The method of characteristics

Mean Field Games for dummies

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THE METHOD OF CHARACTERISTICS

Consider a quasilinear system of nonlinear PDEs of the first order for the K functions $u^k(t,x)$ of time t and the N variables $\left(x^1,\dots,x^N\right)$:

$$
\begin{aligned}
0 &= \frac{\partial u^k}{\partial t} + \sum_{n=1}^{N} a^n(t,x,u)\frac{\partial u^k}{\partial x^n} + f^k(t,x,u) \\
0 &\leq t \leq T, \quad u(T,x) = u_T(x)
\end{aligned}
$$

Note that the coefficients a^n are the same for all the K equations (always true when $K=1$)

The characteristics are the solutions of the following system of $N+K$ ODEs:

$$
\begin{aligned}
\frac{d\xi^n}{dt} &= a^n(t,\xi,\eta), \quad 1 \leq n \leq N \\
\frac{d\eta^k}{dt} &= -f^k(t,\xi,\eta), \quad 1 \leq k \leq K
\end{aligned}
$$

Let $(\xi(t), \eta(t))$ be a characteristic. Along (ξ, η) we have:

$$\frac{d}{dt} u^k(t, \xi(t)) = \frac{\partial u^k}{\partial t} + \frac{\partial u^k}{\partial x} \cdot \frac{d\xi}{dt} = -f^k = \frac{d\eta^k}{dt}$$

Hence:

$$u^k(t, \xi(t)) = \eta(t) - \eta(t_0) + u^k(t_0, \xi(t_0))$$

- if $u(t_0, x)$ is known, we set $x(t_0) = x$ and $\eta(t_0) = u(t_0, x)$, and we recover $u(t, x)$ for all (t, x)
- the method is simple and effective
- requires that the field of characteristics is nice (no folds)

Hamilton-Jacobi-Bellman

The royal road of Caratheodory

Classical HJB

Write a simple HJB with convex Hamiltonian $h(x, u)$, not depending on current value V

$$\frac{\partial V}{\partial t} + h\left(x, \frac{\partial V}{\partial x}\right) + f(x) = 0, \quad V(x, T) = V_T(x)$$

This is not a quasi-linear equation.
To get one, we differentiate wrt x^n. Setting $\frac{\partial V}{\partial x^n} = u_n$ and using equality of cross-derivatives

$$\frac{\partial u_n}{\partial t} + \frac{\partial h}{\partial x^n}(x, u) + \sum \frac{\partial h}{\partial u_k}(x, u) \frac{\partial u_n}{\partial x^k} + \frac{\partial f}{\partial x^n} = 0, \quad 1 \leq n \leq N$$

The latter is a system of quasi linear equations. Since the coefficients $\frac{\partial h}{\partial u_k}(x, u)$ do not depend on n, we have a notion of characteristics. To keep matters simple, we will assume $h(u)$ does not depend on x

Characteristics

The characteristics are defined by:

$$\begin{aligned}\frac{d\xi^n}{dt} &= \frac{\partial h}{\partial u_n}\left(u\left(t,\xi\right)\right), \quad 1 \leq n \leq N \\ \frac{d\eta_n}{dt} &= -\frac{\partial f}{\partial x^n}\left(\xi\left(t\right)\right), \quad 1 \leq n \leq N\end{aligned}$$

If we want to find the characteristic going through x_0 at time t_0, we have to solve the boundary-value problem:

$$\begin{aligned}\xi\left(t_0\right) &= x_0 \\ \eta\left(T\right) &= V_T'\left(\xi\left(T\right)\right)\end{aligned}$$

Note that this is not a Cauchy (initial-value) problem.

The characteristics are maximizers

Let (ξ, η) solve the preceding BVP. Let $\zeta(t)$ be another path starting from x_0 at time t_0. Since V solves HJB:

$$\begin{aligned}
-f(\zeta) + h^*\left(\frac{d\zeta}{dt}\right) &= \frac{\partial V}{\partial t}(t,\zeta) + h\left(\frac{\partial V}{\partial x}(t,\zeta)\right) + h^*\left(\frac{d\zeta}{dt}\right) \\
&\geq \frac{\partial V}{\partial t}(t,\zeta) + \left(\frac{\partial V}{\partial x}(t,\zeta), \frac{d\zeta}{dt}\right) = \frac{d}{ds}V(\zeta(s))
\end{aligned}$$

Integrating, we get:

$$\int_{t_0}^{T}\left[f(\zeta) - h^*\left(\frac{d\zeta}{dt}\right)\right]ds + V_T(\zeta(T)) \leq V(t_0, x_0)$$

If $\zeta = \xi$, then equality is achieved:

$$\int_{t_0}^{T}\left[f(\xi) - h^*\left(\frac{d\xi}{dt}\right)\right]ds + V_T(\xi(T)) = V(t_0, x_0)$$

The royal road of Caratheodory

We have proved that the characteristics (ξ, η), with appropriate boundary conditions, solve the problem:

$$\max_{\zeta} \left\{ \int_{t_0}^{T} \left[f(\zeta) - h^* \left(\frac{d\zeta}{dt} \right) \right] ds + V_T(\zeta(T)) \mid \zeta(t_0) = x_0 \right\}$$

$\xi(t)$ is the optimal path, and $\eta(t)$ is the adjoint (Pontrjagin) variable along the path. So the problem can be solved either (a) by solving HJB, with the terminal condition $V(T, x) = V_T(x)$ or (b) by solving the two-point boundary value problem for the characteristics. The latter is often more difficult than the former

Mean Field Games
(J.M. Lasry, P.L. Lions)

MFG

We are looking at a continuous family of identical individuals, with total mass 1, trying to place themselves on R^N. Each of them controls his position by a diffusion, where the trend $a(t)$ is his control and the volatility σ is constant and given:

$$dx = a_t dt + \sigma dW$$

All BM are independent. At time t, the law of the family is $\mu(t)$ (the crowd). It will be supposed to have a density $m(t,x) > 0$
Each individual, starting at $x(t_0) = x_0,\ \mu(t_0) = \mu_0$.is striving to maximise the criterion:

$$\mathcal{V}(t_0, x_0, \mu_0) = \max_a \mathrm{E}\left[V_T(x(T), \mu(T)) + \int_t^T (h(x_t, \mu_t) - g(x_t, a_t))\, ds\right]$$

Note that this criterion (a) is the same for everyone, but (b) depends on where a particular individual is, and where the others are. The simplifying assumption here is that it depends only on the distribution of individuals on R.

Individual rationality

This is the anonymity assumption: there is no strategic interaction between individuals, there is interaction of each individual with the crowd. The other simplyfing assumption is that each individual knows he is too small to influence the distribution μ. So there is no strategic behaviour, and each individual takes the evolution of $\mu(t)$ as given.
$\mathcal{V}(t, x, \mu)$ must satisfy the HJB equation, where

$$\begin{aligned} 0 &= \frac{\partial \mathcal{V}}{dt} + \max_{a(x)} \left\{ \frac{\partial \mathcal{V}}{\partial x} \cdot a + \frac{\sigma^2}{2} \Delta_x \mathcal{V} + h(x, \mu) - g(x, a) + \frac{\partial \mathcal{V}}{\partial \mu} \cdot \frac{d\mu}{dt} \right\} \\ &= \frac{\partial \mathcal{V}}{dt} + \frac{\sigma^2}{2} \Delta_x \mathcal{V} + h(x, \mu) + g^* \left(x, \frac{\partial \mathcal{V}}{\partial x} \right) + \frac{\partial \mathcal{V}}{\partial \mu} \cdot \frac{d\mu}{dt} \end{aligned}$$

so that:

$$\bar{a}(t, x, \mu) = \frac{\partial g^*}{\partial v} \left(\frac{\partial \mathcal{V}}{\partial x} \right)$$

The master equation

Note the additional term. It is given by the Fokker-Planck equation:

$$\frac{dm}{dt} = \mathrm{div}_x\left(m\bar{a}\right) + \frac{\sigma^2}{2}\Delta_x m, \quad m\left(0\right) = m_0$$

Substituting, we get the master equation :

$$\begin{aligned} 0 \quad = \quad & \frac{\partial \mathcal{V}}{dt} + \frac{\sigma^2}{2}\Delta_x \mathcal{V} + h\left(x, \mu\right) + g^*\left(x, \frac{\partial \mathcal{V}}{\partial x}\right) \\ & + \frac{\partial \mathcal{V}}{\partial \mu} \cdot \left(\mathrm{div}_x\left(m \frac{\partial g^*}{\partial v}\left(\frac{\partial \mathcal{V}}{\partial x}\right)\right) + \frac{\sigma^2}{2}\Delta_x m\right) \end{aligned}$$

This can be seen as quasi-linear system of first-order equation for $\mathcal{V}\left(t, \mu\right)$ (one equation for each x) There are boundary conditions:

$$\mathcal{V}\left(T, x, \mu\right) = V_T\left(x, \mu\right)$$

Characteristics

The characteristics are given by:

$$\begin{aligned}\frac{\partial m}{\partial t} &= \mathrm{div}_x\left(m\bar{a}\right) + \frac{\sigma^2}{2}\Delta_x m \\ \frac{\partial u}{\partial t} &= -h\left(x,\mu\right) - g^*\left(x, \frac{\partial \mathcal{V}}{\partial x}\right) - \frac{\sigma^2}{2}\Delta_x m\end{aligned}$$

for $m\left(t,x\right)$ and $u\left(t,x\right)$, with the boundary conditions:

$$\begin{aligned}\mu\left(0\right) &= \mu_0 \\ u\left(T,x\right) &= V_T\left(x,\mu\left(T\right)\right)\end{aligned}$$

This is the famous forward-backward system of MFG. This is now subsumed in the direct study of the master equation

When is MFG $=$ HJB ?

(The invisible hand)

The planner's problem

The planner has the means to control the individual decisions $a_t(x)$ and has a collective criterion. Starting from $m(t_0, x) = m_0(x)$, the value is:

$$
\begin{aligned}
V(t_0, m_0) &= \max \left\{ \int_0^T \left(f(\mu_t) - \int_X g(a_t(y))\, d\mu_t \right) dt + V_T(m_T) \right\} \\
\frac{\partial m}{\partial t} &= \operatorname{div}_x (m \bar{a}_t) + \frac{\sigma^2}{2} \Delta_x m, \text{ with } \bar{a} = \frac{\partial g^*}{\partial v} \left(\frac{\partial \mathcal{V}}{\partial x} \right)
\end{aligned}
$$

HJB is then:

$$
\begin{aligned}
0 &= \frac{\partial V}{\partial t} + f(\mu) + \\
&\quad \max_a \left\{ \frac{\partial V}{\partial m} \cdot \left(\operatorname{div}_x (ma) + \frac{\sigma^2}{2} \Delta_x m \right) - \int_X g(a(y))\, d\mu \right\}
\end{aligned}
$$

Integrating by parts, we get the planner's HJB:

$$
0 = \frac{\partial V}{\partial t} + f(\mu) + \frac{\partial V}{\partial m} \cdot \frac{\sigma^2}{2} \Delta_x m + \int_X g^* \left(-\frac{\partial}{\partial x} \frac{\partial}{\partial m} V \right) d\mu
$$

Finding the characteristics

$$\mathcal{V} := \frac{\partial V}{\partial \mu}$$

Differentiating HJB, we get:

$$\begin{aligned} 0 \quad = \quad & \frac{\partial \mathcal{V}}{\partial t} + f'(\mu) + \frac{\sigma^2}{2}\Delta_x \mathcal{V} + \frac{\partial \mathcal{V}}{\partial m} \cdot \frac{\sigma^2}{2}\Delta_x m + \\ & \frac{\partial}{\partial m} \max_a \left\{ \int_X \left[-\left(\frac{\partial}{\partial x}\frac{\partial}{\partial m} V \right) a - g(a) \right] m dx \right\} \end{aligned}$$

Using the envelope theorem, the last term becomes:

$$\begin{aligned} \int_X \left[\left(\frac{\partial \mathcal{V}}{\partial m} \right) \mathrm{div}\,(\bar{a}m) - \left(\frac{\partial \mathcal{V}}{\partial x} \right) \bar{a} - g(\bar{a}) \right] dx \quad & = \\ \int_X \left[\left(\frac{\partial \mathcal{V}}{\partial m} \right) \mathrm{div}\,(\bar{a}m) \right] dx + g^* \left(-\frac{\partial \mathcal{V}}{\partial x} \right) \quad & = \end{aligned}$$

The comparison

We finally get the equation:

$$\begin{aligned} 0 &= \frac{\partial \mathcal{V}}{\partial t} + f'\left(\mu\right) + \frac{\sigma^2}{2}\Delta_x \mathcal{V} + \frac{\partial \mathcal{V}}{\partial m} \cdot \frac{\sigma^2}{2}\Delta_x m + \\ &\quad + g^* \left(-\frac{\partial \mathcal{V}}{\partial x} \right) + \int_X \left[\left(\frac{\partial \mathcal{V}}{\partial m} \right) \mathrm{div}\left(\bar{a} m \right) \right] dx \end{aligned}$$

the characteristics of which are the optimal trajectories for the planner. Compare this with the master equation for the MFG which we obtained earlier:

$$\begin{aligned} 0 &= \frac{\partial \mathcal{V}}{dt} + \frac{\sigma^2}{2}\Delta_x \mathcal{V} + h\left(x, \mu\right) + g^* \left(x, \frac{\partial \mathcal{V}}{\partial x} \right) \\ &\quad + \frac{\partial \mathcal{V}}{\partial \mu} \cdot \left(\mathrm{div}_x \left(m \frac{\partial g^*}{\partial v} \left(\frac{\partial \mathcal{V}}{\partial x} \right) \right) + \frac{\sigma^2}{2} \Delta_x m \right) \end{aligned}$$

Theorem

The solution to the planner's problem, starting from $\mu(0) = \mu_0$

$$\max \left\{ \int_0^T \left(f(\mu_t) - \int_X g(a_t(y))\, d\mu_t \right) dt + V_T(m_T) \right\}$$

can be implemented by the MFG:

$$\max_a \mathrm{E} \left[V_T(x(T), \mu(T)) + \int_t^T (f'(\mu_t) - g(-a_t))\, ds \right]$$

An example

Take $f(\mu) = -\frac{1}{2}\int m^2 dx$ and $g(a) = \frac{1}{2}a^2$. Then $h(\mu) = -m$. The (stochastic) individual programs:

$$\min_a \mathrm{E}\left[V_T\left(x(T), \mu(T)\right) + \int_t^T \left(m_t(x_t) + \frac{1}{2}a_t^2 \right) dt \right]$$
$$dx = a_t dt + \sigma dW_t, \quad \frac{dm}{dt} = \mathrm{div}_x (m\bar{a}) + \frac{\sigma^2}{2}\Delta_x m$$
$$x(t_0) = x_0, \ \mu(t_0) = \mu_0$$

realize the optimum of the (deterministic) collective program:

$$\min\left\{ \frac{1}{2}\int_0^T \left(\int_X m_t^2 dx + \int_X a_t^2 m_t dx \right) dt + V_T(m_T) \right\}$$
$$m(t_0, x) = m_0(x)$$