On singular perturbation problems for systems of PDEs

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The problem

We want to solve a system of PDEs:

$$S(\varepsilon, u, Du) = f$$

- near $S(0,0,0) = 0$,
- the linearized operator $v \longrightarrow S_u(\varepsilon, u, Du)v + S_{Du}(\varepsilon, u, Du)\, Dv$ has a right-inverse $L_\varepsilon(u)$ for (ε, u, Du) small
- $\|L_\varepsilon(u)\| \longrightarrow \infty$ when $\varepsilon \longrightarrow 0$

The linear case

Consider on the 2-dimensional torus the linear differential operator

$$\partial_\omega = \omega_1 \frac{\partial}{\partial \theta_1} + \omega_2 \frac{\partial}{\partial \theta_2}$$

where $\omega := (\omega_1, \omega_2)$ satisfies the Diophantine condition:

$$\left| \frac{\omega_1}{\omega_2} - \frac{p}{q} \right| \geq \frac{K}{|q|^{2+\alpha}},\ \alpha > 0$$

Clearly ∂_ω maps H^k into H^{k-1}, that is, it loses one derivative. It also has an inverse:

$$\begin{aligned} u &= \sum_n u_n \exp\left(\omega_1 n_1 \theta_1 + \omega_2 n_2 \theta_2\right) \\ \partial_\omega u &= f \Longleftrightarrow \forall n \in \mathbb{Z}^d, \quad u_n = f_n \left(\omega_1 n_1 + \omega_2 n_2\right)^{-1} \end{aligned}$$

Because of the Diophantine condition, $|u_n| \leq K^{-1} \omega_2^{-1} \|n\|^{1+\alpha} |f_n|$. So ∂_ω^{-1} sends H^k into $H^{k-1-\alpha}$, that is, it loses many derivatives

The spaces

Typically, for PDEs, both the operator and its inverse lose derivatives, so the inverse does not send you back into the initial space. For this reason one must introduce a scale of Banach spaces $(X_s, \|\cdot\|_s), s_0 \leq s < \infty$

$$s_0 \leq s_1 \leq s_2 \leq S \Longrightarrow [X_{s_2} \subset X_{s_1} \quad \text{and} \quad \|\cdot\|_{s_1} \leq \|\cdot\|_{s_2}]$$

For instance, $X_s = H^s(\Omega)$ (Sobolev space) and $C^\infty = \cap_s H^s$ with $S = \infty$. So, in the preceding example, where Ω is the torus,

$$\begin{aligned} \partial_\omega &: H^s \rightarrow H^{s-1} \\ \partial_\omega^{-1} &: H^s \rightarrow H^{s-1-\alpha} \end{aligned}$$

Polynomials

We shall need an additional structure. Let E_N, $N \geq 0$ be an increasing sequence of finite-dimensional spaces, the union being dense in all the X_s

$$X_s = \overline{\cup_{n \geq 0} E_N}$$

The projection $\Pi_N : X_0 \to E_N$ is assumed to satisfy:
(Polynomial growth): *for all nonnegative numbers* s, d *satisfying* $s + d \leq S$, *and all* $x \in E_N$, *there is a constant* C_S such that:

$$\|x\|_{s+d} \leq C_S N^d \|x\|_s$$
$$\|(1 - \Pi_N)x\|_s \leq C_S N^{-d} \|x\|_{s+d}$$

(Interpolation): for $0 \leq t \leq 1$:

$$\|x\|_{ts_1+(1-t)s_2} \leq C_S \|x\|_{s_1}^{t} \|x\|_{s_2}^{1-t} .$$

Nonlinear operators

Nonlinear operators from C^∞ into itself, such as $S(u, Du)$ arise from maps $S : R^d \times R^{nd} \longrightarrow R^m$ through some basic operations. The behaviour of the Sobolev norms H^s is ruled by the so-called Hölder estimates:

- **(Product)** If u and v are in $L^\infty \cap H^s$ then there is a constant m_s depending only on s such that:

$$\|uv\|_s \leq m_s \left(\|u\|_s \sup |v| + \|v\|_s \sup |u|\right)$$

- **(Composition)** Let $f : R \longrightarrow R$ be C^∞. If $u \in L^\infty \cap H^s$ then $f(u) \in L^\infty \cap H^s$ and there is a constant ℓ_s depending on s, f,and $\sup |u|$ such that:

$$\|f(u)\|_s \leq \ell_s \left(1 + \|u\|_s\right)$$

Towards a statement

Let $\left(Y_s, \|\cdot\|'_s\right)$, $0 \leq s \leq S$, be another scale of Hilbert spaces with the same properties. We consider a map F from the unit ball $B_{s_0+\max\{m.\ell\}}$ into Y_0, where s_0, m, ℓ are given. We normalize by setting $F(0) = 0$ and we want to solve $F(u) = v$. Assume F is continuous and Gâteaux-differentiable.

Let m, m', ℓ, ℓ' be given. We assume that for any S and any $s \leq S$, there are constants a_S and b_S and a linear map $L(x) : Y \to X$ satisfying, for all u with $\|u\|_{s_0+\max\{m,\ell\}} \leq 1$

$$
\begin{aligned}
\left\|F'(u)\,h\right\|'_s &\leq a_S\left(\|u\|_{s+m} + \|u\|_{s_0}\|h\|_{s+m}\right)\\
F'(u)\,L(u)\,k &= k\\
\|L(u)\,k\|_s &\leq b_S\left(\|k\|_{s+\ell'}{}' + \|k\|'_{s_0}\|u\|_{s+\ell}\right)
\end{aligned}
$$

A local surjection theorem

Theorem

Suppose:

$$s > s_0 + \max\{m, \ell\}$$
$$\delta > s + \ell'$$

and S is large enough. Then, there exists some $r > 0$ and some $k > 0$ such that, if $\|v\|_\delta < r$, the equation $F(u) = v$ can be solved with

$$\begin{aligned} \|u\|_{s_0+\max\{m,\ell\}} &\leq 1 \\ \|u\|_s &\leq k\|v\|'_\delta \end{aligned}$$

For any $\varepsilon > 0$, we can solve $F(u) = v$ with:

- $v \in Y^\delta$ for any $\delta = s_0 + \max\{m, \ell\} + \ell' + \varepsilon$
- $u \in X^s$ with $s = \delta - (\ell' + \varepsilon)$

The linearized operator yields $\varepsilon = 0$, so the nonlinear operator does almost as well

Comparison with earlier results

There is a long history of results of this type, following the seminal papers of Arnol'd, Nash and Moser, the sharpest results to date being due to Hörmander. Most of them use smoothing operators instead of projections.

- We are not aware that any of these papers gets the optimal value for δ (lowest regularity of the RHS)
- Every single one of them require an additional tame estimate on the second derivative:

$$\begin{aligned} \left\| \left(F''(x)\, u, v \right) \right\|_s' \leq\ & c_1 \left(\|u\|_{s+p_1} \|v\|_{p_2} + \|u\|_{p_1} \|v\|_{s+p_2} \right) \\ & + c_2 \|x\|_{s+p_3} \left(\|u\|_{s+p_4} \|v\|_{p_5} + \|u\|_{p_4} \|x\|_{s+p_5} \right) \end{aligned}$$

They then use second-order expansion to get quadratic convergence:

$$\|x_{n+1} - x_n\| \leq \frac{1}{2} MK \|x_n - x_{n-1}\|^2 = \frac{1}{2} MK \|x_n - x_{n-1}\| \|x_n - x_{n-1}\|$$

with $\|F''(x)\| \leq K$. This is Newton's method. However the range of convergence is extremely small: $\frac{1}{2} MK \|x_n - x_{n-1}\| \leq 1$, so $\|x_1\| \leq 2(MK)^{-1}$. This is the key to the following

The Inverse Function Theorem in Banach spaces

Theorem (Newton-Kantorovitch)

Assume F is C^2 with $F(0) = 0$, and for $\|x\| \leq R$ we have:

$$\left\|F'(x)^{-1}\right\| \leq M \text{ and } \|F''(x)\| \leq K$$

For every y such that $\|y\| \leq \min\left\{\frac{1}{M^2K}, \frac{R}{M}\right\}$ there is unique solution to $F(x) = y$ in the ball $\|x\| \leq \min\left\{\frac{1}{MK}, R\right\}$

Theorem ((Wazewski 1947, IE 2011))

Assume F is G-differentiable with $F(0) = 0$, and for $\|x\| \leq R$ there is a map $L(x)$ such that

$$F'(x)\,L(x) = I_Y \quad \text{and} \quad \sup_{\|x\| \leq R} \|L(x)\| < M$$

Then for every y with $\|y\| \leq \frac{R}{M}$ there is a solution with $\|x\| \leq R$.

First approach to singular perturbations

Suppose we want to solve $F(\varepsilon, u) = v$ with $F(\varepsilon, 0) = 0$ and for ε and u small enough:

$$\begin{aligned} \left\| D_u F(\varepsilon, u)^{-1} \right\| &\leq \varepsilon^{-1} M \\ \left\| D^2_{uu} F(\varepsilon, u) \right\| &\leq K \end{aligned}$$

Then the two preceding theorems give the extimates:

$$\begin{aligned} \text{(NK)} \quad \|v\| &\leq \frac{\varepsilon^2}{M^2 K}, \quad \|u\| \leq \frac{\varepsilon}{KM} \\ \text{(WE)} \quad \|v\| &\leq \frac{\varepsilon R}{M}, \quad \|u\| \leq R \end{aligned}$$

We gain one order of magnitude, but we lose uniqueness

A singular perturbation result.

We have $F(\varepsilon, u)$ with $F(\varepsilon, 0) = 0$. We assume

$$\begin{aligned} \| DF_\varepsilon(u) h \|_s' &\leq a_S \left(\|u\|_{s+m} + \|u\|_{s_0} \|h\|_{s+m} \right) \\ DF_\varepsilon(u) L(u) k &= k \\ \| L(u) k \|_s &\leq b_S \varepsilon^{-g} \left(\|k\|_{s+\ell'}{}' + \|k\|_{s_0}' \|u\|_{s+\ell} \right) \end{aligned}$$

Theorem

Suppose $s > s_0 + \max\{m, \ell\}$, $\delta > s + \ell'$ $g' > g$, and S is large enough. Then, for some $r > 0$ and $k > 0$, whenever $\|v\|_\delta' < r\varepsilon^{g'}$ there exists for every $\varepsilon > 0$ some u_ε such that:

$$\begin{aligned} F(\varepsilon, u_\varepsilon) &= v \\ \|u_\varepsilon\|_{s_0+\max\{m,\ell\}} &\leq 1 \\ \|u_\varepsilon\|_s &\leq k\varepsilon^{-g'} \|v\|_\delta' \end{aligned}$$

A singular perturbation problem

Rauch and Métivier (2010) and Texier and Zumbrun (2013) have studied the following sysem of nonlinear coupled Schrödinger equations in $v = (u, \bar{u})$ for $j, k \leq K$ and $n \leq N$

$$\begin{aligned}\left(\frac{\partial}{\partial t} + i\lambda_j \Delta\right) v_j &= \sum_{n=1}^{N} \left\{ b_{j_n}^{k}(v) \frac{\partial v_k}{\partial x_n} + c_{jn}^{k}(v) \frac{\partial \bar{v}_k}{\partial x_n} \right\} \\ v_k &\in \mathbb{C},\ \lambda_j \in \mathbb{R},\ j \neq j' \Longrightarrow \lambda_j \neq \lambda_{j'}\end{aligned}$$

The coefficients $b_{j_n}^{k}(v)$ and $c_{j_n}^{k}(v)$ must satisfy a growth condition near the origin:

$$\exists\ p \geq 2 : \forall a = (a_1, \quad, a_N) \in \mathbb{N}^d, \quad \left| \frac{\partial b_{j_n}^{k}}{\partial v^a} \right| + \left| \frac{\partial c_{j_n}^{k}}{\partial v^a} \right| \leq C_{|a|} \left| v \right|^{(p-|a|)_+}$$

and we also need some nonresonance conditions:

$$\begin{aligned}\lambda_j + \lambda_{j'} &= 0 \Longrightarrow c_{jj'} - c_{j'j} = 0 \\ \operatorname{Im} b_{jj} &= 0\end{aligned}$$

The initial condition

This system arises in optics, gives rise to several kinds of solutions, according to the initial condition:

$$v_\varepsilon(0,x) = \varepsilon^\sigma \left(a_\varepsilon(x), \bar{a}_\varepsilon(x)\right)$$

There are two important classes of solutions:

$$\begin{aligned} a_\varepsilon(x) &= a^0\left(\frac{x}{\varepsilon}\right) && \text{(concentration)} \\ a_\varepsilon(x) &= a^0(x)\exp\left(\frac{i}{\varepsilon}x\cdot\xi_0\right) && \text{(oscillation)} \end{aligned}$$

Here g will be an (explicit) function of σ, p and d.Applying our result, we can solve the problem for:

$$\sigma > \frac{1-\sigma_a}{p+1} + \frac{d}{2}\frac{p}{(p+1)(p-1)}$$

This is an improvement on earlier results.

The results

We seek a solution on $[0, \ T]$, where T is independent of ε.
For σ large enough, namely:

$$\sigma > \begin{cases} 1 + \frac{N}{2} & \text{(oscillating case)} \\ 1 & \text{(concentrating case)} \end{cases}$$

this is not a singular perturbation problem. Metivier and Rauch, using the standard inverse function theorem in Banach spaces, prove that solutions exist for all d and p.
The case when σ is smaller than these bounds has been investigated by Texier and Zumbrun using a Nash-Moser theorem. We will compare our results to theirs

Comparison: two space dimensions

- Concentrating case
  - TZ require $p \geq 4$ and $\frac{9}{2(p+1)} < \sigma < 1$.
  - ES gives any $p \geq 3$, and $\frac{2p-1}{(p+1)(p-1)} < \sigma < 1$
- Oscillating case
  - TZ require $p \geq 3$ and $\frac{5}{p+1} < \sigma < 2$
  - ES gives any $p \geq 2$ and $\frac{3p-2}{(p+1)(p-1)} < \sigma < 2$

Comparison: three space dimensions

- Concentrating case
  - TZ requires $p \geq 4$ and $\frac{4}{p+1} < \sigma < 1$.
  - ES gives any $p \geq 3$, and $\frac{4p-1}{2(p+1)(p-1)} < \sigma < 1$
- Oscillating case
  - TZ gives any $p \geq 2$ and $\frac{11}{2(p+1)} < \sigma < \frac{5}{2}$
  - ES gives any $p \geq 2$ and $\frac{7p-4}{2(p+1)(p-1)} < \sigma < \frac{5}{2}$

Proof of the Theorem

We introduce positive numbers $\alpha > 1$, $\theta < 1, \sigma < S$ and β with $\alpha\beta < \sigma$. We choose an integer $N_0 \geq 2$ and we define two infinite sequences of integres N_n and M_n by $N_n = \left\lfloor N_0^{\alpha^n} \right\rfloor$ and $M_n = \left\lfloor N_n^{\theta} \right\rfloor$, so that $N_n \to \infty$ very fast and $N_{n-1} \subset M_n \subset N_n$ with $\dim(M_n) << \dim(N_n)$. We then construct by induction a sequence $u_n \in E_{N_n}$ such that:

$$\Pi'_{M_n} F(u_n) = \Pi'_{M_{n-1}} v \text{ for } n \geq 1$$

$$\forall n \geq 1, \quad \|u_n - u_{n+1}\|_{s_0} \leq C N_n^{\alpha\beta - \sigma + s_0} \|v\|'_d$$

$$\forall n \geq 1, \quad \|u_n - u_{n+1}\|_{\sigma} \leq C N_n^{\alpha\beta} \|v\|'_d$$

Since $\beta - \sigma/\alpha < 0$ the sequence x_n is Cauchy in H_{s_0}. So it converges to some $u \in H_{s_0}$ and by continuity $F(u) = v$. Using the interpolation inequality, we see that u_n will also converge in all norms $s < \sigma - \alpha\beta$

Initialization

Introduce a norm $\mathcal{N}_1$ on E_{N_1} and E'_{N_1} :

$$\mathcal{N}_1(z) = \|z\|_\delta + N_1^{-\frac{\theta}{\alpha}(\sigma-\delta)} \|z\|_\sigma$$

Set $\mathcal{B}_1 := \{z \in E_{N_1} \mid \mathcal{N}_1(z) \leq 1\}$ and define a function $f_1 : \mathcal{B}_1 \to E'_{M_1}$ by:

$$f_1(u) = \Pi'_{M_1} F(u)$$

Lemma

If $(1-\theta)(\sigma-\delta) > \theta m + \max\{\ell, \theta\ell'\} + g/\eta$, then $Df_1(u)$ has a right inverse $L_1(u)$ and we have:

$$\|L(u)\|_{\mathcal{N}_1} \leq C\varepsilon^{-g}\left(M_1^{\ell'} + N_1^{\ell}\right)$$

We then apply the local surjection theorem to solve $f_1(u_1) = \Pi_{M_0} v$, provided:

$$\|v\|'_\delta \leq C\varepsilon^{g}\left(M_1^{\ell'} + N_1^{\ell}\right)^{-1}$$

The induction

Introduce a norm $\mathcal{N}_n$ on E_{N_n} :

$$\mathcal{N}_n(z) = \|z\|_{s_0} + N_{n-1}^{-\sigma+s} \|z\|_\sigma$$

Set $\mathcal{B}_n := \left\{ z \in E_{N_n} \mid \mathcal{N}_n(z) \leq \varepsilon^{-g} N_{n-1}^{\alpha\beta-\sigma+s_0} \|v\|'_\delta \right\}$ and define a function $f_n : \mathcal{B}_n \to E_{N_n}$ by:

$$f_n(z) = \Pi'_{M_n} \left(F(u_{n-1} + z) - F(u_{n-1}) \right)$$

Taking into account the induction assumption $\Pi'_{N_{n-1}} F(u_{n-1}) = \Pi'_{N_{n-2}} v$, this can be rewritten as:

$$\begin{aligned} f_n(z) &= e_n + h_n \\ h_n &= \left(\Pi'_{M_{n-1}} - \Pi'_{M_{n-2}} \right) v \in E'_{M_{n-1}} \\ e_n &= \left(\Pi'_{M_{n-1}} - \Pi'_{M_n} \right) F(u_{n-1}) \in E'_{M_n} \end{aligned}$$

The induction

Lemma

Assume:

$$\begin{aligned} (1+\alpha-\theta\alpha)(\sigma-s_0) &> \alpha\beta+\alpha(m+\ell)+\ell'+\frac{g}{\alpha\eta} \\ (1-\theta)(\sigma-s_0) &> m+\theta\ell'+\frac{g}{\alpha\eta} \end{aligned}$$

Then $Df_n(z)$ *has a right-inverse* $L_n(z)$ *on the ball* $\mathcal{N}_n(z) \leq \varepsilon^{-g} N_{n-1}^{\alpha\beta-\sigma+s_0} \|v\|'_\delta$ *and :*

$$\|L_n(z)\|_{\mathcal{N}_n} \leq \varepsilon^{-g}\left(\frac{N_n^{\beta+\ell} N_{n-i}^{-\sigma+s_0+\ell'}}{M_1^{\ell'}+N_1^{\ell}}+M_n^{\ell'}\right)$$

The induction

We have to solve $f_n(z) = e_n + h_n$ with $\mathcal{N}_n(z) \leq \varepsilon^{-g} N_{n-1}^{\alpha\beta-\sigma+s_0} \|v\|'_\delta$. By the local surjection theorem, this is possible provided

$$\|L_n(z)\|_{\mathcal{N}_n} \left(\|e_n\|_{\mathcal{N}_n} + \|h_n\|_{\mathcal{N}_n}\right) \leq \varepsilon^{-g} N_{n-1}^{\alpha\beta-\sigma+s_0} \|v\|'_\delta .$$

Lemma

Assume:

$$\delta > s_0 + \frac{\alpha}{\vartheta}\left(\sigma - s_0 - \alpha\beta + \ell"\right)$$

$$(\alpha - 1)\beta > (1-\vartheta)(\sigma - s_0) + \vartheta m + \ell" + \frac{g}{\eta}$$

$$\ell" = \max\left\{(\alpha-1)\ell + \ell', \vartheta\ell'\right\}$$

Then the local surjection theorem obtains, and u_n can be found

Showing that the values of the parameters are compatible

α, β, ϑ and σ must satisfy, with $\ell" = \max\left\{(\alpha-1)\,\ell + \ell', \vartheta\ell'\right\}$

$$\frac{1}{\alpha} < \vartheta < 1$$

$$(1-\vartheta)(\sigma-\delta) > \vartheta m + \max\left\{\ell, \vartheta\ell'\right\} + \frac{g}{\eta}$$

$$\beta > \max\left\{\ell, \vartheta\ell'\right\} + \frac{\vartheta}{\alpha}(\sigma-\delta)$$

$$\sigma > \alpha\beta + s$$

$$(1+\alpha-\vartheta\alpha)(\sigma-s_0) > \alpha\beta + \alpha(m+\ell) + \ell' + \frac{g}{\eta}$$

$$(1-\vartheta)(\sigma-s_0) > m + \vartheta\ell' + \frac{g}{\alpha\eta}$$

$$\delta > s_0 + \frac{\alpha}{\vartheta}\left(\sigma - s_0 - \alpha\beta + \ell"\right)$$

$$(\alpha-1)\,\beta > (1-\vartheta)(\sigma-s_0) + \vartheta m + \ell" + \frac{g}{\eta}$$

Continuous sections

Theorem (Eric Séré)

Let X and Y be Banach spaces, $B \subset X$ the unit ball, and $F : B \to Y$ with $F(0) = 0$. Suppose F is continuous and Hadamard-differentiable on X. Assume there are a map $L : B \to \mathcal{L}(X, Y)$ and a constant $a < 1$ such that, for every $(x, v) \in B \times Y$ there exists some $\varepsilon > 0$ with:

$$\left\| DF\left(x'\right) \circ L(x)\, v - v \right\| \leq a \|v\|$$

Assume moreover that $\sup \{\|L(x)\| \mid x \in B\} < M$. Denote by $B' \subset Y$ the ball of radius $(1 - a)\, rM^{-1}$. Then there exists a continuous map $G : B' \to B$ such that $F \circ G = I_Y$

So, there is no uniqueness in solving $F(\varepsilon, x) = y$, but we can ask for continuous dependence on the right-hand side. Our plan is to extend this to the singular perturbation problem with loss of derivative and show continuous dependence on v and ε.

THE INVERSE FUNCTION THEOREM
IN BANACH SPACES

A variational principle

Theorem (Ekeland, 1972)

Let (X, d) be a complete metric space, and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous map, bounded from below:

$$\begin{aligned} &\{(x, a) \mid a \geq f(x)\} \text{ is closed in } X \times \mathbb{R} \\ &\inf f > -\infty \end{aligned}$$

Suppose $f(x_0) < \infty$. Then for every $R > 0$, there exists some $\bar{x}$ such that:

$$\begin{aligned} f(\bar{x}) &\leq f(x_0) \\ d(\bar{x}, x_0) &\leq R \\ f(x) &\geq f(\bar{x}) - \frac{f(x_0) - \inf f}{R} d(x, \bar{x}) \quad \forall x \end{aligned}$$

The proof relies on the Baire theorem: if $F_n, n \in N$, is a sequence of closed bounded subsets of a complete metric space, with $F_{n+1} \subset F_n$ and diam $F_n \to 0$, then their intersection is a singleton

First-order version

Definition

We shall say that f is *Gâteaux-differentiable* at x if there exists a continous linear map $Df(x) : X \to X^*$ such that

$$\forall \xi \in X, \quad \lim_t \frac{1}{t}\left[f(x+t\xi) - f(x)\right] = \langle Df(x), \ \xi \rangle$$

In other words, the restriction of f to every line is differentiable (for instance, partial derivatives exist). Note that a G-differentiable function need not be continuous.

Theorem

Let (X,d) be a complete metric space, and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous map, bounded from below. Then for every $x_0 \in X$ and $R > 0$, there exists some $\bar{x}$ such that:

$$f(\bar{x}) \leq f(x_0),$$
$$\|\bar{x} - x_0\| \leq R$$

Proof

For simplicity, take $x_0 = 0$ and $\inf f = 0$. Apply EVP to $x = \bar{x} + tu$ and let $u \longrightarrow 0$. We get:

$$f(\bar{x} + tu) \geq f(\bar{x}) - \frac{f(0)}{R} t \|u\| \quad \forall (t, u)$$

$$\lim_{t \to +0} \frac{1}{t} \left(f(\bar{x} + tu) - f(\bar{x}) \right) \geq -\frac{f(0)}{R} \|u\| \quad \forall u$$

$$\langle Df(x), u \rangle \geq -\frac{f(0)}{R} \|u\| \quad \forall u, \text{ or } \|Df(x)\|^* \leq \frac{f(0)}{R}$$

A non-smooth inverse function theorem

Theorem

Let X and Y be Banach spaces. Let $F : X \to Y$ be continuous and Gâteaux-differentiable, with $F(0) = 0$. Assume that the derivative $DF(x)$ has a right-inverse $L(x)$, uniformly bounded in a neighbourhood of 0:

$$\forall v \in Y, \quad DF(x)\,L(x)\,v = v$$
$$\sup\{\|L(x)\| \mid \|x\| \le R\} < M$$

Then, for every $\bar{y}$ such that

$$\|\bar{y}\| \le \frac{R}{M}$$

there is some $\bar{x}$ such that:

$$\|\bar{x}\| \le M\,\|\bar{y}\|$$
$$F(\bar{x}) = \bar{y}$$

Proof

Take any $\bar{y}$ with $\|\bar{y}\| \leq \frac{R}{M}$. Consider the function $f : X \rightarrow R$ defined by:

$$f(x) = \|F(x) - \bar{y}\|$$

It is continuous and bounded from below, so that we can apply EVP. We can find $\bar{x}$ with:

$$f(\bar{x}) \leq f(0) = \|\bar{y}\| \leq \frac{R}{M}$$

$$\|\bar{x}\| \leq M\|\bar{y}\| \leq R$$

$$\forall x, \quad f(x) \geq f(\bar{x}) - \frac{f(0)}{R}\|x - \bar{x}\| = f(\bar{x}) - \frac{1}{M}\|x - \bar{x}\|$$

I claim $F(\bar{x}) = \bar{y}$.

Proof (ct'd)

Assume $F(\bar{x}) \neq \bar{y}$. The last equation can be rewritten:

$$\forall t \geq 0,\ \forall u \in X, \qquad \frac{f(\bar{x}+tu) - f(\bar{x})}{t} \geq -\frac{1}{M}\|u\|$$

Simplify matters by assuming X is Hilbert. Then:

$$\left(\frac{F(\bar{x}) - \bar{y}}{\|F(\bar{x}) - \bar{y}\|}, DF(\bar{x})\,u\right) = \langle Df(\bar{x}), u\rangle \geq -\frac{1}{M}\|u\|$$

We now take $u = -L(\bar{x})(F(\bar{x}) - \bar{y})$, so that $DF(\bar{x})\,u = -(F(\bar{x}) - \bar{y})$. We get a contradiction:

$$\|F(\bar{x}) - \bar{y}\| \leq \frac{\|L(\bar{x})\|}{M}\|F(\bar{x}) - \bar{y}\| < \|F(\bar{x}) - \bar{y}\|$$