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Discounting and divergence of opinion ☆

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Abstract

Agents impatience rate and their anticipations about the future of the economy, are two essential determinants of the equilibrium discount rate, as illustrated by the Ramsey formula. Heterogeneity in time preference rates and in anticipations is widely acknowledged. Our objective is to determine the equilibrium discount rate when this heterogeneity is taken into account. Among others we tackle the following questions: As an additional risk or uncertainty, can dispersion in agents' characteristics lead to lower discount rates? What is the asymptotic behavior of the discount rate in such a setting? More generally, what is the shape of the yield curve?

JEL classification: D61; E45; G12

Keywords: Discounting; Yield curve; Divergence of opinion; Gamma discounting; Costs and benefits analysis; Ramsey formula; Discount rate

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1. Introduction

The concept of a discount rate is central to economic analysis, as it allows effects occurring at different future times to be compared by converting each future dollar amount into equivalent present dollars. The problem of the determination of a discount rate has acquired renewed relevance when analysing environmental projects or activities with effects spread out over hundreds of years. The evaluation by Costs and Benefits Analysis (CBA) is then very sensitive to the discount rate. For example, concerning climate change, it has been argued that the strong conclusions of the Stern Review were essentially driven by the low assumed discount rate (see, e.g. [32] or [42]). More generally, to determine the appropriate discount rate for public sector CBA is an important issue.

In the short term, the observed risk free rate provides a useful tool to analyse the tradeoff between consumption today and consumption tomorrow. The analysis is less easy when costs and benefits of the set of current potential actions are expected to last to the medium or the long run. For example, greenhouse gas emitted today yields very long term costs like global warming. One can either pay to reduce CO_2 emissions now or pay for dikes to keep the rising ocean from flooding coastal cities. Financial markets do not provide a guideline for investing in technologies that prevent this kind of long-lasting risk. Liquid financial instruments with long durations do not exist. US treasury bonds have time horizons that do not exceed 30 years. We must thus rely on the use of an economic model to make a prediction of the real interest rates over very long horizons and of future real interest rates.

The Ramsey formula gives us the expression for the socially efficient discount rate to use for CBA. Letting R denote the consumption discount rate, the Ramsey formula gives the relation $R = \rho + (1/\eta)g$, where ρ is the pure time preference rate, g is the growth rate of the economy and $(1/\eta)$ is the elasticity of marginal utility, or equivalently the degree of relative risk aversion. This means that there are essentially two determinants of the level of the discount rate. The first determinant is related to pure time preference, leading people to value one unit of consumption more today than tomorrow. The second determinant is related to a wealth effect.[1] The higher the level of consumption tomorrow and the higher the elasticity of marginal utility, the lower the value of one unit of consumption tomorrow and the higher the discount rate. From a practical point of view, one only needs to specify values for the triple $(\rho, 1/\eta, g)$ to derive the discount rate. For instance, the UK government[2] recommends a discount rate of 3.5% for CBA (for use across all departments and all projects) based upon the following figures: $\rho = 1\%$, $\eta = 1$ and $g = 2.5\%$. The Stern Review proposes a discount rate of 1.4% using $\rho = 0.1\%$, $\eta = 1$ and $g = 1.3\%$.

An important drawback of the Ramsey formula, illustrated by the previous examples, is lack of consensus on the value of the time preference rate and on the value of the growth rate of consumption. As [40] states "there does not now exist, nor has ever existed, anything remotely resembling a consensus even (...) among the experts on this subject." Different possible values for the parameters lead to very different values for the discount rate, which in turn lead to very different conclusions, as the results are very sensitive to the discount rate being used. In order to evaluate a cost or a benefit of £1,000,000 in 100 years, the rate of the Green Book leads to a

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[1] When the growth rate is not deterministic, there is a third determinant of the discount rate that takes into account the degree of uncertainty σ^2 and that leads to an extended form of the Ramsey formula, namely $R = \rho + \frac{1}{\eta}g - \frac{1}{2\eta}(1 + \frac{1}{\eta})\sigma^2$. The last term corresponds to the so-called precautionary saving effect.

[2] In the 'Green Book: Appraisal and Analysis in Central Government'.

present value of £32,000 while the rate of the Stern Review leads to a present value of £250,000. What is the appropriate discount rate to adopt when one group of experts, like those of the Green Book, proposes a given rate and another group of experts, like those of the Stern Review, proposes another rate?

Moreover, this lack of consensus among experts about the *right* values for the growth rate and the time preference rate reveals another more fundamental problem, namely the fact that agents differ in their time preference rates as well as in their anticipations about the future of the economy. The Ramsey formula has been derived under the assumption of homogeneous agents (same time preference rate ρ and same anticipated growth rate of the economy g). This raises the following question. To what extent does the Ramsey formula remain valid once heterogeneity in time preference rates and heterogeneity in anticipations about the future are taken into account. More precisely, what is the analogue of the Ramsey formula in such a context? In the setting of our example, what is the value of the socially efficient discount rate if half of the agents are endowed with the characteristics ($\rho_1 = 1\%$, $\eta_1 = 1$, $g_1 = 2.5\%$) assumed by the experts of the Green Book and half of the agents are endowed with the characteristics ($\rho_2 = 0.1\%$, $\eta_2 = 1$, $g_2 = 1.3\%$) assumed by the experts of the Stern Review?

Note that there are several reasons why individuals may be heterogeneous in their preference for the present as well as in their anticipations. Indeed, agents (or experts) currently do not have a complete understanding of the determinants of long term economic evolution. Forecasting for the coming year is already a difficult task and it is then natural that forecasts for the next century or millennium are subject to potentially enormous divergence. The debate on the notion of sustainable growth is an illustration of the degree of possible divergence of opinion about the future of society. Some will argue that the effects of improvement in information technology have yet to be realized and the world faces a period of more rapid growth. In another view, those who emphasize the effects of natural resource scarcity will expect lower growth rates in the future. Some even suggest a negative growth in per capita GDP in the future, due to the deterioration of the environment, population growth and decreasing returns to scale. As far as the rates of pure time preference are concerned, they may reflect different levels of impatience. In a setting with long-lived agents that represent present and future generations, these rates may also reflect divergence of opinion about the importance granted to the welfare of future generations relative to the present. The debate among economists (and also among philosophers) on the notion of intergenerational equity is an illustration of this possible divergence. Some will argue that intergenerational choices should be treated as intertemporal individual choices leading to a high weight on present welfare. Others will argue that fundamental ethics require intergenerational neutrality and that the only ethical basis for placing less value on the welfare of future generations is the uncertainty about whether or not the world will exist and whether or not these generations will be present. To sum up, as underlined by [40], “these and many more are fundamentally matters of judgment or opinion, on which fully informed and fully rational individuals might be expected to differ.” Heterogeneity in time preference rates and in anticipations about the future of the economy are critical features that must be taken into account in order to determine the socially efficient discount rate.

The aim of this paper is to determine and understand the expression of the socially efficient discount rate in a model which, unlike the Ramsey Equation, takes into account heterogeneity in agents’ anticipations and time preference rates.

The first issue we tackle is to what extent the heterogeneous setting is fundamentally different from the homogeneous one. If we consider N agents with individual time preference and belief parameters (ρ_i, g_i), we want to analyse if it is possible to aggregate these individual parameters into consensus ones, such that the Ramsey formula remains valid when replacing the homogeneous parameters of the standard setting by the consensus parameters. If this is the case, what are the expressions of the consensus parameters? In our example, is it possible to aggregate g_1 and g_2 (ρ_1 and ρ_2) into a consensus parameter $\bar{g}$ ($\bar{\rho}$) such that the discount rate R is given by $R = \bar{\rho} + (1/\eta)\bar{g}$? Substantively, does heterogeneity in beliefs, as a source of additional risk or uncertainty, lead to lower discount rates than in the homogeneous setting? How do discount rates vary with dispersion of beliefs? A closely related issue is the problem of aggregating discount rates. If we let R^i denote the discount rate that would prevail in the homogeneous economy populated by agent i only, how does the discount rate R relate to the R^is? In the example above, if one group proposes a discount rate R^1 and another group proposes a discount rate R^2, how can we aggregate them into a socially efficient discount rate? This is the problem of a social planner, who has consulted a group of experts about the discount rate to apply for CBA, and wants to aggregate the proposed discount rates into a socially efficient discount rate. It is underlined in [40] that one should not average the discount rates (R^1 and R^2 in our example, which would lead to a flat discount rate of 2.49%) but the present values ($A^1 = 32{,}000$ and $A^2 = 250{,}000$ in our example) and derive from there the socially efficient discount rate (which leads to a discount rate of 1.98% for a 100 years horizon, in our example). It is alternatively suggested in [20] that an equilibrium analysis, as for models of the term structure of interest rates, might be the cost to be paid to make policy recommendations that have economic sense.

We adopt such an equilibrium approach. First we find that the setting with heterogeneity is fundamentally different from the homogeneous setting: except in very specific settings, it is not possible to aggregate individual parameters into consensus ones such that the Ramsey formula remains valid with these consensus parameters. These specific settings are first logarithmic utility functions, and second deterministic heterogeneity in pure time preference rates and no dispersion of beliefs. In a general setting, it is possible to aggregate individual parameters into consensus ones only with the introduction of an aggregation bias, induced by heterogeneity in time preference rates and beliefs. The expression of this bias is directly related to dispersion in beliefs and in time preference rates. This bias can force rates higher or lower depending on the relative position of η with respect to 1. This result is consistent with the interpretation of belief and time preference heterogeneity as an additional source of risk or uncertainty in the future, leading agents to value more or less future consumption (with respect to present consumption) depending on their relative level of prudence and risk aversion. We show on examples that the impact of this aggregation bias on the socially efficient discount rate can be quite significant. Due to this bias, the discount factor[3] A (discount rate R) does not always lie in the range bounded by $\inf A^i$ and $\sup A^i$ ($\inf R^i$ and $\sup R^i$) where A^i (R^i) is the discount factor (discount rate) that would prevail in the economy populated by agent i.

Moreover, the right concept of average to consider in the aggregation of the individual discount rates R^i is the concept of 'η average'. It corresponds to the arithmetic average only in the case of logarithmic utility functions. Finally, the average is a weighted average, the weights being related to the shadow price of the agents' intertemporal budget constraint. Applying these results to the problem of aggregating experts' proposed discount rates, we find that the certainty equivalent approach of [39,40] which uses the (unweighted) arithmetic average of the proposed discount factors (or present values) is compatible with an equilibrium approach only in a very

[3] The discount factor A^t ($A^{t,i}$) is defined as the price of a zero coupon bond maturing at time t in the heterogeneous economy (in the economy populated by agent i).

specific setting (logarithmic utility functions, no heterogeneity in time preference rates). We show on examples that there can be an important difference between our socially efficient discount rate and the certainty equivalent discount rate.

Note that the problem of aggregating heterogeneous *utility* discount rates (or time preference rates) has already been studied by e.g. [23,26]. In their setting, the consensus *utility* discount rate is an η average of the individual ones. By contrast to dollar costs and benefits, when the future costs are utility costs, simple discounting at the utility discount rate applies. In this paper, we are interested in the *consumption* discount rate which is the natural concept to consider for real dollar costs and benefits analysis and which, as highlighted by the Ramsey formula, takes into account not only the *utility* discount rate but also anticipations about the future of the economy.

The second issue we tackle is to determine whether it is socially efficient to reduce the discount rate per year for more distant horizons when there is belief heterogeneity. In other words, is the discount rate in our setting with belief heterogeneity decreasing for long horizons? This question is of particular interest, since there is a wide agreement that discounting at a constant positive rate for long time horizons (as suggested by the standard Ramsey Equation) is problematic, irrespective of the particular discount rate employed; with a constant rate, the costs and benefits accruing in the distant future appear relatively unimportant in present value terms. Hence decisions made today on this basis may expose us to catastrophic consequences in a distant future. This is succinctly summarised in [39]: "To think about the distant future in terms of standard discounting is to have an uneasy intuitive feeling that something is wrong somewhere." A recently proposed solution to this problem is to use a discount rate which declines over time. It is clear that using a declining discount rate could make an important contribution towards the goal of sustainable development. But what formal justifications exist for using a declining discount rate? Justifications of different natures have been provided recently in the literature. In a deterministic world, decreasing discount rates can arise as a result of known changes in the growth rate. Additional motivation emerges once uncertainty is considered. For example, [39,40] considers uncertainty about the discount rate itself and obtains decreasing discount rates. More generally, [18] shows that from today's perspective the only relevant limiting scenario is the one with the lowest interest rate.[4] Decreasing discount rates are also obtained with Bayesian learning in [41], while [21] shows that serial correlation in the growth rates leads to downward sloping yield curves when the representative agent is prudent. Decreasing discount rates also emerge from the specification of a sustainable welfare function à la [9] and [27]. Lastly, considerable empirical and experimental evidence show that individuals are frequently hyperbolic discounters, see e.g. [28,29].

In this paper, we examine if heterogeneity (in beliefs and in time preference rates) can be a justification for the use of declining discount rates in a general equilibrium framework. In the setting with no dispersion of beliefs, deterministic time preference rates, and no uncertainty, the socially efficient discount factor is an average of the individual discount factors. This easily leads, due to the power of compound discounting, to decreasing discount rates converging to the lowest discount rate, which is the discount rate of the most patient agent (see e.g., [23,26], among others). In our setting, the socially efficient discount factor is not an average of the individual discount factors and the presence of the aggregation bias makes the result a priori unclear.

[4] This result is in the spirit of [16].

We obtain that the bias due to dispersion of beliefs vanishes in the long run and the asymptotic discount rate is essentially given by the lowest asymptotic individual discount rate among all agents. This means that the asymptotic discount rate is determined by the agent who values future consumption the most. Note that this lowest asymptotic individual discount rate does not necessarily correspond to the rate of the most patient agent as in the homogeneous beliefs setting. For example, in the case of homogeneous time preference rates, the asymptotic discount rate is given by the discount rate of the agent who is the most pessimistic about the future of the economy. Surprisingly, this remains true even if the most pessimistic agent is the most irrational agent, hence, as shown by [44], does not survive in the long run.[5] Survival and price impact are different concepts. In general, both the distributions of time preference rates and of pessimism are necessary to determine the asymptotic discount rate. In the setting of a CBA, this leads to discounting long term costs and benefits at the lowest individual discount rate inducing a bias towards the optimal policy of the agent/expert who values the most future consumption in the long term. This provides us with a guideline for long term CBA. Indeed, while the observed risk free rate provides a useful tool for CBA in the short term, financial markets are not very helpful when costs and benefits of the set of current potential actions are expected to last to the medium or the long run.

Finally, we examine the impact of belief and time preference heterogeneity on the expression of the discount rate as well as on the relationship between the discount rate and the time horizon (the possible shapes of the yield curve). Increased dispersion of beliefs leads to a decrease of discount rates when η is larger than 1. We show that aggregate pessimism as well as aggregate patience reduce the rates. Since these aggregate levels are given by stochastic, time-varying (risk-tolerance) weighted averages of the individual levels of pessimism and patience, possible correlation effects are induced. We show that heterogeneity in time preference rates and in beliefs leads to a richer class of possible shapes for the yield curve. We can obtain yield curves that are decreasing in the long run, but increasing in the short and medium term. In particular, our model can fit observed behavior of the yield curve in financial markets.

The paper is organized as follows. Section 2 presents the theoretical framework. Section 3 deals with the determination of the socially efficient discount factor and discount rate. Section 4 is devoted to the behavior of the discount rate for long horizons. In Section 5, we analyse in more detail the shape of the yield curve in specific settings. The conclusion summarizes the main contributions of the paper. Proofs of aggregation results and other extensions of [24] are in Appendix A. Appendix B consists of the proofs of all other results.

2. The theoretical framework

We want to determine the expression of the equilibrium discount rate in a pure exchange economy with agents who differ in their anticipations about the future of the economy and in their time preference rate. Our theoretical framework builds upon [24], generalizing it to take into account heterogeneous time preference rates and infinite horizon. Note that while the aim of [24] was to provide an aggregation procedure, the aim of this paper is to analyse the properties of the discount rates.

We consider a continuous-time Arrow–Debreu economy with an infinite horizon. A filtered probability space $(\Omega, F, (F_t), P)$ is given, which represents future uncertainty. Each agent in-

[5] In such a case, the asymptotic discount rate is determined by the most irrational (most pessimistic) agent while the asymptotic risk free rate is determined by the most rational (surviving) agent.

dexed by $i = 1, \dots, N$, has an adapted endowment process denoted by $(e_t^{*^i})$. We assume that the exogeneously given aggregate endowment/consumption process $e^* \equiv \sum_{i=1}^N e^{*^i}$ is an Itô process satisfying the following stochastic differential equation

$$de_t^* = \mu_t e_t^* \, dt + \sigma_t e_t^* \, dW_t, \quad e_0^* = 1,$$

where W denotes a standard unidimensional $((F_t), P)$-Brownian motion and where the parameters μ_t and σ_t respectively stand for $\mu(t, \omega)$ and $\sigma(t, \omega)$, which means that they might depend upon time and states of the world.[6] We assume that markets are complete in the sense that all Arrow–Debreu securities can be traded. As shown by [14], this assumption is equivalent to the existence of few long lived securities that span the different sources of risk. When F is the Brownian filtration, the existence of a short term rate and of a stock whose dividends are proportional to the total endowment are sufficient to ensure market completeness. Even if such an asset does not exist, bonds with different maturities constitute a sufficient set of securities to complete the market, as underlined[7] by [43].

In our model, agents differ in their anticipations as well as in their time preference rate. More precisely, agents try to maximize a von Neumann–Morgenstern utility function for future consumption (c_t) of the form $E^{Q^i}[\int_0^\infty \exp^{-\int_0^t \rho^i(s,\omega)\,ds} u(c_t(\omega))\,dt]$, where Q^i is a probability measure on (Ω, F), corresponding to the subjective anticipations of agent i, and (ρ_t^i) is an (F_t)-adapted process corresponding to the time preference rate of agent i. The utility function u can represent the individual's utility function but may also incorporate the preferences of his descendants. For tractability and in order to focus on the impact of heterogeneity in beliefs and in time preference rates, we restrict our analysis to homogeneous utility functions of the power type, i.e. we suppose that agents share the same CRRA utility function[8] for consumption, of the form $u'(x) = x^{-1/\eta}$.

As far as belief heterogeneity is concerned, the unique assumption we essentially make is the equivalence of the probability measures Q^i. In other words, we assume that the agents have the same set of possible events (i.e. events with a positive subjective probability[9]). More precisely, letting (M^i) denote the positive density process of Q^i with respect to P we assume that

$$dM_t^i = \delta_t^i M_t^i \, dW_t, \quad M_0^i = 1,$$

where the process (δ_t^i) represents the instantaneous subjective belief of agent i. For agent i, aggregate endowment is an Itô process[10] with diffusion parameter $\sigma(t, \omega)$ and with subjective drift parameter $\mu^i(t, \omega) \equiv \mu(t, \omega) + \sigma(t, \omega)\delta^i(t, \omega)$, which means that in our model, agents differ in their expected instantaneous growth rate of aggregate endowment. They have different anticipations about the future of the economy. For instance, pessimistic (optimistic) agents anticipating lower (higher) growth rates in the future will be represented by nonpositive (nonneg-

6 There is no Markovian assumption; the coefficients μ and σ may depend on the entire past history of the economy. We only assume that for all T, $\int_0^T |\mu_t|\,dt < \infty$ and $\int_0^T |\sigma_t^2|\,dt < \infty$ almost surely.

7 Since the aim of the paper is to determine the discount rates that would prevail at the equilibrium for all possible horizons or equivalently the prices of bonds with all possible maturities, it is natural to assume their existence.

8 Our approach can be extended to the case with HARA utility functions of the form $u'(x) = (\theta + x)^{-1/\eta}$.

9 Note that this is a natural assumption for the existence of an equilibrium. Otherwise some agents will consider as possible some events that are considered as impossible by others and optimal demand in the associated Arrow–Debreu assets will be (positively or negatively) infinite.

10 More precisely, letting $W_t^i \equiv W_t - \int_0^t \delta^i\,ds$, we obtain through Girsanov Theorem that W^i is a Brownian motion under Q^i and $de_t^* = (\mu_t + \sigma_t \delta^i) e_t^* \, dt + \sigma_t e_t^* \, dW_t^i$.

ative) parameters (δ_t^i). Note that even if the agents agree on the volatility[11] $\sigma(t,\omega)$, the fact that $\mu^i(t,.)$ is stochastic permits disagreement among agents about the level of risk. For instance, agents for which the stochastic process μ^i exhibits positive (negative) serial dependence will weigh more (less) extreme events (see [21]). Even in the extreme case where there is no time dependence, where σ is constant and where μ^i is a discrete random variable (independent of t and of W), the variance of the logarithmic return for agent i between date 0 and date T is equal to $\sigma^2 T + \mathrm{Var}(\mu^i)T^2$ allowing for divergence of opinion among agents about the level of risk.

Our model of belief heterogeneity permits to take into account disagreement about the future growth rates, or about their distribution and dynamics (since we allow for stochastic growth rates). For example, agents may have heterogeneous growth rates forecasts, they may have different subjective probabilities of occurrence of a boom or of a crash, they may disagree about the possible impact of economic activity on the climate or about the estimation of the possible economic damages induced by climate change. In our model, agents may also disagree about the pure time preference rate. This may reflect different levels of preference for the present. With long-lived agents, who represent present and future generations, this may reflect different conceptions of intergenerational equity or disagreement about the probability that the world will exist at a given future date.[12] Note that the processes ρ^i and δ^i are very general time varying and (F_t)-adapted stochastic processes. In particular, they may be updated continuously according to the available information. The individual beliefs δ^i we consider might then result from Bayesian updating as in e.g. [10] and [45] or from adaptative learning as in [6].

Letting $D_t^i \equiv \exp^{-\int_0^t \rho^i(s,\omega)\,ds}$ denote the individual pure time preference discount factor at time t, the utility function of agent i, for future consumption can be written in the form $E[\int_0^\infty M_t^i(\omega)D_t^i(\omega)u(c_t(\omega))\,dt]$. In our general framework, an Arrow–Debreu equilibrium is defined, as usual, by a positive price process q^* and a family of optimal consumption plans $(y^{*^i})_{i=1,\dots,N}$ such that markets clear, i.e.

$$\begin{cases} y^{*^i} = y^i(q^*, M^i, D^i, e^{*^i}), \\ \sum_{i=1}^N y^{*^i} = e^* \end{cases}$$

where $y^i(q,M,D,e) = \arg\max_{E[\int_0^\infty q_t(y_t^i - e_t)\,dt]\leqslant 0} E[\int_0^\infty M_t D_t u(c_t)\,dt]$. Note that since our utility functions satisfy Inada conditions, all equilibria are interior, hence for a given Arrow–Debreu equilibrium $(q^*, (y^{*^i})_{i=1,\dots,N})$, there exist positive Lagrange multipliers $(\lambda_i)_{i=1,\dots,N}$ such that for all i, the equality $M_t^i D_t^i u'(y_t^{*^i}) = \lambda_i q_t^*$ holds for all t. We let $\gamma_i \equiv \frac{(1/\lambda_i)^\eta}{\sum_{j=1}^N (1/\lambda_j)^\eta}$.

3. Discount factors and discount rates

We start from an Arrow–Debreu equilibrium $(q^*, (y^{*^i})_{i=1,\dots,N})$. We want to better understand how agents individually and collectively arbitrate between present and future costs and benefits and, more generally, between costs and benefits occurring at different dates. This arbitrage is reflected by the discount rate, whose role is to characterize the tradeoff between certain future

[11] This is due to the continuous time (Itô process) setting. This feature of the model is consistent with the common argument that mean returns are a lot harder to estimate than volatilities [30].

[12] See the Stern report for the link between this probability and the pure time preference rate.

consumption and today's consumption. Our goal is to determine the discount rate to apply to all projects, independent of their risk, and to analyse its properties.

We define the socially efficient discount factor A^t between date 0 and date t as the price at date 0 of a zero-coupon bond maturing at time t; it is given by $A^t \equiv E[q_t^*]$. The (average) socially efficient discount rate R^t is then defined as $R^t \equiv -\frac{1}{t}\log A^t$. It determines at which rate (common to all projects) the future cash flows will be discounted. For instance, if we consider a project with deterministic cash flows (F_t), the present value of the project is given by $\int \exp(-R^t t) F_t\,dt = \int A^t F_t\,dt$. For risky projects (with stochastic cash flows), it suffices to replace the cash flows (F_t) by their certainty equivalent payoffs as in the CCAPM[13] of [5].

We will first analyse the link between the socially efficient discount rate R^t and the individual discount rates. The individual discount factor $A^{t,i}$ of agent i is defined as the discount factor that would prevail in the economy if agent i had all the aggregate endowment. It is also the discount factor that would prevail in a standard homogeneous economy in which all agents would share the characteristics (ρ^i, δ^i) of agent i. It corresponds to the price of a zero-coupon bond if all agents had the same time preference rate ρ^i and the same anticipations δ^i. It is given by $A^{t,i} \equiv E[q_t^{*^i}]$, where $q_t^{*^i} \equiv M_t^i D_t^i u'(e_t^*)$ represents the Arrow–Debreu price process that would prevail if agent i had all the endowment (or if all the agents had the same characteristics as agent i). The individual discount rate is then defined by $R^{t,i} \equiv -\frac{1}{t}\log A^{t,i}$. Since $A^{t,i}$ and $R^{t,i}$ represent equilibrium quantities in economies with homogeneous beliefs and homogeneous time preference rates, their expression is well known and given by the standard homogeneous model, with the characteristics (ρ^i, δ^i). In the specific setting of time and state independent parameters μ, σ, ρ^i and $\mu^i \equiv \mu + \sigma\delta^i$, the expression of the individual discount rate $R^{t,i}$ of agent i is given by the standard (extended) Ramsey formula with parameters (ρ^i, μ^i), i.e. $R^{t,i} = \rho^i + \frac{\mu^i}{\eta} - \frac{1}{2}\frac{1}{\eta}(1 + \frac{1}{\eta})\sigma^2$. The individual discount rate is the discount rate proposed by an individual (or a group of experts), who consults a standard equilibrium model and calibrates it with her own characteristics. For instance, the experts of the Green Book propose a discount rate of $R^{GB} = 3.5\%$ based on the figures $(\rho^{GB}, \mu^{GB}) = (1\%, 2.5\%)$ whereas the experts of the Stern Review propose a discount rate of $R^{\text{Stern}} = 1.4\%$ based on the figures $(\rho^{\text{Stern}}, \mu^{\text{Stern}}) = (0.1\%, 1.3\%)$. More generally, each individual/expert has his own view about what a correct discount rate should be. Weitzman has collected in [40] data among 2160 economists about such recommended discount rates, and has obtained a large range of proposed individual discount rates (the sample mean was $m = 3.96\%$ with a standard deviation $s = 2.94\%$).

The first issue we tackle in this section is to determine the link between the individual discount rates and the socially efficient discount rate. Can we infer the socially efficient discount rate from data on individual discount rates? Can we aggregate experts' proposed discount rates? More precisely, can the socially efficient discount factor A^t, which represents the equilibrium price of a zero-coupon bond in an economy with agents endowed with heterogeneous anticipations and time preference rates (ρ^i, μ^i), be represented as an average of the equilibrium prices $A^{t,i}$, which represent the equilibrium prices in economies, with homogeneous time preference and anticipations parameters (ρ^i, μ^i)? In particular, is the certainty equivalent approach of [39,40], which considers the arithmetic average of the individual discount factors, socially efficient?

[13] In particular, if, from the representative agent point of view, the risk of the project (F_t) is not correlated with macroeconomics risk then the certainty equivalent payoffs (B_t) are simply given by $B_t = E[F_t]$ and the valuation of F_t is given by $\exp(-R^t t)E[F_t]$ [2].

The second issue we tackle in this section is to determine if it is possible to define consensus time preference rates and beliefs such that the expression of the socially efficient discount rate remains the same as in the homogeneous setting, replacing the homogeneous parameters by the consensus ones. Or is the expression of the socially efficient discount rate in the setting with heterogeneous beliefs *fundamentally* different from its expression in the homogeneous setting? In that case, what are its properties?

The answer to the first issue is provided in the next proposition.

Proposition 3.1.

1. *If there is no belief heterogeneity and if time-preference rates are deterministic, i.e., if* $\delta^i(s,\omega) \equiv \delta(s,\omega)$ *and* $\rho^i(s,\omega) \equiv \rho^i(s)$, *then the socially efficient discount factor is an* η*-average of the individual discount factors, more precisely*

$$A^t = \left[\sum_{i=1}^{N} \gamma_i \left(A^{t,i}\right)^\eta\right]^{1/\eta}.$$

2. *In the general setting,*
   - *If* $\eta = 1$, *then* $A^t = \sum_{i=1}^N \gamma_i (A^{t,i})$.
   - *Otherwise, we have*

$$A^t \leqslant \left[\sum_{i=1}^{N} \gamma_i \left(A^{t,i}\right)^\eta\right]^{1/\eta} \quad \text{for } \eta < 1$$

and

$$A^t \geqslant \left[\sum_{i=1}^{N} \gamma_i \left(A^{t,i}\right)^\eta\right]^{1/\eta} \quad \text{for } \eta > 1,$$

with equality holding only when the divergence in individual characteristics $N^i \equiv M^i D^i$ *is deterministic, i.e. if* N^i/N^j *is deterministic for all* (i,j).

Proposition 3.1 means first that, except in very specific settings, it is not possible to recover the socially efficient discount factor as an average of the individual discount factors. These specific settings are first, the setting with deterministic heterogeneity in pure time preference rates and no dispersion of beliefs[14] (which includes[15] the deterministic setting with rational agents and deterministic time preference rates of [23]), and second, the setting with logarithmic utility functions. In a general setting, there is an aggregation bias. The price at date 0 of a zero-coupon bond maturing at date t is lower (higher) than the weighted η-average of the individually anticipated prices for $\eta < 1$ ($\eta > 1$). Note that this bias is not observed at the Arrow–Debreu prices level. Indeed, as shown in Appendix A (proof of Proposition 3.1), we have $q_t^* = [\sum_{i=1}^N \gamma_i (q_t^{*^i})^\eta]^{1/\eta}$ for all t and all η.

[14] Note that no dispersion of beliefs does not mean that individuals are rational; they can all share the same subjective belief. Analogously, all time preference rates ρ^i need not be deterministic but they need to be written in the form $\rho^i(t,\omega) = \rho(t,\omega) + a^i(t)$ where ρ is a common term and a^i is a deterministic process.

[15] For $\delta^i(t,\omega) = \sigma(t,\omega) \equiv 0$, and deterministic ρ^i.

As far as the magnitude of these biases are concerned, we shall see in Section 5 that the difference between the socially efficient discount rate and the rate associated with the η-verage of the individual discount factors can be significant. When it is large enough, the discount factor A does not lie in the interval $[\inf_i A^i, \sup_i A^i]$.

Let us elaborate on why these biases (with respect to the η-average) are in opposite directions depending on the position of η with respect to 1. The interpretation of η as a degree of relative risk tolerance is not enlightening for our purpose. It seems more meaningful to observe that the condition $\eta \geqslant 1$ is equivalent, in our setting, to the condition that prudence is larger than twice absolute risk aversion. This last condition[16] appears crucial in intertemporal choices analysis; It is shown in [22], in a standard portfolio problem, that the opportunity to invest in a risky asset raises (reduces) the aggregate saving if and only if absolute prudence is larger (smaller) than twice absolute risk aversion. Moreover, [17] studies the problem of the optimal use of a good whose consumption can produce damages in the future and shows that scientific progress providing information on the distribution of the intensity of damages induces earlier prevention effort only if prudence is larger than twice risk aversion. Hence, a possible interpretation of the central role of $\eta = 1$ is the following. Interpret belief and time preference heterogeneity in a stochastic setting as additional risk or as less information (or more uncertainty) about the future. According to [22] or [17], this should lead agents to value more future consumption in the case $\eta > 1$ and less future consumption in the case $\eta < 1$, which agrees with the result of Proposition 3.1.

Proposition 3.1 also means that the right concept of average to consider for discount factors (in the case of power utility functions) is an η-average, which is an arithmetic average only in the case of logarithmic utility functions. This implies that the approach that considers an arithmetic average of the individual discount factors is compatible with a general equilibrium approach as far as agents have logarithmic utility functions. Moreover, this η-average is not an equally weighted average (as in the certainty equivalent approach of [40]), but a weighted average, with weights given by the parameters γ_i. These weights are deterministic and will be analysed in more detail in Section 5. We will also show, in specific settings, that the difference between the rate associated with the η-average and the rate associated with the arithmetic average of the discount factors can be significant.

We now more precisely analyse the expression of the socially efficient discount rate and, in particular, we compare it with the standard setting. For this purpose, let us first recall some results about the instantaneous risk free rate. In the standard setting with rational beliefs and homogeneous time preference rate ρ, we know that the instantaneous risk free rate is given by

$$r^f(stdd) = \rho + \frac{\mu}{\eta} - \frac{1}{2}\frac{1}{\eta}\left(1 + \frac{1}{\eta}\right)\sigma^2 \tag{3.1}$$

where all parameters ρ, μ, σ hence r^f may depend upon t and ω. This is an extension of the Ramsey Equation to a stochastic setting, which illustrates the patience effect, the wealth effect as well as the precautionary saving effect on the risk free rate. In our setting with heterogeneous time preference rates and beliefs, we obtain (see Proposition A.2 in Appendix A) that the risk free rate is given by

$$r^f = \rho_D + \frac{\mu + \delta_M \sigma}{\eta} - \frac{1}{2}\frac{1}{\eta}\left(1 + \frac{1}{\eta}\right)\sigma^2 + \rho_B \tag{3.2}$$

[16] This condition has been thoroughly studied by [17] and has appeared in different contexts [13,8,35,12].

with $\rho_D \equiv \sum_{i=1}^N \tau_i \rho^i$, $\delta_M \equiv \sum_{i=1}^N \tau_i \delta^i$, and $\rho_B \equiv \frac{1}{2}(1-\eta)\operatorname{Var}^\tau(\delta)$ where $\tau_i = \frac{y^{*i}}{e^*}$ represents agent i risk tolerance and where $\operatorname{Var}^\tau(\delta)$ represents the variance of the δ^is across the agents when agent i is endowed with a weight τ_i. In other words, the homogeneous time preference rate ρ of the Ramsey formula is replaced by the average time preference rate ρ_D and the objective growth rate μ is replaced by the average subjective growth rate $\mu + \delta_M \sigma$. Furthermore, there is an additional term ρ_B that is directly related to dispersion of beliefs and the impact is towards an increase or a decrease of the risk free rate depending on the position of η with respect to 1. When this effect is important enough, r^f does not lie in the interval $[\inf_i r_i, \sup_i r_i]$. This is a generalization of [24] which takes into account heterogeneous time preference rates. Note that "the precautionary saving term" is the same in Eqs. (3.1) and (3.2) since in our model, all agents necessarily agree on the volatility level σ (as a consequence of the equivalence of the subjective probability measures Q^i). Notice that even when all parameters ρ^i, δ^i, μ and σ are constants, the risk tolerance weighted averages ρ_D and δ_M as well as the variance term ρ_B are time-varying and stochastic, hence the risk free rate is also time-varying and stochastic (which is not the case in the standard setting).

Comparing Eqs. (3.1) and (3.2), it is easy to see that there are essentially three possible ways through which heterogeneity in beliefs and in time preference rates may lead to lower risk free rates; first, a negative correlation between impatience and risk tolerance or a low level of impatience; second, a positive correlation between pessimism and risk tolerance or a high level of pessimism; third, a negative dispersion of beliefs effect that corresponds to the case $\eta > 1$.

The following proposition enables us to show that analogous results are obtained for the socially efficient discount rate. We adopt the same notation as in Eq. (3.2) and we adopt the usual notation $\mathcal{E}_t(\delta) \equiv \exp(\int_0^t \delta_s \, dW_s - \frac{1}{2}\int_0^t (\delta_s)^2 d_s)$.

Proposition 3.2. *The socially efficient discount rate R^t is given by*

$$R^t = -\frac{1}{t} \log E\left[D_t M_t B_t u'\left(e_t^*\right)\right] \tag{3.3}$$

$$= -\frac{1}{t} \log E^{\overline{Q}_t}\left[\exp - \int_0^t r_s^f \, ds\right] \tag{3.4}$$

with $B_t = \exp(-\int_0^t \rho_B(s)\, ds)$, $D_t = \exp(-\int_0^t \rho_D(s)\, ds)$, $M_t = \mathcal{E}_t(\delta_M)$ and $\frac{d\overline{Q}_t}{dP} = \mathcal{E}_t(\delta_M - \frac{\sigma}{\eta})$.

Notice first through Eq. (3.4) that when r^f is deterministic (or when r^f and $\frac{d\overline{Q}_t}{dP}$ are independent), all that we have just said about the impact of beliefs and time preference rates heterogeneity on r^f is true of R^t. In particular, aggregate patience, aggregate pessimism, as well as dispersion of beliefs when $\eta > 1$ induce a socially efficient discount rate that is lower than in the standard setting.

More generally, the comparison of Eq. (3.3) with the expression of the socially efficient discount rate in the standard setting, which is given by $R^t(stdd) = -\frac{1}{t}\log E[\exp(-\rho t)u'(e_t^*)]$, highlights three determinants of the impact on the discount rate of heterogeneity in beliefs and in time preference rates: first, the consensus time preference factor D; second, the consensus belief (density) M; and third, the aggregation bias B. The analysis of the effect of these three factors on the discount rate (between date 0 and t) is less immediate than for the (instantaneous) risk free rate at a given date t. However, the main conclusions remain valid. Indeed, excess patience at the aggregate level, in the form of an average time preference rate $\int_0^t \rho_D(s)\, ds$ that is lower than the standard time preference rate $\int_0^t \rho(s)\,ds$ leads to a higher D hence, ceteris paribus, to a lower discount rate R^t. Beliefs dispersion for $\eta \geqslant 1$ leads to a nonpositive parameter ρ_B and to a factor B that is greater than 1, hence, ceteris paribus, to a discount rate that is lower than in the standard setting. Analogously, increased dispersion of beliefs in the form of a higher $\mathrm{Var}^\tau(\delta)$ leads to an increase in B for $\eta \geqslant 1$ hence, ceteris paribus, to a decrease in the discount rate. As far as aggregate pessimism is concerned, intuitively, a pessimistic belief increases the expected value of a decreasing function of the total endowment e^*, hence should lead to a lower discount rate R. The case with deterministic parameters μ_t and σ_t illustrates this intuition. It can be seen (Appendix A, Proposition A.3) that, if the consensus belief is neutral or pessimistic, i.e., when $\delta_M \leqslant 0$, then

$$-\frac{1}{t}\log E\left[M_t u'\left(e_t^*\right)\right] \leqslant -\frac{1}{t}\log E\left[u'\left(e_t^*\right)\right],$$

hence the effect of pessimism only is toward a lower discount rate. In fact, the impact of belief heterogeneity is towards a lower (resp. higher) socially efficient discount rate if the consensus belief is neutral or pessimistic, i.e. $\delta_M \leqslant 0$, and when $\eta \geqslant 1$ (resp. the consensus belief is neutral or optimistic, i.e. $\delta_M \geqslant 0$, and when $\eta \leqslant 1$). This result is in the spirit of the findings of [15] that obtains, in a specific sentiment framework, that “whenever risk aversion is [an integer] greater than 1, an increase in the variance of sentiment reduces the expected values of all the future stochastic discount factors.”

4. Long term considerations

We now turn to long term considerations. In particular, is it socially efficient, when diversity of opinion is taken into account, to reduce the discount rate per year for far distant horizons? Does the socially efficient discount rate converge to the lowest individual discount rate? If the socially efficient discount factor were an average of the individual discount factors, as is the case for utility discount rates (or pure time preference rates), such a property would be immediate, due to the power of compound discounting (see e.g., [23,39,33,31]). In our setting, from the previous section, the socially efficient discount factor cannot be expressed as an average of the individual ones; there is an aggregation bias, related to dispersion of beliefs, which makes the analysis more complex.

We obtain the following result.

Proposition 4.1. *We suppose that for all i, the asymptotic individual discount rate $R^{\infty,i} \equiv \lim_{t\to\infty} R^{t,i}$ exists.*[17] *The asymptotic socially efficient discount rate exists and is given by the lowest asymptotic individual discount rate i.e.*

$$R^\infty \equiv \lim_{t\to\infty} R^t = \inf\left\{R^{\infty,i},\ i=1,\ldots,N\right\}.$$

Let us remark that if we think of $R^{t,i}$ as the discount rate proposed by individual/expert i, the existence of $R^{\infty,i}$ for all i, means that each individual/expert is able to propose an asymptotic discount rate.

[17] These limits can be replaced by limits along sequences, i.e. $R^{\infty,i} = \lim_{n\to\infty} R^{t_n,i}$ for some sequence t_n such that $\lim t_n = \infty$. In this case, the asymptotic socially efficient discount rate would be defined along the same sequences.

Proposition 4.1 first shows that the bias, due to belief heterogeneity, exhibited in the previous section, vanishes in the long run. The socially efficient discount rate behaves asymptotically as the discount rate associated to an average of the individual discount factors. Indeed, under the conditions of the proposition, the socially efficient discount rate R^t converges to the lowest asymptotic individual discount rate as does the rate associated to any of the considered averages of the individual discount factors. The fact that the dispersion term vanishes in the long term may seem counterintuitive according to Eqs. (3.2) and (3.3). The bias between R and an average of the R^is is represented (up to a constant) by the variance of the δ^is and it is not clear whether or not this variance term is negligible in the long run. In particular, the unweighted variance of the δ^is is exogeneously given in our model and does not necessarily converge to zero. The fact that the weighted variance vanishes is then directly related to the dynamics of the stochastic weights τ_i. In the simple setting with two agents and constant parameters, we will show in Section 5 that agents "share" the states of the world in the long run (i.e., τ_1 is near 1 for some states of the world and τ_2 is near 1 in the other states of the world) leading to a vanishing weighted variance. However, we emphasise that the bias vanishes only asymptotically, and we show, in the next section, that we may have to consider very far horizons (hundreds of years) before observing this asymptotic behavior.

Proposition 4.1 proves foremost that, from today's perspective, among the possible anticipated asymptotic behaviors $R^{\infty,i}$, the only relevant asymptotic behavior is the one with the lowest discount rate. In other words, in a setting with heterogeneous agents, only the agent with the lowest anticipated discount rate matters in the long run. The intuition is as follows. The agent who values the most future consumption (either because she is very pessimistic about the future or because she is very patient or any combination of these two possibilities) makes the market for long-term bonds and therefore imposes her price. Asymptotically, the value of the socially efficient discount factor is then given by the discount factor that would prevail in an economy made of the agent with the lowest rate only (or equivalently, if that agent concentrated all aggregate endowment). In the case of homogeneous beliefs and heterogeneous time preference rates, this implies that the asymptotic discount rate is given by the rate associated with the lowest rate of impatience. In particular, the result of [23] on the asymptotic discount rate with heterogeneous time preference rates[18] remains valid in a stochastic setting. With homogeneous time preference rates and heterogeneous beliefs, Proposition 4.1 implies that the asymptotic socially efficient discount rate is given by the rate of the most pessimistic agent. More generally, both the distributions of time preference rates and of pessimism are necessary to determine the asymptotic discount rate (which is the lowest anticipated discount rate). In particular, with constant parameters, the asymptotic socially efficient discount rate is given by the individual discount rate of the agent with the lowest $\rho^i + \frac{\mu^i}{\eta}$. When the instantaneous growth rate is not a constant but a discrete random variable, independent of t and of W, taking values $\mu_1, \dots, \mu_J$ with probability $p_1, \dots, p_J$, the asymptotic individual discount rate is given by the rate associated to the worst possible scenario, i.e., $R^{\infty,i} = \rho^i + \frac{1}{\eta}(\inf_{j=1,\dots,J} \mu_j + \sigma\delta^i) - \frac{1}{2}\frac{1}{\eta}(1+\frac{1}{\eta})\sigma^2$ and the socially efficient discount rate is then given by the lowest possible rate among the agents, i.e. $R^\infty = \inf_{i=1,\dots,N}(\rho^i + \sigma\frac{\delta^i}{\eta}) + \frac{\inf_{j=1,\dots,J}\mu_j}{\eta} - \frac{1}{2}\frac{1}{\eta}(1+\frac{1}{\eta})\sigma^2$.

In the setting of a CBA, Proposition 4.1 leads to discount long-term costs and benefits at the lowest individual rate inducing a bias towards the optimal policy of the agent who values future consumption the most in the long term.

[18] See also [4,27].

In order to determine the whole shape of the yield curve, we must determine explicit formulas for the socially efficient discount rates. As seen in Proposition 3.2, we need to analyse how the individual risk tolerances τ_i evolve over time, since they are key features in our analysis. Besides, we have seen that the covariance between individual risk tolerances and individual belief and taste parameters play an important part. This means that it is necessary to consider specific settings in order to analyse the shape of the yield curve.

5. Specific settings and the shape of the yield curve

We consider the setting with constant parameters. In such a setting, we know that in the standard model, with homogeneous and rational anticipations μ and homogeneous time preference rates ρ, the yield curve is flat and the socially efficient discount rate is given for all t by $R^t = \rho + \frac{\mu}{\eta} - \frac{1}{2}\frac{1}{\eta}(1 + \frac{1}{\eta})\sigma^2$. The aim of this section is to analyse the impact of heterogeneity in beliefs and in time preference rates on the shape of the yield curve.

We assume that the parameters μ and σ of the aggregate endowment process are constant as well as the individual belief and time preference parameters δ^i and ρ^i. We suppose that for all i, the endowment process of agent i satisfies $e^{*^i} = w_i e^*$ for some constant w_i. Moreover, we assume that, for all i,

$$\rho^i + \left(\frac{1}{\eta} - 1\right)\left(\mu + \sigma\delta^i\right) - \frac{1}{2}\frac{1}{\eta}\left(\frac{1}{\eta} - 1\right)\sigma^2 > 0, \tag{5.1}$$

which is a necessary condition for the individual optimization problems to be well defined. Note that assuming constant δ^is may seem incompatible with learning. However, we consider the case with constant parameters as an approximation of the situation where all the parameters are stochastic and where learning is regularly compensated by new shocks on the drift μ.

For all i, the individual discount rate $R^{t,i}$ is time and state independent and given by $R^i = \rho^i + \frac{\mu+\sigma\delta^i}{\eta} - \frac{1}{2}\frac{1}{\eta}(1 + \frac{1}{\eta})\sigma^2$. We recall that even in this setting the consensus characteristics $\rho_D \equiv \sum_{i=1}^N \tau_i \rho^i$ and $\delta_M \equiv \sum_{i=1}^N \tau_i \delta^i$, as well as the aggregation bias $\rho_B \equiv \frac{1}{2}(1-\eta)\operatorname{Var}^\tau(\delta)$ are time-varying, stochastic processes.

5.1. Logarithmic utility functions

As described in Section 3, the case of logarithmic utility functions is very specific. The socially efficient discount factor can be expressed as a weighted arithmetic average of the individual discount factors, $A^t = \sum_i \gamma_i A^{t,i}$ (see Proposition 3.1). There is no aggregation bias B in the expression of the socially efficient discount rate (see Eq. (3.3)).

Note that in the logarithmic setting, condition (5.1) amounts to the condition that all time preference rates be positive. We obtain the following result.

Proposition 5.1. *In the case of logarithmic utility agents with positive time preference rates ρ^i, we have*

1. *The weight γ_i of each individual discount factor in the socially efficient discount factor is given by $\gamma_i = \frac{w_i \rho^i}{\sum_{j=1}^N w_j \rho^j}$, $i = 1, \ldots, N$.*

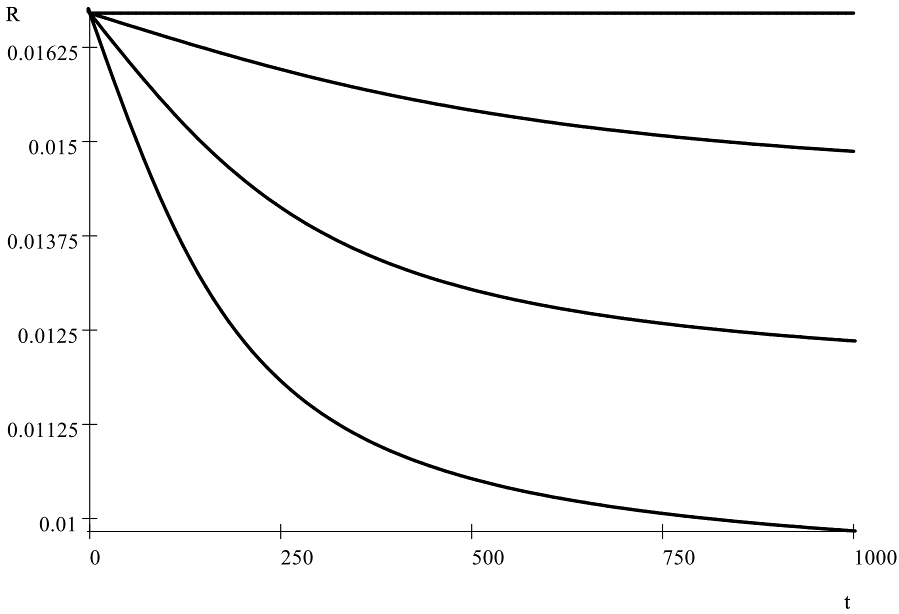

Fig. 1. This figure represents R^t as a function of t in the case of logarithmic utility functions, same time-preference rates ($\rho^1 = \rho^2$), same relative initial endowment levels ($w_1 = w_2$), and for three different levels of beliefs dispersion δ_j, $j = 1, 2, 3$ (agent 1 has a belief δ_j and agent 2 has a belief $-\delta_j$). The straight line represents the rational rate. We take $\mu = 1.8\%$, $\sigma = 3.6\%$ and ρ near 0. We take $\delta_1 = 0.07$, $\delta_2 = 0.14$, $\delta_3 = 0.21$. The three curves are decreasing (see Proposition 5.1). For all t, the discount rate R^t decreases with the level of beliefs dispersion (see Table 1), hence the three curves do not cross and the lowest curve corresponds to the highest level of beliefs dispersion. The three curves start from the rational rate. Each curve $\mathcal{C}_j$ converges asymptotically to the pessimistic discount rate $\rho + \mu - \sigma^2 - \sigma\delta_j$.

2. *The socially efficient discount rate satisfies*

$$R^t = \mu - \sigma^2 - \frac{1}{t}\log\left[\sum_{i=1}^{N}\gamma_i \exp\bigl(-\bigl(\rho^i + \delta^i\sigma\bigr)t\bigr)\right] \quad \textit{for all } t > 0,$$

$$R^0 = \mu - \sigma^2 + \sum_{i=1}^{N}\gamma_i\bigl(\rho^i + \sigma\delta^i\bigr), \qquad R^\infty = \mu - \sigma^2 + \inf_{i=1,\dots,N}\bigl(\rho^i + \sigma\delta^i\bigr).$$

The yield curve $(R^t)_t$ *is downward sloping.*

We recall that in the standard setting with logarithmic utility functions, the yield curve is flat and the rational discount rate R^t is given by $R^t = \rho + \mu - \sigma^2$ for all t. When beliefs and time preference rates are heterogeneous, Proposition 5.1 shows that the yield curve, i.e. the socially efficient discount factor as a function of time, is always downward sloping. The behavior of the socially efficient discount factor as a function of dispersion of beliefs is more complex since it depends on the correlation between individual characteristics (w_i, ρ^i, δ^i).

In order to focus on the impact of dispersion of beliefs, Figs. 1 and 2 represent the yield curve in a two agent setting with no pessimism/optimism on average, i.e. $\delta^1 + \delta^2 = 0$.

Fig. 1 represents the yield curve in the particular setting with $w_1 = w_2$ and $\rho^1 = \rho^2$. In this case, the discount factor is an equally weighted arithmetic average of the individual discount fac-

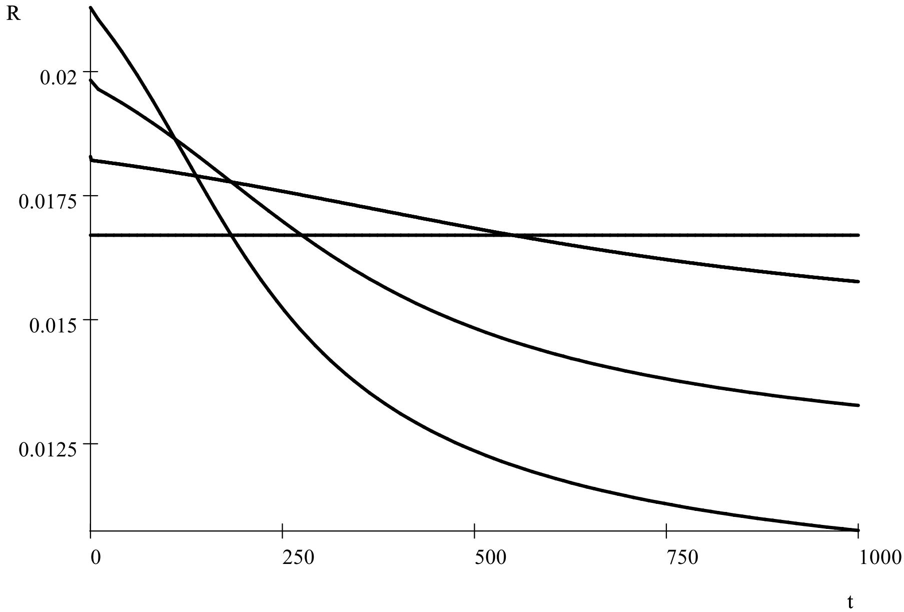

Fig. 2. This figure represents R^t as a function of t for three different levels of beliefs dispersion δ_j, $j = 1, 2, 3$ in the case of logarithmic utility functions, equal time-preference rates ($\rho^1 = \rho^2$) and a positive correlation between wealth and optimism. The straight line represents the rational rate. We take $\mu = 1.8\%$, $\sigma = 3.6\%$ and ρ near 0; $\delta_1 = 0.07$, $\delta_2 = 0.14$, $\delta_3 = 0.21$ (agent 1 has a belief δ_j and agent 2 has a belief $-\delta_j$); $w_1 = 0.8$, $w_2 = 0.2$. The three curves are decreasing (Proposition 5.1), each curve $\mathcal{C}_j$ converging asymptotically to the pessimistic discount rate $\rho + \mu - \sigma^2 - \sigma\delta_j$. Since $w_1 > w_2$, for small t the discount rate R^t increases with the level of beliefs dispersion (Proposition 5.1), and for larger t, it decreases with the level of beliefs dispersion, hence the three curves cross. Since $w_1 \neq w_2$, the three curves start from different points, which are given by $\rho + \mu - \sigma^2 + (w_1 - w_2)\sigma\delta_j$. The lowest curve at $t = 1000$ corresponds to the highest level of beliefs dispersion.

tors, as in [39,40]. The short-term rate is the rational rate and the long-term rate is the pessimistic rate. Moreover, for all t, the discount rate R^t decreases with dispersion of beliefs.

The setting with possibly different endowment levels (but still with $\rho^1 = \rho^2$) is illustrated in Fig. 2. The consensus discount factor is an endowment-weighted arithmetic average of the individual discount factors. The yield curve is still downward sloping. The short-term rate is an endowment weighted average of the individual short-term rates, which differs from the rational rate if $w_1 \neq w_2$. If there is a positive correlation between optimism and initial endowment, the short-term discount rate is higher than in the rational setting and an increase in belief heterogeneity increases the short-term rate. The long-term rate is still given by the pessimistic rate and an increase in dispersion of beliefs always decreases the long-term rate. More generally, an increase in dispersion of beliefs lowers the yield curve when there is a negative correlation between optimism and initial endowment and rotates it clockwise when there is a positive correlation between optimism and initial endowment. Finally, an increase in the initial relative wealth of the optimistic agent induces a higher short-term discount rate and a greater spread between the short-term discount rate R^0 and the long-term discount rate R^∞, the spread being always positive.

Table 1 sums up the possible results with two agents who are on average rational, i.e. $\delta^1 + \delta^2 = 0$. We obtain similar results in the setting with N agents that are on average rational.

Table 1
Yield curve properties in the case of logarithmic functions when agents are on average rational i.e. $\delta_1 = \delta > 0$ and $\delta_2 = -\delta$.

	γ_i	A^t	R^t	R^0	R^∞	$R^t(\delta)$ as a function of δ
$w_1 = w_2$ and $\rho_1 = \rho_2 = \rho$	$\gamma_i = \frac{1}{2}$	$\frac{1}{2}A^{t,1} + \frac{1}{2}A^{t,2}$	$\searrow$	$\rho + \mu - \sigma^2$ rational rate	$\rho + \mu - \sigma^2 - \delta\sigma < R^0$ pessimistic rate	$\searrow$
$\rho_1 = \rho_2 = \rho$	$\gamma_i = w_i$	$w_1 A^{t,1} + w_2 A^{t,2}$	$\searrow$	$\rho + \mu - \sigma^2 + (w_1 - w_2)\delta\sigma$	$\rho + \mu - \sigma^2 - \delta\sigma \leqslant R^0$ pessimistic rate	If $w_1 \leqslant w_2$, $\searrow$. If $w_1 > w_2$, $\nearrow$ for small t and $\searrow$ for large t.
General case	$\gamma_i = \frac{w_i \rho^i}{\sum_j w_j \rho^j}$	$\gamma_1 A^{t,1} + \gamma_2 A^{t,2}$	$\searrow$	$\mu - \sigma^2 + w_1(\rho^1 + \delta\sigma) + w_2(\rho^2 - \delta\sigma)$	$\mu - \sigma^2 + \inf(\rho^i + \delta^i \sigma)$ lowest rate	

The yield curve is always downward sloping. The behavior of $R^t(\delta)$, the discount rate for a given maturity t as a function of beliefs dispersion δ, depends upon individual initial endowment and time preference rates distribution.

If there is a (pessimistic or optimistic) bias on average, i.e. if $\frac{1}{N}\sum_{i=1}^{N}\delta^i = \bar{\delta} \neq 0$, then there is an additional optimism (when $\bar{\delta}$ is positive) or pessimism (when $\bar{\delta}$ is negative) effect on the discount rate.

We can also consider settings with a continuous distribution on the individual beliefs (δ^i) as a limit of a setting with a large number of agents. If we suppose that initial endowment is equally distributed and that all agents have the same time preference rate, then we obtain as an easy extension of the discrete setting that the discount factor is an average of the individual discount factors, i.e. $A^t = E^i[A^{t,i}]$, where E^i is the expectation operator associated to the distribution of agents' characteristics. Assuming a Gamma distribution on the individual discount rates (a normal distribution on individual beliefs), we retrieve the expression of the discount rate in [40] ([33]). In [25], more general distributions on the individual characteristics (ρ^i, δ^i) are considered and calibrations on [40] data are provided.

It is interesting to notice that while the socially efficient discount rate always converges in the long run to the lowest individual discount rate, the future short-term rates are stochastic and may remain higher than the rational rate and may even converge in the long run to the highest individual risk free rate.[19] Consider, for instance, the setting with $\rho^1 = \rho^2$, $\delta^1 + \delta^2 = 0$ and $w_1 = w_2$. It is then easy to obtain that for all t, $\tau_1(t)$ and $\tau_2(t)$ have the same distribution, which means that none of the agents "wins" and the short-term rate remains equal on average to the rational rate. If one agent is pessimistic (or optimistic) with $\delta^1 < 0$ (or $\delta^1 > 0$) and the other is rational with $\delta^2 = 0$, the weights γ_i are the same as in the previous setting since they do not depend upon individual beliefs. However, agent 1 is "wrong" while agent 2 is "right" and it can be shown that[20] agent 2 is the only surviving agent, i.e. $\tau_1(t) \to_{t\to\infty} 0$ and $\tau_2(t) \to_{t\to\infty} 1$, a.s. Hence, by Eq. (3.2), the future risk free rate converges in the long run to the rational rate. More generally, when one agent is "more wrong" than the other agent, future short-term rates converge to the short-term rate that would prevail in the economy made of the agent that is less wrong alone, while the socially efficient discount rate converges to the lowest anticipated discount rate.

5.2. Power utility functions

We now consider the case of power utility functions. As previously discussed, there are essentially two different settings, $\eta < 1$ or $\eta > 1$, for which the impact of dispersion of beliefs is opposite.

5.2.1. The case $\eta < 1$

Let us start by considering the specific case $\eta = 1/2$. In this case, we recall that in the standard setting the yield curve is flat and, for all t, $R^t = \rho + 2\mu - 3\sigma^2$. We consider two agents who are rational on average. Note that condition (5.1) is equivalent in this setting to the condition that $\mu - \sigma^2 + \inf_i(\rho^i + \sigma\delta^i) > 0$.

Proposition 5.2. *Consider the case of power utility functions with $\eta = 1/2$. Suppose that $\delta^1 = \delta > 0$, $\delta^2 = -\delta$, $w_1 = w_2$, $\rho^1 = \rho^2 = 0$ and that $\mu - \sigma^2 - \sigma\delta > 0$.*

[19] The individual risk free rate of agent i is the risk free rate that would prevail in an homogeneous economy in which all agents would share the characteristics of agent i or equivalently in the initial economy if agent i concentrated all aggregate endowment.

[20] See [44] for related issues.

1. *The ratio* $(\frac{\gamma_1}{\gamma_2})$ *is given by*

$$\frac{\gamma_1}{\gamma_2} = \sqrt{\frac{\mu - \sigma^2 + \sigma\delta}{\mu - \sigma^2 - \sigma\delta}}.$$

2. *The discount rate is a decreasing function of t and is given by*

$$R^t = 2\mu - 3\sigma^2 - \frac{1}{t}\ln\big((\gamma_1)^2 e^{-2\delta\sigma t} + (\gamma_2)^2 e^{2\delta\sigma t} + 2\gamma_1\gamma_2 e^{-\frac{1}{2}(\delta)^2 t}\big),$$
$$R^0 = 2\mu - 3\sigma^2 + 2\delta\sigma(\gamma_1 - \gamma_2) - (\delta)^2\gamma_1\gamma_2,$$
$$R^\infty = 2\mu - 3\sigma^2 - 2\sigma\delta.$$

According to the first point of the proposition, the relative weight of the optimistic agent is greater than the relative weight of the pessimistic agent, i.e., $\gamma_1 > \gamma_2$: there is an optimistic bias at the aggregate level. This result is valid in the general setting with $\eta < 1$ (see Appendix A, Proposition A.4). This implies in particular that an increase in dispersion of beliefs leads to an increase of the short-term rate. Notice that in the long term an increase in dispersion of beliefs leads to a decrease of the socially efficient discount rate since it corresponds to a more pessimistic belief for the more pessimistic agent.

The main result we obtain is the fact that the yield curve is decreasing. We have already seen that the bias between the discount rate R and an average of the R^is is represented (up to a constant) by the dispersion of beliefs term $\frac{1}{4}\operatorname{Var}^\tau(\delta)$. In the case of logarithmic utility functions, there is no dispersion of beliefs term and, as already seen, the socially efficient discount rate R^t decreases with t and converges to the more "pessimistic" individual discount rate. In the case $\eta = \frac{1}{2}$, we have $\frac{\gamma_2}{\gamma_1}(\frac{\tau_1}{\tau_2}) = (\frac{M_t^2}{M_t^1})^{1/2} \sim \ln\mathcal{N}(0, \delta\sqrt{t})$. In particular, this implies that, for large t, "τ_1 is large with respect to τ_2 or τ_2 is large with respect to τ_1" with a probability near 1. Loosely speaking, there are two kinds of states of the world, those for which τ_1 vanishes for large t and those for which τ_2 vanishes for large t; the dispersion of beliefs term $\frac{1}{4}\operatorname{Var}^\tau(\delta) = \tau_1\tau_2(\delta)^2$ vanishes asymptotically. The socially efficient discount rate curve is then globally decreasing and converges, as in the logarithmic case, to the most pessimistic rate. Everything works as if we had two scenarios, one with the optimistic rate and one with the pessimistic rate. The asymptotic socially efficient discount rate is associated to the worst scenario as in [18,19,39,40]. This reasoning is valid for general $\eta < 1$. Indeed, we then have $\frac{\gamma_2}{\gamma_1}(\frac{\tau_1}{\tau_2}) = (\frac{M_t^2}{M_t^1})^\eta \sim \ln\mathcal{N}(0, 2\eta\delta\sqrt{t})$, and as in the case $\eta = 1/2$, the dispersion of beliefs term $\frac{1}{2}(1-\eta)\operatorname{Var}^\tau(\delta)$ vanishes asymptotically.

In Fig. 3 we represent the socially efficient yield curve as well as the rates associated to the η-average and to the arithmetic average of the individual discount factors. All these curves converge asymptotically to the rate associated to the most pessimistic belief but the "η" one is a much better approximation of the yield curve than the "arithmetic" one. The distance between the yield curve and the "η" curve measures the impact of the bias due to dispersion of beliefs. The variance term increases the short rate but its impact decreases with t and vanishes asymptotically. However this impact can remain nonnegligible for centuries.

We have assumed so far that both agents have the same initial endowment. If we relax this assumption, we still obtain decreasing yield curves converging to the most pessimistic rate. However, when the more optimistic agent has a larger initial endowment, she has a greater weight in the average formula and the impact of her optimism lasts longer. The yield curve then has a higher starting point at $t = 0$ and its initial slope is smaller. When the more pessimistic agent has

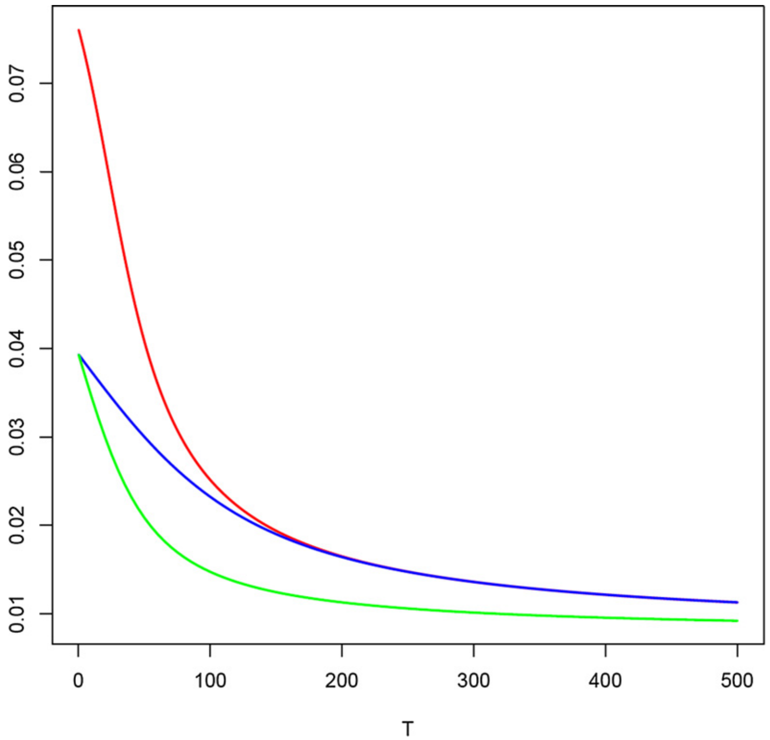

Fig. 3. This figure represents the socially efficient discount rate for $\gamma_1 = \gamma_2$, $\eta = 0.4$, $\rho = 0$, $\mu = 0.018$, $\sigma = 0.036$, and $\delta = 0.35$ as well as the curves associated to the η-average (resp. arithmetic average) of the individual discount factors. The discount rate curve dominates the η-average that dominates the arithmetic average. With these parameters values, the rational rate is equal to 3.2%.

a larger initial endowment, she has a larger weight. The starting point of the yield curve is lower and the convergence to its asymptotic rate is more rapid.

5.2.2. *The case $\eta > 1$*

As already underlined, the socially efficient discount rate exhibits both an average belief/time preference effect that is measured by $\rho_D \equiv \sum_{i=1}^N \tau_i \rho^i$ and $\delta_M \equiv \sum_{i=1}^N \tau_i \delta^i$ and a dispersion of beliefs effect that is measured by the variance term. As in the case $\eta < 1$, the average effect induces a decrease of R^t when t increases (since the associated rate converges asymptotically to the lowest rate) and the variance term decreases and vanishes asymptotically. However, in the case $\eta > 1$, the dispersion of beliefs term $\frac{1}{2}(1-\eta)\operatorname{Var}^\tau(\delta)$ is negative. This leads then to two opposite effects when t increases, the average effect inducing a decrease of R^t and the dispersion effect inducing an increase of R^t. Depending on the relative size of these effects, we may obtain decreasing curves as in the case $\eta < 1$ as well as increasing then decreasing curves as in Fig. 4. The case $\eta > 1$ leads then to a richer family of possible shapes and is compatible with the fact that long-term rates in bonds markets (i.e. $t = 30$) are usually higher than short-term rates. In the case of an initially increasing yield curve, this initial shape results from the dispersion of beliefs and is not replicated when we approximate the yield curve by the rates associated to the η-average or the arithmetic average of the individual discount factors as can be seen in Fig. 4.

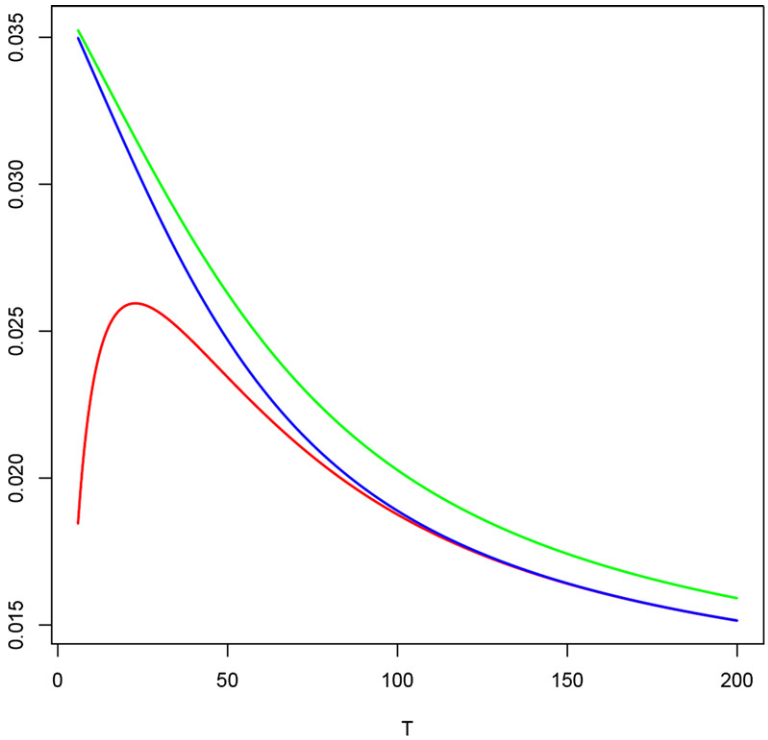

Fig. 4. This figure represents the socially efficient discount rate for $\gamma_1 = 0.6$, $\eta = 1.2$, $\mu = 0.04$, $\sigma = 0.036$, $\delta = 0.7$ as well as the curves associated to the η-average (resp. arithmetic average) of the individual discount factors. The arithmetic average dominates the η-average that dominates the discount rate curve.

6. Conclusion

When public investment projects entail costs and benefits in the very long run, a question arises about the selection of the relevant discount rate for the Costs and Benefits Analysis. Financial markets do not provide a guideline in this case. In this paper we provide an equilibrium analysis of the effect of heterogeneity in beliefs and in time preference rates on the socially efficient discount rate in a general stochastic setting.

First, we show that, in general, the socially efficient discount rate does not reduce to an average of the individual discount rates. There is an additional bias induced by beliefs and time preference rates dispersion. Moreover, the right concept of average to consider is not the arithmetic (unweighted) average as in [39,40] but a (weighted) η-average.[21] We show, by examples, that both effects (η-average instead of arithmetic average and additional dispersion bias) significantly impact the socially efficient discount rate.

We obtain that heterogeneity in beliefs and in time preference rates impacts the socially efficient discount factor as would an additional source of risk. Increased dispersion of beliefs and/or

[21] We retrieve the approach of [39,40] if we consider logarithmic utility functions and homogeneous time preference rates.

time preference rates leads to lower socially efficient discount rates when the relative risk aversion is less than one.

In the short and medium term, belief heterogeneity leads to a rich class of possible shapes for the yield curve. We may obtain increasing yield curves for the first 50 years as it is often the case in financial markets.

In the long run, we show that the dispersion effect disappears and the asymptotic socially efficient discount rate is the lowest asymptotic individual discount rate, the expression of which depends on both distribution of patience and pessimism across agents. This is another element supporting the use of decreasing discount rates in the long run, which is a necessary condition for long term effects to be taken into account in a Costs and Benefits Analysis.

These results are derived under the assumption of homogeneous utility functions.[22] This assumption has been made both for tractability reasons and in order to focus on the impact of heterogeneity in anticipations and in time preference rates. In the case of heterogeneous levels of risk aversion, it is possible to derive aggregation formulas for the (instantaneous) short term rate in the spirit of Eq. (3.2) (see e.g. [11,3]). However, as far as discount rates are concerned, the formulas we obtain lack analytical tractability and, at this stage, can only be solved numerically.

As in [18,19,32,39,40,42], we have considered an exchange economy with exogeneous growth in order to focus on the trade-off between current and future wealth, in particular for CBA. Even though there is no production in our model, it can shed some light on the analysis of the trade-off between current production and the environmental and economic welfare of future generations. Our results permit the characterisation of the marginal cost we are willing to invest in any technology that can marginally impact future aggregate endowment. A more comprehensive analysis of the socially efficient discount rates in a production economy with endogeneous growth is left for future research.

Appendix A. Aggregation of individual beliefs and time-preferences

The aim of this appendix is to extend the results of [24] to our setting with an infinite horizon and heterogeneous time preference rates. We deal with aggregation issues in the spirit of [1,7,34, 36,37].

Proposition A.1. *We let N^i denote the individual composite characteristic $M^i D^i$.*

1. *The individual characteristics N^i can be aggregated into a consensus characteristic N such that*

$$q_t^* = N_t u'\left(e_t^*\right)$$

with

$$N = \left[\sum_{i=1}^{N} \gamma_i \left(N_t^i\right)^{\eta}\right]^{1/\eta}.$$

2. *The consensus characteristic N can be written in the form $N = BDM$ where M is a consensus probability belief, D is a pure time consensus discount factor and B is an aggregation bias, related to dispersion of beliefs. More precisely, the martingale process M and the finite*

[22] An analysis of the term structure of interest rates with heterogeneous utility functions can be found in [38].

variation processes D and B satisfy $dM_t = \delta_M M_t\, dW_t$, $dD_t = -\rho_D D_t\, dt$, $dB_t = -\rho_B B_t\, dt$ with $M_0 = D_0 = B_0 = 1$ and

$$\delta_M = \sum_{i=1}^{N} \tau_i \delta^i, \qquad \rho_D = \sum_{i=1}^{N} \tau_i \rho^i,$$

$$\rho_B = \frac{1}{2}(1-\eta)\left[\sum_{i=1}^{N} \tau_i \left(\delta^i\right)^2 - \delta_M^2\right] = \frac{1}{2}(1-\eta)\,\mathrm{Var}^\tau(\delta).$$

Proof. 1. Since q^* is an interior equilibrium price process, we know that there exist Lagrange multipliers (λ_i) such that for all i and for all t,

$$\frac{1}{\lambda_i} N_t^i u'\left(y_t^{i^*}\right) = q_t^*. \tag{A.1}$$

Since $\sum_{i=1}^N y_t^{i^*} = e_t^*$, we get

$$q_t^* = N_t u'\left(e_t^*\right) \quad \text{with } N_t = \left[\sum_{i=1}^{N} \gamma_i \left(N_t^i\right)^\eta\right]^{1/\eta}. \tag{A.2}$$

2. We can write that $dy_t^{i^*} = a_i(t)\,dt + b_i(t)\,dW_t$ for processes (a_i) and (b_i) such that $\sum_{i=1}^N a_i(t) = \mu_t e_t^*$ and $\sum_{i=1}^N b_i(t) = \sigma_t e_t^*$. Analogously, we introduce the processes μ_N and δ_N such that $dN_t = \mu_N(t) N_t\, dt + \delta_N(t) N_t\, dW_t$. We apply Itô's Lemma to both sides of Eq. (A.2). Identifying the diffusion and drift parts and after simple computations, we obtain

$$\delta_N = \delta_M = \sum_{i=1}^{N} \tau_i \delta^i,$$

$$\mu_N = \frac{1}{2}(\eta - 1)\left[\sum_{i=1}^{N} \tau_i \left(\delta^i\right)^2 - \delta_M^2\right] - \sum_{i=1}^{N} \tau_i \rho^i.$$

It is easy to check then that N is of the form $N = BDM$. □

Proposition A.2. *The risk free rate is given by*

$$r^f = \rho_D + \frac{1}{\eta}[\mu + \delta_M \sigma] - \frac{1}{2}\frac{1}{\eta}\left(1 + \frac{1}{\eta}\right)\sigma^2 + \rho_B \tag{A.3}$$

with $\delta_M = \sum_{i=1}^N \tau_i \delta^i$, $\rho_D = \sum_{i=1}^N \tau_i \rho^i$, $\rho_B = \frac{1}{2}(1-\eta)[\sum_{i=1}^N \tau_i (\delta^i)^2 - \delta_M^2] \equiv \frac{1}{2}(1-\eta)\,\mathrm{Var}^\tau(\delta)$.

Proof. We adopt the notations of Proposition A.1. We let μ_{q^*} (σ_{q^*}) denote the drift (diffusion) parameter of the process q^*, i.e. $dq_t^* = \mu_{q^*} q_t^*\, dt + \sigma_{q^*} q_t^*\, dW_t$. Since q^* is a state price density, we obtain as in the standard setting that $r^f = -\mu_{q^*}$. We know from the proof of Proposition A.1 that $q^* = Nu'(e^*)$, hence we get through Itô's Lemma that

$$\mu_{q^*} = \mu_{u'(e^*)} + \mu_N + \delta_M\left(\frac{u''(e^*)}{u'(e^*)}\sigma e^*\right)$$

where $\mu_{u'(e^*)}$ denotes the drift parameter of the process $u'(e^*)$. We easily deduce that

$$r^f = \sum_{i=1}^{N} \tau_i \rho^i + \frac{1}{\eta}\left[\mu + \left(\sum_{i=1}^{N} \tau_i \delta^i\right)\sigma\right] - \frac{1}{2}\frac{1}{\eta}\left(1 + \frac{1}{\eta}\right)\sigma^2 - \frac{1}{2}(\eta - 1)\operatorname{Var}^\tau(\delta). \qquad \square$$

Proposition A.3. *If μ and σ are deterministic and $\delta_M \leqslant 0$, then $-\frac{1}{t}\log E[M_t u'(e_t^*)] \leqslant -\frac{1}{t}\log E[u'(e_t^*)]$. If we further assume that $\eta > 1$ then $-\frac{1}{t}\log E[M_t B_t u'(e_t^*)] \leqslant -\frac{1}{t}\log E[u'(e_t^*)]$.*

Proof. The result can be found in [24] for a finite horizon T and homogeneous time preference rates and is adapted here to the case $T = \infty$ and heterogeneous time preference rates. If Q_t is defined on F_t by $\frac{dQ_t}{dP} = M_t$ and letting $W_s^{Q_t} \equiv W_s - \int_0^s \delta_M(u)\,du$, we have

$$\left(e_t^*\right)^{-1/\eta} = \exp\left(-1/\eta\left[\int_0^t \left(\mu_u - \frac{1}{2}\sigma_u^2\right)du + \int_0^t \sigma_u\,dW_u^{Q_t}\right]\right)$$
$$\times \exp\left(-1/\eta \int_0^t \sigma_u \delta_M(u)\,du\right).$$

If $\delta_M \leqslant 0$, then $\exp(-1/\eta \int_0^t \sigma_u \delta_M(u)\,du) \geqslant 1$, hence,

$$\begin{aligned} E\left[M_t\left(e_t^*\right)^{-1/\eta}\right] &\geqslant E^{Q_t}\left[\exp\left(-1/\eta\left[\int_0^t \left(\mu_u - \frac{1}{2}\sigma_u^2\right)du + \int_0^t \sigma_u\,dW_u^{Q_t}\right]\right)\right] \\ &\geqslant E\left[\exp\left(-1/\eta\left[\int_0^t \left(\mu_u - \frac{1}{2}\sigma_u^2\right)du + \int_0^t \sigma_u\,dW_u\right]\right)\right] \\ &\geqslant E\left[\left(e_t^*\right)^{-1/\eta}\right]. \end{aligned}$$

It now suffices to remark that $B_t \geqslant 1$, a.s. for $\eta \geqslant 1$ to conclude. $\square$

Proposition A.4. *Consider the case of power utility functions. Suppose that $\delta^1 = \delta > 0$ and $\delta^2 = -\delta$, $w_1 = w_2$ and $\rho^1 = \rho^2 = 0$, then there is an optimistic bias ($\gamma_1 > \gamma_2$) for $\eta < 1$ and a pessimistic bias ($\gamma_1 < \gamma_2$) for $\eta > 1$.*

Proof. The result can be found in [24] for a finite horizon T and is adapted here to the case $T = \infty$. Since $w_1 = w_2$, the relative weights $(\frac{1}{\lambda_1})^\eta$ and $(\frac{1}{\lambda_2})^\eta$ must satisfy

$$\Xi \equiv E\left[\int_0^\infty \left(e_t^*\right)^{1-1/\eta} \frac{(M^1/\lambda_1)^\eta - (M^2/\lambda_2)^\eta}{\{(M^1/\lambda_1)^\eta + (M^2/\lambda_2)^\eta\}^{1-1/\eta}}\,dt\right] = 0.$$

It is immediate that Ξ can be written in the form $\Xi = \frac{1}{\lambda_1}x^{-1/\eta}g(x)$ with $x^2 = (\frac{\lambda_2}{\lambda_1})^\eta$ and

$$g(x) = E\left[\int_0^\infty \left(e_t^*\right)^{1-1/\eta} \frac{x(M_t^1)^\eta - \frac{1}{x}(M_t^2)^\eta}{\{x(M_t^1)^\eta + \frac{1}{x}(M_t^2)^\eta\}^{1-1/\eta}}\,dt\right].$$

For $\eta < 1$, we show that $\lambda_1 \leqslant \lambda_2$. We prove 1) that $g(1) \leqslant 0$, and 2) that g is increasing with x, which implies that for $\lambda_1 > \lambda_2$ we would have $\Xi < 0$, which is impossible. We have

$$g'(x) = E\left[\int_0^\infty \frac{x((M_t^1)^\eta + (M_t^2)^\eta/x^2)^2 - x((M_t^1)^\eta - (M_t^2)^\eta/x^2)^2(1-1/\eta)}{(x(M_t^1)^\eta + (M_t^2)^\eta/x)^{2-1/\eta}}\big(e_t^*\big)^{1-1/\eta}\,dt\right],$$

which is positive for $\eta < 1$ and proves 2). Now, $g(1) = \int_0^\infty E[(e_t^*)^{1-1/\eta}\frac{(M^1)^\eta-(M^2)^\eta}{\{(M^1)^\eta+(M^2)^\eta\}^{1-1/\eta}}]\,dt$. With deterministic coefficients, $(M^1)^\eta$ and $(M^2)^\eta$ can be written in the form $(M^1)^\eta(t) = Z(t)(e^*(t))^{\frac{\eta\delta}{\sigma}}$ and $(M^2)^\eta(t) = Z(t)(e^*(t))^{-\frac{\eta\delta}{\sigma}}$ for some deterministic process Z. It is easy to see that $\frac{(M^1)^\eta-(M^2)^\eta}{\{(M^1)^\eta+(M^2)^\eta\}^{1-1/\eta}}$ is increasing in e^*, hence decreasing in $(e^*)^{1-1/\eta}$, leading to

$$E\left[\big(e_t^*\big)^{1-1/\eta}\frac{(M_t^1)^\eta-(M_t^2)^\eta}{\{(M_t^1)^\eta+(M_t^2)^\eta\}^{1-1/\eta}}\right] \leqslant E\big[\big(e_t^*\big)^{1-1/\eta}\big]E\left[\frac{(M_t^1)^\eta-(M_t^2)^\eta}{\{(M_t^1)^\eta+(M_t^2)^\eta\}^{1-1/\eta}}\right].$$

Since $E[\frac{(M_t^1)^\eta-(M_t^2)^\eta}{\{(M_t^1)^\eta+(M_t^2)^\eta\}^{1-1/\eta}}] = 0$, we obtain that $g(1) \leqslant 0$. □

Appendix B. Proofs

Proof of Proposition 3.1. We let $q_t^{i^*} \equiv M_t^i D_t^i u'(e_t^*)$. By Proposition A.1, we have

$$q_t^* = \left[\sum_{i=1}^N \gamma_i\big(M_t^i D_t^i\big)^\eta\right]^{1/\eta} u'\big(e_t^*\big).$$

This implies that $q^* = [\sum_{i=1}^N \gamma_i (q_t^{i^*})^\eta]^{1/\eta}$.

1. If $\delta^i \equiv \delta$ and if $\rho^i(s,\omega) \equiv \rho^i(s)$, we have $M^i \equiv \widetilde{M}$ for all i and $A^{t,i} = D_t^i E[\widetilde{M}_t u'(e_t^*)]$. We then have

$$\begin{aligned}
A^t &= E\left[\left(\sum_{i=1}^N \gamma_i\big(\widetilde{M}_t D_t^i u'\big(e_t^*\big)\big)^\eta\right)^{1/\eta}\right] \\
&= \left(\sum_{i=1}^N \gamma_i\big(D_t^i\big)^\eta\right)^{1/\eta} E\big[\widetilde{M}_t u'\big(e_t^*\big)\big] \\
&= \left(\sum_{i=1}^N \gamma_i\big(D_t^i E\big[\widetilde{M}_t u'\big(e_t^*\big)\big]\big)^\eta\right)^{1/\eta} \\
&= \left[\sum_{i=1}^N \gamma_i\big(A^{t,i}\big)^\eta\right]^{1/\eta}.
\end{aligned}$$

2. In the case $\eta = 1$, we have $A^t = E[\sum_{i=1}^N \gamma_i q_t^{i^*}] = \sum_{i=1}^N \gamma_i E[q_t^{i^*}] = \sum_{i=1}^N \gamma_i A^{t,i}$.
In the case $\eta < 1$, we have

$$\begin{aligned}
\big(A^t\big)^\eta &= \left\| \sum_{i=1}^N \gamma_i\big(q_t^{i^*}\big)^\eta \right\|_{1/\eta} \\
&\leqslant \sum_{i=1}^N \gamma_i \big\|\big(q_t^{i^*}\big)^\eta\big\|_{1/\eta} = \sum_{i=1}^N \gamma_i \big\|q_t^{i^*}\big\|_1^\eta = \sum_{i=1}^N \gamma_i\big(A^{t,i}\big)^\eta
\end{aligned}$$

hence $A^t \leqslant [\sum_{i=1}^N \gamma_i (A^{t,i})^\eta]^{1/\eta}$.

In the case $\eta > 1$, using Minkovski's Lemma, we get analogously that

$$E\left\{\left[\sum_{i=1}^{N}\gamma_i\left(q_t^{i^*}\right)^\eta\right]^{1/\eta}\right\}^\eta \geqslant \sum_{i=1}^{N}\gamma_i E\left[q_t^i\right]^\eta,$$

hence $A^t \geqslant [\sum_{i=1}^N \gamma_i (A^{t,i})^\eta]^{1/\eta}$.

When N^i/N^j is deterministic for all (i,j) we get as in 1. that $A^t = [\sum_{i=1}^N \gamma_i (A^{t,i})^\eta]^{1/\eta}$. □

Proof of Proposition 3.2. Eq. (3.3) results from the definition of $R^t = -\frac{1}{t}\log E[q_t^*]$ and from Proposition A.1. For the second equation, we adopt the notation of Appendix A for the drift and diffusion parameters of the process q^*. We then have

$$R^t = -\frac{1}{t}\log E\left[\exp\int_0^t\left(\mu_{q^*} - \frac{(\sigma_{q^*})^2}{2}\right)ds + \int_0^t \sigma_{q^*}\,dW_s\right]$$

$$= -\frac{1}{t}\log E\left[e^{\int_0^t \mu_{q^*}\,ds} e^{\int_0^t \sigma_{q^*}\,dW_s - \int_0^t \frac{(\sigma_{q^*})^2}{2}\,ds}\right].$$

Since $\mu_{q^*} = -r^f$ (see the proof of Proposition A.2), this implies that

$$R^t = -\frac{1}{t}\log E\left[e^{-\int_0^t r^f\,ds} e^{\int_0^t \sigma_{q^*}\,dW_s - \int_0^t \frac{(\sigma_{q^*})^2}{2}\,ds}\right].$$

It is easy to obtain by Proposition A.1 and Itô's Lemma that $\sigma_{q^*} = \delta_M - \frac{\sigma}{\eta}$. Hence $R^t = -\frac{1}{t}\log E^{\overline{Q}_t}[e^{-\int_0^t r_s^f\,ds}]$. □

Proof of Proposition 4.1. We start by identifying bounds for the socially efficient discount rate in 1. and we prove the result for the asymptotic socially efficient discount rate in 2.

1. For $\eta = 1$, we know by Proposition 3.1 that $A^t = \sum_{i=1}^N \gamma_i A^{t,i}$, hence

$$\sup_i\left(\gamma_i A^{t,i}\right) \leqslant A^t \leqslant \sup_i A^{t,i}.$$

For $\eta < 1$, we have seen in Proposition 3.1 that $A^t \leqslant [\sum_{i=1}^N \gamma_i (A^{t,i})^\eta]^{1/\eta}$. Since $\sum_{i=1}^N \gamma_i = 1$, we get that $A^t \leqslant \sup_i A^{t,i}$. Moreover, as seen in the proof of Proposition 3.1 we have

$$A^t = E\left[\left(\sum_{i=1}^N \gamma_i\left(q_t^{i^*}\right)^\eta\right)^{1/\eta}\right],$$

hence for all i we have

$$A^t \geqslant E\left[\left(\gamma_i\left(q_t^{i^*}\right)^\eta\right)^{1/\eta}\right] = E\left[(\gamma_i)^{1/\eta} q_t^{i^*}\right],$$

and

$$A^t \geqslant \sup_i\left((\gamma_i)^{1/\eta} A^{t,i}\right).$$

For $\eta > 1$, we get analogously that $A^t \geqslant \sup_i((\gamma_i)^{1/\eta} A^{t,i})$. Moreover, since for all nonnegative real numbers (a_i) we have, for $\eta > 1$,

$$\left(\sum_{i=1}^N a_i\right)^{1/\eta} \leqslant \sum_{i=1}^N a_i^{1/\eta}$$

we then get that, for $\eta > 1$,

$$A^t = E\left[\left(\sum_{i=1}^N \gamma_i \left(q_t^{i^*}\right)^\eta\right)^{1/\eta}\right]$$
$$\leqslant E\left[\sum_{i=1}^N (\gamma_i)^{1/\eta} q_t^{i^*}\right] = \sum_{i=1}^N (\gamma_i)^{1/\eta} A^{t,i}.$$

2. Consider first the case $\eta = 1$. Since $A^{t,i} = e^{-R^{t,i}t}$, we have $\sup_i A^{t,i} = e^{-\inf_i R^{t,i}t}$. Let us denote by I an agent such that $\inf_i R^{\infty,i} = R^{\infty,I}$, we have then $\inf_i R^{t,i} = R^{t,I}$ for t large enough. Furthermore, it is immediate that $\sup_i(\gamma_i A^{t,i}) \geqslant \gamma_I e^{-R^{t,I}t}$. By 1., we then get, for t large enough,

$$-\frac{1}{t}\log\left(e^{-R^{t,I}t}\right) \leqslant R^t \leqslant -\frac{1}{t}\log\left(\gamma_I e^{-R^{t,I}t}\right)$$

or equivalently

$$R^{t,I} \leqslant R^t \leqslant -\frac{\log(\gamma_i)}{t} + R^{t,I}$$

hence $\lim_{t\to\infty} R^t = R^{\infty,I} = \inf_i R^{\infty,i}$.

Consider the case $\eta < 1$. As in the case $\eta = 1$ and with the same notations, it is easy to verify that, for t large enough, $\sup_i((\gamma_i)^{1/\eta} A^{t,i}) \geqslant (\gamma_I)^{1/\eta} e^{-R^{t,I}t}$ and that $\sup_i A^{t,i} = e^{-R^{t,I}t}$. Hence, we have $R^t \to_{t\to\infty} R^{\infty,I} = \inf_i R^{\infty,i}$.

Consider now the case $\eta > 1$. We have

$$\sum_{i=1}^N (\gamma_i)^{1/\eta} A^{t,i} \leqslant \left(\sum_{i=1}^N (\gamma_i)^{1/\eta}\right) \sup_i A^{t,i} \leqslant N \sup_i A^{t,i}.$$

For t large enough, we then have

$$R^{t,I} - \frac{1}{t}\log N \leqslant R^t \leqslant R^{t,I} - \frac{1/\eta}{t}\log \gamma_I$$

and $R^t \to_{t\to\infty} R^{\infty,I} = \inf_i R^{\infty,i}$. □

Proof of Proposition 5.1. 1. The relative endowment level w_i of agent i must satisfy

$$w_i E\left[\int_0^\infty q_t^* e_t^*\,dt\right] = E\left[\int_0^\infty q_t^* y_t^{*^i}\,dt\right] = E\left[\int_0^\infty \frac{1}{\lambda_i} e^{-\rho^i t} M_t^i\,dt\right] = \frac{1}{\lambda_i}\int_0^\infty e^{-\rho^i t}dt = \frac{1}{\lambda_i \rho^i},$$

hence $\gamma_i = \frac{w_i \rho^i}{\sum w_j \rho^j}$.

2. The expression for R^t is a direct consequence of the fact that $R^t = -\frac{1}{t}\log[\sum_{i=1}^2 \gamma_i e^{-R^i t}]$, obtained in Proposition 3.1. The expression for R^0 is easily obtained by taking the limit when t converges to 0 in the expression of R^t. The expression for R^∞ results from Proposition 4.1. □

Proof of Proposition 5.2. 1. Since $w_1 = w_2$, the relative weights γ_i must solve

$$E\left[\int_0^\infty \left(e_t^*\right)^{-1}\left[\gamma_1\left(M_t^1\right)^{1/2} - \gamma_2\left(M_t^2\right)^{1/2}\right]\left[\gamma_1\left(M_t^1\right)^{1/2} + \gamma_2\left(M_t^2\right)^{1/2}\right]dt\right] = 0.$$

This is equivalent to

$$\int_0^\infty e^{(-\mu+\sigma^2/2-(\delta)^2/2)t} E\big[(\gamma_1)^2 e^{(\delta-\sigma)W_t} - (\gamma_2)^2 e^{(-\delta-\sigma)W_t}\big]\,dt = 0$$

or

$$\int_0^\infty e^{(-\mu+\sigma^2/2-(\delta)^2/2)t} \big((\gamma_1)^2 e^{\frac{(\delta-\sigma)^2}{2}t} - (\gamma_2)^2 e^{\frac{(\delta+\sigma)^2}{2}t}\big)\,dt = 0.$$

This implies that

$$\frac{(\gamma_1)^2}{(\gamma_2)^2} = \frac{(-\mu+\sigma^2-\sigma\delta)}{(-\mu+\sigma^2+\sigma\delta)}.$$

2. We know that $R^t = -\frac{1}{t}\log[\{\sum_{i=1}^2 \gamma_i (M_t^i)^\eta\}^{1/\eta}(e_t^*)^{-1/\eta}]$. It is then easy to obtain that $R^t = \frac{\mu}{\eta} - \frac{\sigma^2}{2\eta} + \frac{(\delta)^2}{2} - \frac{1}{t}\ln E[(\gamma_1 e^{(\eta\delta-\sigma)W_t} + \gamma_2 e^{(-\eta\delta-\sigma)W_t})^{1/\eta}]$. For $\eta = 1/2$, we have $R^t = 2\mu - \sigma^2 + \frac{(\delta)^2}{2} - \frac{1}{t}\ln V_t$ with

$$V_t = E\big[(\gamma_1)^2 e^{2(\delta/2-\sigma)W_t} + (\gamma_2)^2 e^{-2(\delta/2+\sigma)Wt} + 2\gamma_1\gamma_2 e^{-2\sigma W_t}\big].$$

We have then

$$\begin{aligned} V_t &= (\gamma_1)^2 e^{2(\delta/2-\sigma)^2 t} + (\gamma_2)^2 e^{2(\delta/2+\sigma)^2 t} + 2\gamma_1\gamma_2 e^{2\sigma^2 t} \\ &= e^{2\sigma^2 t}\big((\gamma_1)^2 e^{((\delta)^2/2-2\sigma\delta)t} + (\gamma_2)^2 e^{((\delta)^2/2+2\sigma\delta)t} + 2\gamma_1\gamma_2\big), \end{aligned}$$

hence

$$R^t = 2\mu - 3\sigma^2 - \frac{1}{t}\ln\big((\gamma_1)^2 e^{-2\delta\sigma t} + (\gamma_2)^2 e^{2\delta\sigma t} + 2\gamma_1\gamma_2 e^{-\frac{1}{2}(\delta)^2 t}\big).$$

The result on R^∞ comes from Proposition 4.1. The result on R^0 comes from taking the limit when t converges to 0 in the expression of R^t. □

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