of *Girsanov's Theorem*:

Theorem 18 *Suppose W is a K-dimensional BM on $\left(\Omega, F_T^W, P\right)$, and let $\left(\lambda_t^1, ..., \lambda_t^K\right)$ be adapted processes such that:*

$$E_P\left[\exp\left(\frac{1}{2}\int_0^T \sum_{k=1}^K \left(\lambda_t^k\right)^2 dt\right)\right] < \infty$$

Define Z_t by:

$$\frac{dZ_t}{Z_t} = \sum_{k=1}^K \lambda_t^k dW^k$$

and set $dQ = Z_T dP$. Then there is a K-dimensional Brownian motion $\tilde{W}$ on (Ω, F_T, Q) such every P-Ito process S_t:

$$dS_t = \mu_t dt + \sum_{k=1}^K \sigma_t^k dW^k$$

becomes a Q-Ito process:

$$dS_t = \left(\mu_t + \sum_{k=1}^K \lambda_t^k \sigma_t^k\right) dt + \sum_{k=1}^K \sigma_t^k d\tilde{W}^k$$

Corollary 19 *Suppose adapted processes $\left(\lambda_t^1, ..., \lambda_t^K\right)$ can be found such that:*

$$\mu_t + \sum_{k=1}^K \lambda_t^k \sigma_t^k = 0$$

Then Q is a martingale measure.

5.2 Complete markets

We return to our standard model (8). Since asset prices are positive, there is no loss of generality in rewriting it as follows:

$$\begin{array}{c} \frac{dS_t^n}{S_t^n} = \mu_t^n dt + \sum_{k=1}^K \sigma_t^{nk} dW_t^k, \quad 1 \leq n \leq N \\ \frac{dB_t}{B_t} = r_t dt \end{array} \tag{18}$$

or, in discounted prices:

$$\frac{d\tilde{S}_t^n}{\tilde{S}_t^n} = (\mu_t^n - r_t)\, dt + \sum_{k=1}^{K} \sigma_t^{nk} dW_t^k, \quad 1 \leq n \leq N$$

Assume that it is arbitrage-free, so that there is a martingale measure Q, and the discounted values $\tilde{S}^n$ are Q-martingales. By the preceding section, there must be adapted processes $\left(\lambda_t^1, ..., \lambda_t^K\right)$ and a K-dimensional Q-Brownian motion $\tilde{W}$ such that $\tilde{S}^n$ is a Q-Ito process given by:

$$\frac{d\tilde{S}^n}{\tilde{S}^n} = \left(\mu_t^n + \sum_{k=1}^{K} \lambda_t^k \sigma_t^k - r_t\right) dt + \sum_{k=1}^{K} \sigma_t^{nk} d\tilde{W}_t^k, \quad 1 \leq n \leq N \tag{19}$$

Since the $\tilde{S}^n$ are Q-martingale, we must have:

$$\sum_{k=1}^{K} \lambda_t^k \sigma_t^k = r_t - \mu_t^n,\ 1 \leq n \leq N \tag{20}$$

This is a system of N equations with K unknowns for the λ_t^k. If they determine the λ_t^k uniquely, then the martingale measure Q is unique. The λ_t^k are known as the *market prices of risks.*

Definition 20 *The market is* complete *if the martingale measure is unique.*

A rule of thum is that if there are as many (or more) assets N than sources of risk K, then the market is complete. Otherwise, it is called incomplete.

5.3 Pricing European options in complete markets

From now on, we are dealing with a complete market, described by the equations (18)

$$\begin{array}{c} \frac{dS_t^n}{S_t^n} = \mu_t^n dt + \sum_{k=1}^{K} \sigma_t^{nk} dW_t^k, \quad 1 \leq n \leq N \\ \frac{dB_t}{B_t} = r_t dt \end{array}$$

and the martingale measure Q is characterized by the adapted process $\left(\lambda_t^1, ..., \lambda_t^K\right)$ and the K-dimensional Q-Brownian motion $\left(\tilde{W}^1, ..., \tilde{W}^K\right)$.

The P-equations for discounted processes $\tilde{S}^n$ are:

$$\frac{d\tilde{S}_t^n}{\tilde{S}_t^n} = (\mu_t^n - r_t)\, dt + \sum_{k=1}^{K} \sigma_t^{nk} dW_t^k, \quad 1 \leq n \leq N$$

The $\tilde{S}^n$ are Q-martingales, and by Girsanov's theorem„ the Q-equations for the same processes are given by:

$$\frac{d\tilde{S}_t^n}{\tilde{S}_t^n} = \sum_{k=1}^{K} \sigma_t^{nk} d\tilde{W}_t^k, \quad 1 \leq n \leq N \tag{21}$$

Going back to the undiscounted prices, we find the Q-equations:

$$\frac{dS_t^n}{S_t^n} = r_t dt + \sum_{k=1}^{K} \sigma_t^{nk} d\tilde{W}_t^k, \quad 1 \leq n \leq N \tag{22}$$

We ask ourselves what happens if the market authorities introduce an additional asset S_t^{N+1}, to be traded alongside $\left(B_t, S_t^1, ..., S_t^N\right)$. If the market is to remain arbitrage-free, and if no new Brownian motion has been introduced, there should still be a martingale measure $\hat{Q}$ on (Ω, F_T). In other words, we will have:

$$S_t^n = E_{\hat{Q}}\left[\frac{X_T}{B_T} \mid F_t^W\right], \quad 1 \leq n \leq N+1$$

Since the original market $\left(B_t, S_t^1, ..., S_t^N\right)$ was supposed to be complete, the N first equations are enough to determine the market prices of risk, so we must have $Q = \hat{Q}$. In other words, in complete markets, the pricing formula:

$$\tilde{X}_t = E_Q\left[\tilde{X}_T \mid F_t\right] \tag{23}$$

holds for *any* asset, not only for portfolios.

Definition 21 *A* European option *on* $\left(B, S^1, ..., S^N\right)$ *with exercise time* T *is an asset which delivers* $f\left(B, S_T^1, ..., S_T^N\right)$ *at time* T.

At what price P_t should such an option trade ? Clearly, we have introduced no new source of risk, so the preceding results apply. The only arbitrage-free price is given by (23):

$$X_t = B_t E_Q\left[\frac{1}{B_T} f\left(S_T^1, ..., S_T^N\right) \mid F_t\right]$$

It is natural to expect that the current price of the option should depend only on the current prices of the assets and the time to maturity. In other words, we are seeking a (deterministic) function $\varphi\left(t, s^1, ..., s^N\right)$ such that:

$$X_t = \varphi\left(t, B_t, S_t^1, ..., S_t^N\right)$$

We are now in the Q-world. Apply Ito's formula, using the Q-equations (22):

$$\begin{aligned} dX_t &= \left(\frac{\partial \varphi}{\partial t} + r_t \frac{\partial \varphi}{\partial b} B_t + r_t \sum_{n=1}^{N} S_t^n \frac{\partial \varphi}{\partial s^n} + \frac{1}{2} \sum_{n,m=1}^{N} \sum_{k=1}^{K} \sigma_t^{nk} \sigma_t^{mk} S_t^n S_t^m \frac{\partial^2 \varphi}{\partial s^n \partial s^m}\right) dt + \sum_{n} \sum_{k=1}^{K} \sigma_t^{nk} \frac{\partial \varphi}{\partial s^n} S^n d\tilde{W}^k \\ d\tilde{X}_t &= \left(\frac{\partial \varphi}{\partial t} + r_t \frac{\partial \varphi}{\partial b} + r_t \sum_{n=1}^{N} \tilde{S}_t^n \frac{\partial \varphi}{\partial s^n} + \frac{1}{2} B_t \sum_{n,m=1}^{N} \sum_{k=1}^{K} \sigma_t^{nk} \sigma_t^{mk} \tilde{S}_t^n \tilde{S}_t^m \frac{\partial^2 \varphi}{\partial s^n \partial s^m} - r_t \tilde{X}_t\right) dt + \sum_{n} \sum_{k=1}^{K} \sigma_t^{nk} \frac{\partial \varphi}{\partial s^n} \tilde{S} \end{aligned}$$

For $\tilde{X}_t$ to be a Q-martingale, the drift coefficient must vanish. We are left with:

Proposition 22 *Suppose r_t, the μ_t^n and the σ_t^{nk} are deterministic functions of $\left(B_t, S_t^1, ...S_t^N\right)$. The arbitrage-free price of a European option, with terminal value $f\left(B_T, S_T^1, ..., S_T^N\right)$, is given by:*

$$X_t = \varphi\left(t, B_t, S_t^1, ..., S_t^N\right)$$

where the (deterministic) function $\varphi\left(b, s^1, ..., s^n\right)$ satisfies the second-order PDE:

$$\frac{\partial\varphi}{\partial t} + r_t\left(\frac{\partial\varphi}{\partial b} + \sum_{n=1}^{N} s^n \frac{\partial\varphi}{\partial s^n}\right) + \frac{1}{2}\sum_{n,m=1}^{N}\sum_{k=1}^{K}\sigma_t^{nk}\sigma_t^{mk}\frac{\partial^2\varphi}{\partial s^n \partial s^m}s^n s^m - r\varphi = 0 \quad (24)$$

with the boundary condition:

$$\varphi\left(T, b, s^1, ..., s^N\right) = f\left(b, s^1, ..., s^N\right) \quad (25)$$

Finding the *hedging portfolio* is now easy. One seeks a portfolio $\left(h_t^0, h_t^1, ..., h_t^N\right)$ which:

- has the same value as the option X_t, so that:

$$dX_t = r_t dt + \sum_{n}^{N}\sum_{k=1}^{K}\sigma_t^{nk}\frac{\partial\varphi}{\partial s^n}S^n d\tilde{W}^k$$

- is self-financing, so that:

$$\begin{aligned}
dX_t &= h_t^0 r_t B_t dt + \sum_{n}^{N} h_t^n \sum_{k=1}^{K}\sigma_t^{nk}\frac{\partial\varphi}{\partial s^n}S^n d\tilde{W}^k \\
&= \left(h_t^0 B_t + \sum_{n=1}^{N} h_t^n S^n\right) r_t dt + \sum_{n=1}^{N}\sum_{k=1}^{K} h_t^n S_t^n \sigma^{nk} d\tilde{W}_t^k \\
&= rX_t dt + \sum_{n=1}^{N}\sum_{k=1}^{K} h_t^n \sigma^{nk} S_t^n d\tilde{W}_t^k
\end{aligned}$$

Identifying, we get:

$$\begin{aligned}
h_t^n &= \frac{\partial\varphi}{\partial s^n}\left(t, B_t, S_t^1, ..., S_t^N\right), \quad 1 \le n \le N \\
h_t^0 &= \frac{1}{B_t}\left(X_t - \sum_{n=1}^{N} h_t^n S^n\right)
\end{aligned}$$

5.4 The Black and Scholes world

5.4.1 The martingale measure

Consider the simplest possible market (μ, σ, r are positive constants)

$$\begin{aligned} \frac{dS}{S} &= \mu dt + \sigma dW \\ \frac{dB}{B} &= r \end{aligned}$$

Note that the value of S_t can be given explicitly in terms of W_t :

$$S_t = S_0 \exp\left[\sigma W_t + \left(\mu - \frac{1}{2}\sigma^2\right) t\right]$$

The martingale measure Q is found by writing equation (19) in this context:

$$\frac{d\tilde{S}}{\tilde{S}} = (\mu + \lambda\sigma - r)\, dt + \sigma d\tilde{W}$$

Setting the drift to 0 gives $\lambda = -(\mu - r)/\sigma$ (the *market price of risk*) so that the evolution of the risky asset under Q is given by:

$$\frac{d\tilde{S}}{\tilde{S}} = \sigma d\tilde{W}$$

Going back to non-discounted values gives:

$$\frac{dS}{S} = r dt + \sigma dW$$

Note that, by formula (17), we have:

$$Z_t = \exp\left[\int_0^t \lambda dW_s - \frac{1}{2}\int_0^t \lambda^2 ds\right] = \exp\left[\lambda W_t - \frac{1}{2}\lambda^2 t\right] \tag{26}$$

5.4.2 Pricing a call

Consider a *European call* C on asset 1:

$$C_T = \begin{cases} S_T^1 - K \text{ if } S_T^1 \geq K \text{ (exercise price)} \\ 0 \text{ if } S_T^1 \leq K \end{cases}$$

The Black and Scholes formula Its price at time 0 is given by:

$$C_0 = B_t E_Q\left[\frac{\left(S_T^1 - K\right)_+}{B_T}\right] = e^{-r(T-t)} E_Q\left[\left(S_T^1 - K\right)_+\right]$$

This can be evaluated by numerical simulations.

However, equation (24) with the boundary condition (25) provides us with an explicit formula. We have:

$$C_t = \varphi\left(t, S_t\right)$$

and the function φ is given by:

$$\begin{aligned}
\frac{\partial \varphi}{\partial t} + r\frac{\partial \varphi}{\partial s}s + \frac{1}{2}\sigma^2\frac{\partial^2 \varphi}{\partial s^2}s^2 - r\varphi &= 0 \\
\varphi\left(T, s\right) &= \left(s - K\right)_+
\end{aligned}$$

Setting $x = \ln s$, and considering the function $\psi\left(t, x\right) := \varphi\left(t, e^x\right)$, changes the equation to:

$$\frac{\partial \psi}{\partial t} + r\frac{\partial \psi}{\partial x} + \frac{1}{2}\sigma^2\left(\frac{\partial^2 \psi}{\partial s^2} + \frac{\partial \psi}{\partial x}\right) - r\varphi = 0$$

which is linear with constant coefficients (basically the heat equation, with the time direction reversed). So there is an explicit solution, given by the celebrated Black and Scholes formula:

$$C_0 = S_0 N\left(d_+\left(T, S\left(0\right)\right)\right) - Ke^{-rT}N\left(d_-\left(T, S\left(0\right)\right)\right)$$

where N is the distribution function of a standard Gaussian variable, and

$$d_\pm\left(T, s\right) = \frac{1}{\sigma\sqrt{T}}\left[\ln\frac{s}{K} + \left(r \pm \frac{\sigma^2}{2}\right)T\right]$$

Note that it does not depend on the trend μ. Two investors who have different estimates of μ, one believing that the stock will go up, and the other believing that the stock will go down, agree on the price of the call. But the first one will buy the call, and the other will sell it. As we shall see now, it the call trades at $\varphi\left(t, S_t\right)$, the seller can invest the price into a portfolio which will exactly cover his loss, if any. In other words, the seller takes no risk.

The heding portfolio We value of the call is precisely the value of a portfolio consisting of h_t^0 in cash and h_t in risky asset, with:

$$\begin{aligned}
h^t &= \frac{\partial \varphi}{\partial s}\left(t, S_t\right) \\
h_t^0 &= \varphi\left(t, S_t\right) - \frac{\partial \varphi}{\partial s}\left(t, S_t\right)
\end{aligned}$$

This is the *heding portfolio*, or *replicating portfolio*. Whoever sells the call can buy the hedging portfolio, and manage it by simply applying the formula $h^t = \frac{\partial \varphi}{\partial s}\left(t, S_t\right)$. By doing so, the seller incurs no risk: if the call is in the money, the hedging portfolio will have earned exactly enough to cover the loss, if the call is out of the money, the hedging portfolio will have terminal value 0. If the call was traded at any other price than φ, the hedging portfolio would become an arbitrage opportunity.

5.4.3 Solving the terminal-wealth problem

$$\sup E_P \left\{ \left[\ln \left(\tilde{X}_T B_T \right) \right] \mid X_0 = x \right\}$$

The current value of the investment We want to solve the terminal-wealth problem with logarithmic utility. It has been shown that the value of the optimal portfolio is given by:

$$X_t = x \frac{1}{E_P \left[Z \mid t \right]}$$

Using formula (26), this becomes:

$$X_t = x \exp \left[-\lambda W_t + \frac{1}{2} \lambda^2 t \right]$$

The value is given in terms of the current value ot the Brownian motion W_t; it would be easier to express it in terms of the current value of the stock S_t. Since S_t is $\mathcal{F}_t$-measurable, this is possible. We want to write $X_t = \varphi(t, S_t)$. This gives:

$$\ln S_t = \ln S_0 + \sigma W_t + \left(\mu - \frac{1}{2} \sigma^2 \right) t$$

$$\begin{aligned} \ln X_t &= \ln x - \lambda W_t + \frac{1}{2} \lambda^2 t \\ &= \ln x - \frac{\lambda}{\sigma} \left(\ln S_t - \ln S_0 - \left(\mu - \frac{1}{2} \sigma^2 \right) t \right) + \frac{1}{2} \lambda^2 t \end{aligned}$$

Writing $\lambda = -\frac{\mu - r}{\sigma}$ in the last equation, we find:

$$\begin{aligned} \ln X_t &= \ln x + \frac{\mu - r}{\sigma^2} \left(\ln S_t - \ln S_0 - \left(\mu - \frac{1}{2} \sigma^2 \right) t \right) + \frac{1}{2} \left(\frac{\mu - r}{\sigma} \right)^2 t \\ &= \ln x + \frac{\mu - r}{\sigma^2} \ln \frac{S_t}{S_0} + \frac{\mu - r}{2\sigma^2} \left(\mu - r + \sigma^2 \right) t \end{aligned}$$

The final answer is $X_t = \varphi(t, S_t)$ with:

$$\begin{aligned} \varphi(t, s) &= \exp \left[\ln x + \frac{\mu - r}{\sigma^2} \left(\ln \frac{s}{S_0} + \frac{1}{2} \left(\mu - r + \sigma^2 \right) t \right) \right] \\ &= x \left(\frac{s}{S_0} \right)^{\frac{\mu - r}{\sigma^2}} \exp \left[\frac{\mu - r}{2\sigma^2} \left(\mu - r + \sigma^2 \right) t \right] \end{aligned}$$

The hedging portfolio We now get the hedging portfolio, that is the optimal quantity h_t of risky asset that must be held at time t if the price is S_t :

$$h_t = \frac{\partial \varphi}{\partial s} = \frac{\mu - r}{\sigma^2} \frac{1}{S_t} \varphi\left(S_t\right) = \frac{\mu - r}{\sigma^2} \frac{Xt}{S_t}$$

Note that:

$$\xi_t := \frac{h_t S_t}{X_t} = \frac{\mu - r}{\sigma^2}$$

This is known as *Merton's rule*: in the optimal portfolio, the proportion of wealth held in the risky asset (and hence, the proportion of wealth held in the non-risky asset) must be constant. This rule holds, not only for logarithmic utility, but also for all utilities of the form $u\left(x\right) = \frac{1}{p} x^p, x > 0, p \in] - \infty, +1[$