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SYMPOSIUM

Notes on optimal transportation

Ivar Ekeland

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Abstract These are introductory lectures to the mathematical theory of optimal transportation, and its connections with the economic theory of incentives.

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1 Introduction

These notes are based on a course I taught at the University of British Columbia. They provide an introduction to the mathematical theory of optimal transportation for students and researchers in economics. This theory has deep roots in the past, since it originates with the French geometer Gaspard Monge (1746–1818), who asked the following question: what is the most economical way of transferring mass from one place to another? More precisely, given two shapes Θ and X with equal volume, find a measure-preserving map $\xi : \Theta \longrightarrow X$ which minimizes the integral:

$$\int_{\Theta} \|\xi(\theta) - \theta\| \, d\theta.$$

In fact, Monge's problem is quite difficult (although it does have a solution) and will not be solved in these notes. We will, however, be able to solve the same problem for the integral:

I. Ekeland (✉)
Canada Research Chair in Mathematical Economics, University of British Columbia,
Vancouver, BC V6T 1Z2, Canada
e-mail: ekeland@math.ubc.ca

$$\int_{\Theta} \|\xi(\theta) - \theta\|^{\alpha} \, d\theta$$

where α can take any positive value except, of course, $\alpha = 1$. The original Monge problem was stated in 1781 (Monge 1781), and had to wait over two hundred years for a solution: the existence of a solution for $\alpha = 1$ was first proved by Sudakov (1979), using probability theory. Since then other proofs have been given by Evans and Gangbo (1999) on one side, and Trudinger and Wang (2001) on the other.

Meanwhile the original problem of Monge had been revived by Kantorovich (1942): he found a very interesting dual formulation of the problem, which enabled him to prove the existence of a generalized solution: the Kantorovich solution is not a map, that is it does not associated with every θ a point x in X, it is a conditional probability, that is, it associates with every θ a probability P_θ on X. This idea, and notably the dual formulation, turned out to be extremely fruitful. The breakthrough came with the work of Brenier (1991) in 1991, who shifted attention to the case of non-linear transport costs, of the type $\|x - \theta\|^{\alpha}$ with $0 < \alpha < 1$ (concave case) or $\alpha > 1$ (convex case), and more generally $u(\theta, x)$, and proved some remarkable results: he showed that instead of looking for a measure-preserving map $\xi(\theta)$, it was enough to look for a function $V(\theta)$ with some special property related to convexity. $V(\theta)$ is usually referred to as the potential, and once the potential has been found, the transportation map can be derived from its gradient V'. This was a considerable simplification: finding a measure-preserving map from a d-dimensional domain to another means finding d functions of d variables satisfying some very complicated conditions on their derivatives (the Jacobian determinant should be equal to 1), while finding the potential means finding a single function of d variables, satisfying some simple inequalities.

Remarkably, economic theory had been progressing on parallel lines. Theory had moved away from the Arrow-Debreu paradigm of complete information, and exploring the consequences of informational asymmetry. It is by now well-known that this splits into two branches, moral hazard and adverse selection. Because of the mathematical difficulties, work quickly focused on the case when there are finitely many types (typically two, the high type $\bar{\theta}$ and the low type $\underline{\theta}$), or when there is a one-dimensional continuum of types, $\theta \in [\underline{\theta}, \bar{\theta}]$ (see for instance Laffont and Tirole (1993) and the references therein). A few pioneers, however, had made inroads on the case when there is a d-dimensional continuum of types, notably J.C. Rochet (see Rochet and Stole (2003) and the references therein). The connection between their work and the mathematical theory of optimal transportation was first noticed by myself in the late nineties, after listening to a series of lectures by Robert McCann in Paris. The purpose of the present notes is to bring the two theories together in a readable way.

These notes are structured as follows. In the first section, we will explain the optimal transportation problem and its connection with the adverse selection problem in the discrete case: there are no mathematical difficulties, and the potential function $V(\theta)$ is conjured up relatively easily.[1] The potential function satisfies a special

[1] Once the idea is there, of course! I do not know where this idea originates: I first met it in the PhD thesis of J.C. Rochet, who attributes it to R.T. Rockafellar.

property, call u-convexity, which boils down to standard convexity in the case when $u(\theta, x) = \sum \theta_i x^i$, and the second section is devoted to the analysis of u-convex functions. Finally, the third section studies the continuous case: we proves Brenier's main results for the optimal transportation problem, and we describe Rochet and Choné's model for a principal facing a d-dimensional continuum of agents.

These note lay no claim to be exhaustive. We refer to Villani (2003) for optimal transportation and to Bolton and Dewatripont (2005) for contract theory, including adverse selection, but there are many other books and papers.

2 The discrete case

2.1 Optimal transportation

We are given two finite sets Θ and X, with the same number of elements, and a function $u : \Theta \times X \longrightarrow \mathbb{R}$. The elements $\theta \in \Theta$ and $x \in X$ are understood as locations. On each $\theta \in \Theta$ is stored one unit of a homogeneous good, which is to be transporte to some new location $x \in X$ at minimal cost. Each location in Θ and X can hold at most one unit of the good (so that, in fact, the transportation map from Θ to X will be a bijection) and $u(\theta, x)$.is the cost of transporting one unit from θ to x. The total cost is the sum of the individual costs.

Denote by $\mathcal{B}$. the set of all bijections from Θ to X. A bijections $\xi \in \mathcal{B}$ is *optimal* if it minimize:

$$\sum_{\theta\in\Theta} u(\theta, \xi(\theta)) \leq \sum_{\theta\in\Theta} u(\theta, \xi'(\theta)) \quad \text{for all} \quad \xi' \in \mathcal{B}.$$

If ξ is optimal, it will often be called the *transportation map*. Since the set of bijections is finite (albeit large), it is obvious that such an optimal map exists. We will now try to characterize it.

An N*-chain* is a sequence $\{\theta_0, \dots, \theta_N\} \subset \Theta$. It is *closed* if $\theta_0 = \theta_N$ so that it folds back upon itself. A closed N-chain is also called an N*-cycle*. A *chain* is an N-chain, for some finite N.

Proposition 1 *A bijection $\xi : \Theta \to X$ is optimal iff, for every cycle $\{\theta_0, \dots, \theta_N = \theta_0\}$, we have:*

$$\sum_{n=0}^{N-1} \left[u(\theta_n, \xi(\theta_n)) - u(\theta_n, \xi(\theta_{n+1}))\right] \geq 0. \tag{A}$$

Proof (Sufficiency) Let ξ be a bijection satisfying condition (A), and let ζ be another bijection from Θ to X. Pick any point $\theta_0 \in \Omega$, and define the θ_n, $n \geq 1$, recursively by $\zeta(\theta_n) = \xi(\theta_{n+1})$. Since Θ is a finite set, the θ_n cannot all be different. Let m be the lowest integer such that $\theta_m = \theta_N$ for some $N > m$. If $m \neq 0$, we have:

$$\zeta(\theta_{m-1}) = \xi(\theta_m) = \xi(\theta_N) = \zeta(\theta_{N-1}).$$

By the definition of m, it follows that $\theta_{m-1} \neq \theta_{N-1}$, and the above equality contradicts the fact that ζ is a bijection. So we must have $m = 0$, and $\Gamma_1 = \{\theta_0, \dots, \theta_N\}$ is a cycle. Because of condition (A), we have:

$$\sum_{\theta \in \Gamma_1} [u(\theta, \xi(\theta)) - u(\theta, \zeta(\theta))] = \sum_{n=0}^{N-1} \left[u(\theta_n, \xi(\theta_n)) - u(\theta_n, \xi(\theta_{n+1}))\right] \geq 0.$$

If Γ_1 is the whole of Θ, this concludes the proof. If not, pick another point $\theta_0' \notin \Gamma_1$, and go through the same procedure. We get another cycle Γ_2, which has no common point with Γ_1. Proceeding in this way, we can partition Θ in a disjoint union of cycles Γ_k, $1 \leq k \leq K$, and:

$$\sum_{\theta \in \Theta} [u(\theta, \xi(\theta)) - u(\theta, \zeta(\theta))] = \sum_{k=1}^{K} \sum_{\theta \in \Gamma_k} [u(\theta, \xi(\theta)) - u(\theta, \zeta(\theta))]$$

and the right-hand side is non-negative by condition (A). This means that ξ is optimal. □

Proof (Necessity) Assume a bijection ξ is optimal. Pick any cycle $\Gamma = \{\theta_0, \dots, \theta_N\}$ in Θ. Define a new map $\zeta : \Theta \to X$ as follows:

$$\zeta(\theta_n) = \xi(\theta_{n+1}) \quad \text{for } 0 \leq n \leq N-1$$
$$\zeta(\theta) = \xi(\theta) \quad \text{if } \theta \notin \Gamma.$$

Then ζ is still a bijection, and since ξ is optimal we must have:

$$\sum_{\theta \in \Gamma} u(\theta, \xi(\theta)) \geq \sum_{\theta \in \Gamma} u(\theta, \zeta(\theta)).$$

Replacing $\zeta(\theta)$ by its values, we get exactly condition (A). □

Note for future reference that condition (A) can be rewritten as follows:

$$\sum_{n=1}^{N} \left[u(\theta_n, \xi(\theta_n)) - u(\theta_{n-1}, \xi(\theta_n))\right] \geq 0, \quad \theta_0 = \theta_N.$$

2.2 Potentials

In the following, we fix a point $\bar{\theta} \in \Theta$.

Proposition 2 *Suppose a map $\xi : \Theta \to \Theta$ (not necessarily a permutation) satisfies condition* (A). *Then the formula*:

$$f(\theta) := \inf\left\{\sum_{n=1}^{N}\left[u(\theta_n,\xi(\theta_n)) - u(\theta_{n-1},\xi(\theta_n))\right] \Big| \{\theta_0,\ldots,\theta_N\}\in\Omega,\ \theta_0=\bar{\theta},\ \theta_N=\theta\right\} \tag{1}$$

defines a real-valued function f on Ω with the following properties:

1. $f(\bar{\theta}) = 0$
2. $f(\theta') \geq f(\theta) + u(\theta', x(\theta)) - u(\theta, x(\theta))\ \ \forall\theta,\theta'$.

Any such function f is called a potential *associated with the map x.*

Proof Since any chain starting and ending at $\bar{\theta}$ must be closed, we have:

$$f\left(\bar{\theta}\right) \geq \inf\left\{\sum_{n=1}^{N}\left[u(\theta_n, x(\theta_n)) - u(\theta_{n-1}, x(\theta_n))\right] \mid \{\theta_0,\ldots,\theta_N\} \in \Omega_0\right\}.$$

The right-hand side is non-negative by condition (A), so $f\left(\bar{\theta}\right) \geq 0$. On the other hand, taking the trivial 1-path $\theta_0 = \bar{\theta} = \theta_1$ yields $f\left(\bar{\theta}\right) \leq 0$, so we have $f\left(\bar{\theta}\right) = 0$.

Formula (1) defines f as function with values in $\mathbb{R}\cup\{-\infty\}$. Pick two points θ and θ'. We shall consider special chains connecting $\bar{\theta}$ to θ, those whose last leg is $\left\{\theta',\theta\right\}$. From the definition of f, we have:

$$\begin{aligned}
f(\theta) &\leq \inf\left\{\sum_{n=1}^{N}\left[u(\theta_n, x(\theta_n)) - u(\theta_{n-1}, x(\theta_n))\right] \mid \theta_0 = \bar{\theta},\ \theta_{N-1} = \theta',\ \theta_N = \theta\right\}\\
&= \inf\left\{\sum_{n=1}^{N-1}\left[u(\theta_n, x(\theta_n)) - u(\theta_{n-1}, x(\theta_n))\right] + u(\theta, x(\theta))\right.\\
&\qquad\left. - u\left(\theta', x(\theta)\right) \mid \theta_0 = \bar{\theta},\ \theta_{N-1} = \theta'\right\}\\
&= \inf\left\{\sum_{n=1}^{N-1}\left[u(\theta_n, x(\theta_n)) - u(\theta_{n-1}, x(\theta_n))\right] \mid \theta_0 = \bar{\theta},\ \theta_{N-1} = \theta'\right\}\\
&\quad + u(\theta, x(\theta)) - u\left(\theta', x(\theta)\right)\\
&= f(\theta') + u(\theta, x(\theta)) - u\left(\theta', x(\theta)\right)
\end{aligned}$$

and this inequality holds for every θ and θ'. Applying it to $\theta = \bar{\theta}$, we get:

$$\begin{aligned}
f(\theta') &\geq u(\theta', x(\bar{\theta})) - u(\bar{\theta}, x(\bar{\theta})), \quad \forall\theta'\\
&\geq \inf\left\{u(\theta', x(\bar{\theta})) - u(\bar{\theta}, x(\bar{\theta})) \mid \theta \in \Theta\right\}
\end{aligned}$$

and since the set Θ is finite, the right-hand side is bounded. So the function f is real-valued, and satisfies the desired inequality. □

Note for future use that equality is achieved if $\theta = \theta'$, so that inequality (2) can be rewritten:

$$f(\theta') = \sup_{\theta} \left\{ u(\theta', x(\theta)) - u(\theta, x(\theta) + f(\theta)) \right\} = \sup_{\theta} \left\{ u(\theta', x(\theta)) + t(\theta) \right\}. \quad (2)$$

Finally, note that the potential depends on the initial choice of $\bar{\theta}$: different choices lead to different potentials.

2.3 Adverse selection

Remarkably, there is another economic interpretation, not in terms of transporting goods from one place to another, but in terms of contract theory and adverse selection. We now see Θ as a set of types (note that a several individuals may belong to the same type) and X as a set of tasks; however, we no longer require that Θ and X have the same number of elements. An individual of type θ performing task x and getting paid a sum t will have a total utility of

$$u(\theta, x) + t$$

Tasks are being dispensed by a third party, the *principal*: he/she does so by publishing a list of tasks (also called a menu in the literature), each of which carries a salary; any such pair (x, t), where x is a task and t is a salary, is called a *contract*. Each agent then chooses in the list the contract he/she prefers: denote by $(\xi(\theta), t(\theta))$ the contract chosen by agents of type θ (assuming they all choose the same contract). We thereby get a map:

$$(\xi, t) : \Theta \to X \times \mathbb{R}$$

which is called a *contract line* (or menu). Such a contract line must satisfy the property:

$$u(\theta, \xi(\theta)) + t(\theta) \geq u(\theta, \xi(\theta')) + t(\theta') \quad \forall (\theta, \theta') \quad \text{(IC)}$$

which simply expresses the fact that individuals of type θ do not prefer to their own contract $(\xi(\theta), t(\theta))$ the contract $(\xi(\theta'), t(\theta'))$ that individuals of type θ' have chosen [otherwise they have chosen $(\xi(\theta'), t(\theta'))$ in the first place].

An *allocation* $\xi : \Theta \to X$ will be called *incentive-compatible* if there is a map $t : \Theta \to \mathbb{R}$ such that the contract (x, t) satisfies (IC). It turns out that ξ is incentive-compatible if and only if it satisfies condition (A).

Proposition 3 *A map $\xi : \Theta \to X$ is an incentive-compatible allocation iff it satisfies condition* (A).

Proof (Necessity) Assume ξ is incentive-compatible, and let t be the corresponding payment. Pick any cycle $(\theta_0, \dots, \theta_N)$, so that $\theta_0 = \theta_N$. We have:

$$\begin{aligned} u(\theta_0, \xi(\theta_0)) + t(\theta_0) &\geq u(\theta_0, \xi(\theta_1)) + t(\theta_1) \\ u(\theta_1, \xi(\theta_1)) + t(\theta_1) &\geq u(\theta_1, \xi(\theta_2)) + t(\theta_2) \\ &\cdots \\ u(\theta_{N-1}, \xi(\theta_{N-1})) + t(\theta_{N-1}) &\geq u(\theta_{N-1}, \xi(\theta_N)) + t(\theta_N). \end{aligned}$$

Summing up, and remembering that $t(\theta_0) = t(\theta_N)$, we find that all the terms in t cancel out, and we are left with condition (A). □

Proof (Sufficiency) Assume ξ satisfies condition (A). Let f be the potential associated with x. Set:

$$t(\theta) = f(\theta) - u(\theta, \xi(\theta)).$$

Writing this into inequality (2) in Proposition 2 we get exactly condition (A). □

Corollary 4 *If Θ and X have the same number of elements, then any incentive-compatible allocation solves the optimal transportation problem.*

2.4 An informal approach to the continuous case

The continuous version of the optimal transportation problem reads as follows. Given two subsets Θ and X of $\mathbb{R}^d$, endowed with the Borelian tribe and positive measures μ and ν satisfying $\mu(\Theta) = \nu(X)$, find, among all measure-preserving maps $\xi : \Theta \to X$, the one(s) which minimize the integral:

$$\int_\Theta u(\theta, \xi(\theta))\, \mathrm{d}\theta.$$

Replacing the sum in condition (A) by an integral, we obtain that an optimal map x must be such that:

$$\oint u_\theta(\theta, \xi(\theta))\, \mathrm{d}\theta_i \geq 0$$

along every closed loop. Running the loop in the other direction, we see that the inequality is in fact an equality. By the Poincaré lemma, this means that $u_\theta(\theta, \xi(\theta))$ is a gradient: there is a function $f : \Theta \to \mathbb{R}$ such that:

$$\frac{\partial u}{\partial \theta_i}(\theta, \xi(\theta)) = \frac{\partial f}{\partial \theta_i}.$$

So we find the potential again ! Note that if the map $x \to u_\theta(\theta, x)$ can be inverted, then the map $\xi(\theta)$ can be deduced from the function $f(\theta)$. In other words, instead of looking for a map ξ from $\mathbb{R}^d$ into itself, that is, d functions of d variables, we are looking for a single function f of d variables. In the sequel, we will find it as the solution of a bizarre optimization problem. Note that f has very special properties, which follow from formula (2), and which be useful. We describe them in the next section.

3 *u*-Convex analysis

We are given sets Θ and X (no longer finite) and a map $u : \Theta \times X \to \mathbb{R}$. Points in Θ will be denoted by θ, and points in X by x.

3.1 *u*-Convex functions

We will be dealing with function taking values in $\mathbb{R} \cup \{+\infty\}$. Such a function will be called *proper* if it is not identically $\{+\infty\}$.

A function $f : \Theta \to \mathbb{R} \cup \{+\infty\}$ will be called *u-convex* iff there exists a non-empty subset $A \subset X \times \mathbb{R}$ such that:

$$f(\theta) = \sup_{(x,t)\in A} \{u(\theta.x) + t\}. \tag{3}$$

A function $g : x \to \mathbb{R} \cup \{+\infty\}$ will be called *u-convex* iff there exists a non-empty subset $B \subset \Theta \times \mathbb{R}$ such that:

$$g(x) = \sup_{(\theta,t)\in B} \{u(\theta.x) + s\}.$$

3.2 *u*-Conjugates

Let $f : \Theta \to \mathbb{R} \cup \{+\infty\}$ be a proper function (not necessarily u-convex). We define its u-conjugate f^* by:

$$f^*(x) = \sup_{\theta} \{u(\theta, x) - f(\theta)\}. \tag{4}$$

Let $g : X \to \mathbb{R} \cup \{+\infty\}$ be a proper function (not necessarily u-convex). We define its u-conjugate g^* by:

$$g^*(\theta) = \sup_{x} \{u(\theta.x) - g(x)\}.$$

If f is proper, then f^* takes values in $\mathbb{R} \cup \{+\infty\}$ and is a *u-convex* function on X.

If g is proper, then g^* takes values in $\mathbb{R} \cup \{+\infty\}$ and is a *u-convex* function on Θ.

Example 5 Set $\varphi(\theta) = u(\theta, \bar{x}) + \bar{t}$. Then

$$\varphi^*(\bar{x}) = \sup_{\theta} \left\{u(\theta, \bar{x}) - u(\theta, \bar{x}) - \bar{t}\right\} = -\bar{t}.$$

Example 6 Let $\theta \to x(\theta)$ be a map satisfying condition (A). The associated potential f is u-convex: this follows from condition (2)

Conjugation reverses ordering: if $f_1 \leq f_2$, then $f_1^* \geq f_2^*$. As a consequence, if f is u-convex, then f^* is proper. Indeed, since f is u-convex, we have $f \geq \varphi$ for some function φ of this type, and then $f^*(\bar{x}) \leq -\bar{t} < \infty$.

Proposition 7 (The Fenchel inequality) *For any proper functions* $f : \Theta \to \mathbb{R} \cup \{+\infty\}$ *and* $g : X \to \mathbb{R} \cup \{+\infty\}$, *we have:*

$$f(\theta) + f^*(x) \geq u(\theta, x) \quad \forall(\theta, x)$$
$$g(x) + g^*(\theta) \geq u(\theta, x) \quad \forall(\theta, x).$$

3.3 u-Subgradients

Let $f : \Theta \to \mathbb{R} \cup \{+\infty\}$ be a proper u-convex function. Take some point $\theta \in \Theta$. We shall say that a point $x \in X$ is a *u-subgradient* of f at θ if:

$$f(\theta) + f^*(x) = u(\theta, x). \tag{5}$$

The set of subgradients of f at θ will be called the *subdifferential* of f at θ and denoted by $\partial_u f(\theta)$.

Similarly, let $g : X \to \mathbb{R} \cup \{+\infty\}$ be a proper convex function. Take some point $x \in X$. We shall say that a point $\theta \in \Theta$ is a *u-subgradient* of g at θ if:

$$g^*(\theta) + g(x) = u(\theta, x) \quad \forall(\theta, x).$$

Proposition 8 $x \in \partial_u f(\theta)$ *iff*

$$f(\theta') \geq f(\theta) + u(\theta', x) - u(\theta, x) \;\; \forall(\theta, x). \tag{6}$$

Proof (Necessity) Assume $x \in \partial_u f(\theta)$. Then, by (5), we have:

$$f(\theta') \geq u(\theta', x) - f^*(x) = u(\theta', x) - [u(\theta, x) - f(\theta)].$$

□

Proof (Sufficiency) We have:

$$\begin{aligned} f^*(x) &= \sup_{\theta'} \left\{u(\theta', x) - f(\theta')\right\} \\ &\leq \sup_{\theta'} \left\{u(\theta', x) - f(\theta) - u(\theta', x) + u(\theta, x)\right\} \\ &= u(\theta, x) - f(\theta) \end{aligned}$$

so $f(\theta) + f^*(x) \leq u(\theta, x)$. We have the converse by the Fenchel inequality, so equality holds. □

Example 9 Consider a map $\theta \to x(\theta)$. Then it satisfies condition (A) iff there is a u-convex function f such that $x(\theta)$ is a subgradient of f at θ.

3.4 u-Biconjugates

It follows from the Fenchel inequality that:

$$f^{**}(\theta) = \sup_x \left\{ u(\theta, x) - f^*(x) \right\} \leq f(\theta).$$

Example 10 Set $\varphi(\theta) = u(\theta, \bar{x}) + \bar{t}$. Then

$$\varphi^{**}(\theta) = \sup_x \left\{ u(\theta, x) - \varphi^*(x) \right\} \geq u(\theta, \bar{x}) + \bar{t} = \varphi(\theta)$$

and hence $\varphi^{**}(\theta) = \varphi(\theta)$.

Proposition 11 *For every proper function* $f : \Theta \to \mathbb{R}$, *we have*

$$f^{**}(\theta) = \sup_\varphi \{\varphi(\theta) \mid \varphi \leq f, \ \varphi \ u\text{-convex}\}.$$

Proof Denote by $\bar{f}$ the right-hand side of the above formula. We want to show that $f^{**} = \bar{f}$.

Since $f^{**} \leq f$ and f^{**} is u-convex, we must have $f^{**} \leq \bar{f}$.

To show that $f^{**} \geq \bar{f}$, since $\bar{f}$ is u-convex, it is enough to show that every u-affine function which is less than $\bar{f}$ is also less that f^{**}. Let $(\bar{x}, \bar{t})$ be such that $u(\theta, \bar{x}) + \bar{t} \leq \bar{f}(\theta)$. Since $\bar{f} \leq f$, we also have $u(\theta, \bar{x}) + \bar{t} \leq f(\theta)$. Taking biconjugates, we get $u(\theta, \bar{x}) + \bar{t} \leq f^{**}(\theta)$ as well. □

Corollary 12 *If f is convex, then $f^{**} = f$. and we have:*

$$f(\theta) = \sup_x \left\{ u(\theta, x) - f^*(x) \right\}. \tag{7}$$

Corollary 13 *The following are equivalent:*

1. $x \in \partial_u f(\theta)$
2. $\theta \in \partial_u f^*(x)$
3. $f(\theta) + f^*(x) = u(\theta, x)$.

3.5 Smoothness

In the following, and for the sake of commodity, we shall denote by u_θ the partial derivative of u with respect to the variables θ:

$$u_\theta(\theta, x) := \left(\frac{\partial u}{\partial \theta_1}(\theta, x), \ldots, \frac{\partial u}{\partial \theta_d}(\theta, x) \right)$$

and we shall assume the following:

Condition (C) Θ *is an open bounded subset of* $\mathbb{R}^n$ *and* X *is a compact subset of* $\mathbb{R}^m$. *The functions* u *and* u_θ *are well-defined and continuous, and they extend continuously to* $\bar{\Theta} \times X$.

The family $\left\{u(\theta, \bullet) \;\middle|\; \theta \in \bar{\Theta}\right\}$ is equicontinuous on X. So all u-convex functions on X are continuous on X. Because of the compactness of X, the supremum is attained in condition (7), so that they all u-convex functions on Θ are subdifferentiable everywhere on Θ.

The function $u_\theta(\theta, x)$ is uniformly bounded on $\bar{\Theta} \times X$. It follows that the family $\{u(\bullet, x) \mid x \in X\}$ is uniformly Lipschitz, and therefore all u-convex functions on $\bar{\Theta}$ are Lipschitz on Θ. By a theorem of Rademacher, they are differentiable almost everywhere with respect to the Lebesgue measure.

Proposition 14 *Let* f *be a* u*-convex function* f*, and* θ *a point where it is Frechet-differentiable with derivative* $\nabla f(\bar{\theta})$. *If* $\bar{x} \in \partial_u f(\bar{\theta})$*, we have:*

$$\nabla f(\bar{\theta}) = u_\theta(\bar{\theta}, \bar{x}).$$

Proof By Proposition 8, we have $f(\theta) \geq f(\bar{\theta}) + u(\theta, x) - u(\bar{\theta}, x)$ for all θ. Differentiating at $\theta = \bar{\theta}$, we get the result. □

3.6 Generalized Spence-Mirrlees

Definition 15 We shall say that u satisfies the *generalized Spence-Mirrlees* condition (henceforth GSM) if it satisfies condition (C), and for Lebesgue-a.e. $\theta \in \Theta$, the map $u_\theta(\theta, \bullet)$ is one-to-one:

$$u_\theta(\theta, x_1) = u_\theta(\theta, x_2) \Longrightarrow x_1 = x_2.$$

Example 16 If $\Theta = X = \mathbb{R}^n$, then $u(\theta, x) = \|\theta - x\|^\alpha$ satisfies (GSM) for $\alpha \neq 0$ and $\alpha \neq 1$.

Definition 17 Assume u satisfies GSM. Then, for every u-convex function f, there is a unique map $\theta \to \xi(\theta)$ such that

$$\nabla f(\theta) = u_\theta(\theta, \xi(\theta)) \text{ Lebesgue-a.e.}$$

We shall refer to ξ as the *u-subgradient* map of f.

The original condition formulated by Spence and Mirrlees was for a function u of two real variables θ and x. It required that:

$$\frac{\partial^2 u}{\partial \theta \partial x} < 0$$

so that the function $x \to u_\theta(\theta, x)$ is decreasing, and hence one-to-one. Condition (GSM) is an extension, which I learned from Guillaume Carlier.

4 The continuous case

We now treat the optimal transportation problem in the continuous case. In contrast with the discrete case, the existence of a solution now becomes highly non-trivial. We solve it by finding the potential $f(\theta)$ associated with the optimal transportation map: it is found by solving an optimization problem which we now state.

4.1 A bizarre optimization problem

We are given two Borel subsets $\Theta \subset \mathbb{R}^{d_1}$ and $X \subset \mathbb{R}^{d_2}$, endowed with positive and finite measures μ and ν. We posit the following optimization problem:

$$\inf \left[\int_{\Theta} f(\theta)\, d\mu + \int_X g(x)\, d\nu \right] \tag{8}$$

$$f(\theta) + g(x) \geq u(\theta, x) \quad \forall (\theta, x)\,. \tag{9}$$

Changing f to $f+a$ and g to $g-a$, for any constant a, does not affect the constraint, but changes the criterion by $[\mu(\Theta) - \nu(X)]\, a$. It follows that, for the problem to be meaningful, we must require that:

$$\mu(\Theta) = \nu(X)$$

and that, even then, optimal solutions, if they exist, are not unique. If (f, g) is optimal, so is $(f + a, g - a)$.

Proposition 18 *Assume Θ, X and u satisfy condition* (C) *and that $\mu(\Theta) = \nu(X)$. Then the problem* (8), (9) *has an optimal solution (f, g). If (f, g) is an optimal solution, then there is a u-convex function $\bar{f}$ such that $f = \bar{f}$ μ-a.e. and $g = \bar{f}^* \nu - a.e.$*

Proof Let (f_n, g_n) be a minimizing sequence:

$$\int_{\Theta} f(\theta)\, d\mu + \int_X g(x)\, d\nu \to \inf. \tag{10}$$

Then $f_n^* \leq g_n$ and (f, f_n^*) still satisfies the constraints, so (f, f_n^*) still is a minimizing sequence. Then $f_n^{**} \leq f_n$ and (f_n^{**}, f_n) still satisfies the constraints, so (f_n^{**}, f_n^*) is a minimizing sequence. The sequences f_n^{**} and f_n^* both are equicontinuous, and they will have uniformly convergent subsequences provided they are bounded (Ascoli's theorem).

Set $a_n = \min_{\Theta} f_n^{**}$, and consider the functions $h_n(\theta) = f_n^{**}(\theta) - a_n$, so that the h_n are u-convex and $h_n \geq 0$. Since Θ is compact, there is a point θ_n where $h_n(\theta_n) = 0$.

We have:

$$h_n^*(x) = \sup_\theta \{u(\theta, x) - h_n(\theta)\} \geq u(\theta_n.x) \geq \inf_{\Theta\times X} \{u(\theta, x)\}$$
$$h_n(x) = \sup_x \left\{u(\theta, x) - h_n^*(x)\right\} \leq \sup_{\Theta\times X} \{u(\theta, x)\} - \inf_{\Theta\times X} \{u(\theta, x)\}.$$

So the sequence h_n is uniformly bounded. A similar argument shows that the sequence h_n^* is uniformly bounded. By Ascoli's theorem, we can extract subsequences which converge uniformly to u-convex functions $\bar{f}$ and $\bar{f}^*$. Taking limits in (10), we get the result. □

Theorem 19 *Assume moreover that μ is absolutely continuous with respect to the Lebesgue measure and that u satisfies GSM. Then the subgradient map ξ of f, defined by:*

$$\nabla f(\theta) = \frac{\partial u}{\partial \theta}(\theta, \xi(\theta))$$

is a measure-preserving map from Θ to X :

$$\xi(\mu) = \nu$$

and if (f_1, g_1) and (f_2, g_2) are two optimal solutions, then $\xi(\theta) = \xi_2(\theta)$ μ-a.e.

Proof Let $(f = g^*, g)$ be an optimal solution of problem (8), (9). Take any continuous function φ on X. For any $h > 0$, we have:

$$\int_\Theta (g + h\varphi)^* \mathrm{d}\mu + \int_X (g + h\varphi) \mathrm{d}\nu \geq \int_\Theta g^* \mathrm{d}\mu + \int_X g \mathrm{d}\nu$$

which yields immediately:

$$\frac{1}{h}\int_\Theta \left[(g + h\varphi)^* - g^*\right] \mathrm{d}\mu + \int_X \varphi \mathrm{d}\nu \geq 0. \tag{11}$$

Now, let $\xi(\theta)$ and $\xi_h(\theta)$ be measurable selections of $\partial_u g$ and $\partial_u(g + h\varphi)$, that is, maps $\theta \to \xi(\theta)$ and $\theta \to \xi_h(\theta)$ such that

$$u_\theta(\theta, \xi(\theta)) = \frac{\partial g^*}{\partial \theta}(\theta) \quad \mu\text{-a.e.}$$
$$u_\theta(\theta, \xi_h(\theta)) = \frac{\partial}{\partial \theta}(g + h\varphi)^*(\theta) \quad \mu\text{-a.e.}$$

From the definition of g^*, we have:

$$u(\theta, \xi_h(\theta)) - g(\xi_h(\theta)) \leq g^*(\theta) = u(\theta, \xi(\theta)) - g(\xi(\theta))$$

so that:

$$u(\theta, \xi_h(\theta)) - g(\xi_h(\theta)) - u(\theta, \xi(\theta)) + g(\xi(\theta)) \leq 0. \tag{12}$$

From the definition of $(g + h\varphi)^*$, we have:

$$\begin{aligned} u(\theta, \xi(\theta)) - g(\xi(\theta)) - h\varphi(\xi(\theta)) &\leq (g + h\varphi)^*(\theta) \\ &= u(\theta, \xi_h(\theta)) - g(\xi_h(\theta)) - h\varphi(\xi_h(\theta)) \end{aligned}$$

from which we deduce, taking into account the fact that $\xi(\theta) \in \partial_u g^*(\theta)$:

$$h\varphi(\xi_h(\theta)) - h\varphi(\xi(\theta)) \leq u(\theta, \xi_h(\theta)) - g(\xi_h(\theta)) - u(\theta, \xi(\theta)) + g(\xi(\theta)) \tag{13}$$

$$= (g+h\varphi)^*(\theta) + h\varphi(\xi_h(\theta)) - g^*(\theta). \tag{14}$$

Finally, comparing (12), (13), and (14), we have:

$$-h\varphi(\xi(\theta)) \leq (g + h\varphi)^*(\theta) - g^*(\theta) \leq -h\varphi(\xi_h(\theta)). \tag{15}$$

Let us now use the assumptions. By Rademacher's theorem and GSM, the maps $\xi(\theta)$ and $\xi_h(\theta)$ are uniquely defined, up to a.e. equivalence. Letting $h \to 0$, and fixing θ, we find that, because of the compactness of X, the sequence $\xi_h(\theta)$ must have cluster points. Any cluster point $\xi(\theta)$ of $\xi_h(\theta)$ must satisfy:

$$u_\theta(\theta, \xi(\theta)) = g^*_\theta(\theta)$$

which defines $\xi(\theta)$ uniquely, by GSM. So, $\xi_h(\theta)$ converges to $\xi(\theta)$ almost everywhere.[2]

Dividing by h, taking the limit as $h \to 0$ and using the continuity of φ yields:

$$\lim_{h \to 0} \frac{1}{h} \left[(g + h\varphi)^*(\theta) - g^*(\theta) \right] = -\varphi(\xi(\theta)). \tag{16}$$

Taking limits in equation (11) yields:

$$-\int_\Theta \varphi(\xi(\theta))\, \mathrm{d}\mu + \int_X \varphi \mathrm{d}\nu \geq 0$$

and since this inequality holds both for φ and $-\varphi$, it must be an equality. So ξ sends μ on ν, as announced.

[2] There is a nicety here. We are applying Rademacher's theorem to each $(g + h\varphi)^*$, leaving aside a negligible set N_h which depends on h. We should take $h = 1/k$, with $k \to \infty$ an integer, so that the union $\cup_h N_h =: N$ is still negligible.

As for uniqueness, assume that (f_1, g_1) and (f_2, g_2) both are optimal solutions to problem (8), (9), with ξ_1 and ξ_2 being the corresponding subgradient maps. Then:

$$f_1(\theta) + g_1(\xi_2(\theta)) \geq u(\theta, \xi_2(\theta)) \quad \mu\text{-a.e.} \tag{17}$$

Integrating, and remembering that ξ_2 sends μ to ν, we get:

$$\begin{aligned}\int_\Theta f_1 \mathrm{d}\mu + \int_X (g_1 \circ \xi_2)\, \mathrm{d}\mu &= \int_\Theta f_1 \mathrm{d}\mu + \int_X g_1 \mathrm{d}\nu \\ &\geq \int_\Theta u(\theta, \xi_2(\theta))\, \mathrm{d}\mu(\theta).\end{aligned}$$

Since (f_2, g_2) is another optimal solution, we also have:

$$\begin{aligned}\int_\Theta f_1 \mathrm{d}\mu + \int_X g_1 \mathrm{d}\nu &= \int_\Theta f_2 \mathrm{d}\mu + \int_X g_2 \mathrm{d}\nu \\ &= \int_\Theta u(\theta, \xi_2(\theta))\, \mathrm{d}\mu(\theta).\end{aligned}$$

Comparing this with (17) we find that $f_1(\theta) + g_1(\xi_2(\theta)) = u(\theta, \xi_2(\theta)) \quad \mu$-a.e. So $\xi_1(\theta) = \xi_2(\theta)$ μ-a.e. □

Corollary 20 *If Θ is connected, if (f_1, g_1) and (f_2, g_2) are two solutions of problem (8), (9) then there is a constant a such that $f_2 = f_1 + a$ and $g_2 = g_1 - a$.*

Note that, if GSM does not hold, then Theorem 19 does not apply. The proof carries over up to equation (15). But now there is no reason why the functions $\xi_h(\theta)$ should converge a.e., and no good way to pick $\xi(\theta)$. In fact, as we stated in the Introduction, the original Monge problem, with $u(\theta, x) = \|\theta - x\|$, does have an optimal solution, although u does not satisfy GSM, but there is great difficulty in proving it. Only recently did we get a satisfactory proof.

On the other hand, if GSM does not hold, and/or μ is not continuous with respect to the Lebesgue measure, problem (8), and (9) still has a solution (f, g), which corresponds to generalized solutions of the optimal transportation problem (these are the solutions that were discovered by Kantorovich). To see this, note that (16) becomes:

$$\lim_{h \to 0} \sup \left[(g + h\varphi)^*(\theta) - g^*(\theta)\right] \leq -\min\{\varphi(x) \mid x \in \partial_u g(\theta)\}$$

and hence:

$$\int_\Theta \max\{\varphi(x) \mid x \in \partial_u g(\theta)\}\, \mathrm{d}\mu \geq \int_X \varphi \mathrm{d}\nu \geq \int_\Theta \min\{\varphi(x) \mid x \in \partial_u g(\theta)\}\, \mathrm{d}\mu.$$

It follows that the measure μ can be disintegrated as:

$$\mu = \int_X \pi_x \mathrm{d}\nu$$

where π_x is a probability measure on Θ. The interpretation is that all individuals of the same type do not do the same thing: $\pi_x(\theta)$ is the proportion of individuals of type θ which choose ξ. When GSM is satisfied, all individuals of type θ choose the same $x = \xi(\theta)$, so that $\pi_x(\theta)$ is the Dirac mass at $\xi(\theta)$.

4.2 Optimal transportation

Theorem 21 *Assume that Θ, X and u satisfy condition* (C)*, that Θ is endowed with a positive measure μ, absolutely continuous with respect to the Lebesgue measure, that X is endowed with a positive measure ν such that:*

$$\mu(\Theta) = \nu(X) < \infty$$

and that u satisfies GSM. Then the problem:

$$\max \int_\Theta u(\theta, \xi(\theta)) \, \mathrm{d}\mu(\theta) \tag{18}$$

$$\xi : \Theta \to X, \ \xi(\mu) = \nu \tag{19}$$

has a solution, and any two solutions are equal μ-a.e..

Proof We just consider the problem (8), (9) and apply Theorem 19. Let (f, g) be an optimal solution, and $\bar{\xi}$ be the subgradient map of f. Then $\bar{\xi}$ preserves measure, and if $\xi : \Theta \to X$ is another measure-preserving map, we have, by the Fenchel inequality:

$$\begin{aligned}\int_\Theta u(\theta, \xi(\theta)) \, \mathrm{d}\mu(\theta) &\leq \int_\Theta f(\theta) \, \mathrm{d}\mu(\theta) + \int_X g(\xi(\theta)) \, \mathrm{d}\mu(\theta) \\ &= \int_\Theta f(\theta) \, \mathrm{d}\mu(\theta) + \int_X g(x) \, \mathrm{d}\nu \\ &= \int_\Theta f(\theta) \, \mathrm{d}\mu(\theta) + \int_\Theta g\left(\bar{\xi}(\theta)\right) \mathrm{d}\nu(\xi) \\ &= \int_\Theta u\left(\theta, \bar{\xi}(\theta)\right) \mathrm{d}\mu(\theta)\end{aligned}$$

the latter equality expressing the fact that $\xi(\theta) \in \partial_u f(\theta)$, and the intermediate equalities using the fact that ξ and $\bar{\xi}$ are measure-preserving. □

We shall now give a theorem where θ and x play fully symmetric roles: GSM will be satisfied, not only with respect to θ, but also with respect to x. In that case the mapping x can be inverted (up to a.e. equivalence).

Theorem 22 *Let Θ be an open bounded subset of $\mathbb{R}^{d_1}$ and X be an open bounded subset of $\mathbb{R}^{d_2}$. Let μ and ν be positive measures on Θ and X respectively, both absolutely continuous with respect to the Lebesgue measure, and such that:*

$$\mu(\Theta) = \nu(X) < \infty$$

Let $u : \Theta \times X \to \mathbb{R}$ be continuously differentiable, and assume that all derivatives extend continuously to the closure $\bar{\Theta} \times \bar{X}$. Assume finally that u satisfies GSM with respect to the variables θ and x:

$$u_\theta(\theta, x_1) = u_\theta(\theta, x_2) \Longrightarrow x_1 = x_2$$
$$u_x(\theta_1, x) = u_x(\theta_2, x) \Longrightarrow \theta_1 = \theta_2$$

Then the problem

$$\max_{\xi} \int_{\Theta} u(\theta, s(\theta))\, \mathrm{d}\mu(\theta) \qquad \xi : \Theta \to X,\ \xi(\mu) = \nu \tag{P$_1$}$$

has a solution $\bar{\xi}$, and the problem

$$\max_{\zeta} \int_{X} u(\zeta(x), x)\, \mathrm{d}\nu(x) \qquad \zeta : X \to \Theta,\ \zeta(\nu) = \mu \tag{P$_2$}$$

has a solution $\bar{\zeta}$. These solutions are given by the formulas:

$$f(\xi(\theta)) = \max_x \{u(\theta, x) - g(x)\} \quad a.e.$$
$$g(\zeta(x)) = \max_\theta \{u(\theta, x) - f(\theta)\} \quad a.e.$$

where $f = g^*$ and $g = f^*$ are a suitable pair of u-convex functions. If ξ_1 and ξ_2 are two solutions of (P$_1$)*, then $\xi_1 = \xi_2$ a.e. If $\zeta_1 = \zeta_2$ are two solutions of* (P$_2$)*, then $\zeta_1 = \zeta_2$ a.e. If ξ is a solution of* (P$_1$)*, and ζ is a solution of* (P$_2$)*, then*

$$(\xi \circ \zeta)(x) = x \quad a.e.$$
$$(\zeta \circ \xi)(\theta) = \theta \quad a.e.$$

Proof The existence and uniqueness of ξ and ζ follow from the preceding theorem. For the same reason, we have$\{\xi(\theta)\} = \partial_u f(\theta)$ a.e. and $\{\zeta(x)\} = \partial_u g(x)$ a.e. (the brackets mean that the sets are singletons), with $g = f^*$. By corollary 13, it follows that ξ and ζ are inverse of each other. □

The assumptions will be satisfied if, for instance, Θ and X are open bounded subsets of $\mathbb{R}^d$, endowed with positive measures μ and ν, absolutely continuous with respect to the Lebesgue measure, and if $u(\theta, x) = c(\theta - x)$ satisfying one of the following:

- $u(\theta, x) = c(\theta - x)$ where $c : \mathbb{R}^d \to \mathbb{R}$ is $C^1$and $c' : \mathbb{R}^d \to \mathbb{R}$ is one-to-one.
- Θ and X are disjoint, and $u(\theta, x) = c(\|\theta - x\|)$, where $c' : (0, \infty) \to \mathbb{R}$ is one-to-one.

For instance, if $u(\theta, x) := \|\theta - x\|^\alpha$, the case $\alpha > 1$ (convex cost) falls into the first category, and the case $\alpha < 1$ (concave cost) falls into the second (and so requires Θ and X to be disjoint for the optimal transportation problem to have a solution). As mentioned above, the case $\alpha = 1$ does not satisfy GSM, and a special treatment is required.

4.3 Brenier's theorems

All these results originate with Yann Brenier, who investigated the optimal transportation problem for the special case when $u(\theta, x) = \|\theta - x\|^2$and Θ and X are subsets of $\mathbb{R}^d$, yielding the problem:

$$\min_s \int_\Theta \frac{1}{2} \|\theta - s(\theta)\|^2 \,\mathrm{d}\mu(\theta)$$
$$s : \Theta \to X, \ s(\mu) = \nu.$$

Note that this is equivalent to the seemingly different problem:

$$\max_s \int_\Theta \sum_{i=1}^d \theta^i s_i(\theta) \,\mathrm{d}\mu(\theta)$$
$$s : \Theta \to X, \ s(\mu) = \nu$$

corresponding to $u(\theta, x) = \theta' x$. Indeed, we have:

$$\begin{aligned}
\int_\Theta \frac{1}{2} \|\theta - s(\theta)\|^2 \,\mathrm{d}\mu(\theta) &= \int_\Theta \frac{1}{2} \|\theta\|^2 \,\mathrm{d}\mu(\theta) + \int_\Theta \frac{1}{2} \|s(\theta)\|^2 \,\mathrm{d}\mu(\theta) \\
&\quad - \int_\Theta \frac{1}{2} \theta' s(\theta) \,\mathrm{d}\mu(\theta) \\
&= \int_\Theta \frac{1}{2} \|\theta\|^2 \,\mathrm{d}\mu(\theta) + \int_\Theta \frac{1}{2} \|x\|^2 \,\mathrm{d}\nu(x) \\
&\quad - \int_\Theta \frac{1}{2} \theta' s(\theta) \,\mathrm{d}\mu(\theta)
\end{aligned}$$

because s preserves measure. The first two terms on the right-hand side are constants, and we are left with the third one to optimize.

As above, we assume that condition (C) is satisfied, and that μ is a positive measure on Θ, absolutely continuous with respect to the Lebesgue measure and finite: $\mu(\Theta) < \infty$. Applying theorem 21, we get a series of results.

Proposition 23 *Let ν be a positive measure on X, such that $\mu(\Theta) = \nu(X)$. Then there is a standard convex function f on $\mathbb{R}^d$, and a μ-negligible subset $N \subset \Theta$, such that f is differentiable on $\Theta \backslash N$, and the gradient f' maps $\Theta \backslash N$ into X and μ on ν:*

$$f'(\mu) = \nu.$$

If g is another standard convex function with the same property, then $f' = g'$ a.e.

This is a remarkable theorem, because the set Θ and X are not required to be convex.In fact, they can have any shape at all. Note that the definition of a standard convex function as a supremum of affine functions over Θ implies that it is defined, not only over Θ, but over $\mathbb{R}^d$.

Proposition 24 *Given any Borel map $\xi : \Theta \to X$, there is a standard convex function f on $\mathbb{R}^d$, and a μ-negligible subset $N \subset \Theta$, such that f is differentiable on $\Theta \backslash N$, and the gradient f' has the same distribution as x:*

$$f'(\mu) = \xi(\mu).$$

If g is another standard convex function with the same property, then $f' = g'$ a.e. on Θ.

This follows from the preceding proposition by taking $\nu := \xi(\mu)$. The map $f' : \Theta \backslash N \to X$ is called the *increasing rearrangement* of ξ (see the one-dimensional case). It satisfies the following inequalities:

$$\int_{\Theta} \left\| \theta - f'(\theta) \right\|^2 \mathrm{d}\mu \leq \int_{\Theta} \| \theta - \xi(\theta) \|^2 \mathrm{d}\mu$$

$$\int_{\Theta} \sum \theta_i \frac{\partial f}{\partial \theta_i} \mathrm{d}\mu \geq \int_{\Theta} \sum \theta_i \xi^i(\theta) \, \mathrm{d}\mu.$$

The next result states that we go from $x : \Theta \longrightarrow X$ to its increasing rearrangement by a measure-preserving transformation of the base Θ:

Proposition 25 *Let $X \subset R^d$ be an open bounded set and $\xi : \Theta \to X$ be a Borel map such that $\xi(\mu)$ is absolutely continuous with respect to the Lebesgue measure. Then we have $\xi = f' \circ \varphi$ a.e., where f is a standard convex function on Θ and $\varphi : \Theta \to \Theta$ satisfies $\varphi(\mu) = \mu$.*

Proof There is a standard convex function f such that $f'(\mu) = \xi(\mu)$. There is an inverse $g' : X \to \Theta$, which also preserves measure. Setting $g' \circ \xi := \varphi$ and writing $\xi = f' \circ g' \circ \xi$ gives the desired result. □

This is reminiscent of (but different from) the standard polar decomposition theorem for matrices.

Proposition 26 *Any invertible real matrix M can be written as $M = UA$, where U is orthogonal and A is symmetric and positive definite.*

Proof Take M an $n \times n$ matrix with $\det M \neq 0$. Then $A := (M^* M)^{1/2}$ is a symmetric matrix, positive definite. Set $M = UA$. Then $U = MA^{-1}$ and $U^* = A^{-1}M^*$, so that $UU^* = MA^{-2}M^* = I$. □

4.4 Adverse selection

Let $\Theta \subset \mathbb{R}^{d_1}$ be the space of types characterizing agents, and $X \subset \mathbb{R}^{d_2}$ be the space of actions which the principal wishes the agents to undertake. Let us think of the principal as a monopolist manufacturing cars; cars of quality x are priced at $p(x)$ and cost $c(x)$ to produce. Each agent buys 0 or 1 car, and the principal has to decide what qualities of cars to manufacture and at what prices to sell them in order to maximize his/her profit.

The distribution μ of types is known to the principal. An agent with type θ buying a car of quality x and paying p for it derives utility $u(\theta, x) - p$. If this is less than a certain quantity $\bar{u}(\theta)$ (his reservation utility), he will not undertake the action. A price menu is a map $\theta \to (\xi(\theta), p(\xi(\theta)))$. This menu will be *individually rational* if:

$$u(\theta, \xi(\theta)) - p(\xi(\theta)) \geq \bar{u}(\theta) \quad \forall \theta \tag{IR}$$

and *incentive-compatible* if:

$$u(\theta, \xi(\theta)) - p(\xi(\theta)) \geq u(\theta, \xi(\theta')) - p(\xi(\theta')) \quad \forall (\theta, \theta'). \tag{IC}$$

The expected utility which the principal derives from this price menu is:

$$\int_A [p(\xi(\theta)) - c(\xi(\theta))] \, \mathrm{d}\mu(\theta) \tag{20}$$

where $A \subset \Theta$ is the set of agents which actually buy. The principal's problem consists in maximizing this integral over all individually rational and incentive-compatible contracts, that is, over all maps $\xi : \Theta \to X$ satisfying (IR) and (IC).

The key to solving this problem consists of introducing the function:

$$f(\theta) := \max_x \{u(\theta, x) - p(x)\}.$$

From the point of view of mathematics, this is the *potential function* associated with an optimal transportation problem where the cost is $u(\theta, x)$. From the point of view of economics, this is the *indirect utility* which consumer θ derives from the contract p. We know that f is u-convex; if GSM holds, and μ is absolutely continuous with respect to the Lebesgue measure, the u-subgradient map ξ, given by $\nabla f(\theta) = u(\theta, \xi(\theta))$, is well-defined a.e., and finding an incentive-compatible map $\xi : \Theta \to X$ is equivalent to finding its potential $f : \Theta \to \mathbb{R}$, which is a u-convex function. This is the basic simplification that connects optimal transportation and adverse selection.

Writing the integral (20) in terms of the potential f, we get:

$$\int_A [p(\xi(\theta)) - c(\xi(\theta))] \, d\mu(\theta) = \int_A [u(\theta, \xi(\theta)) - f(\theta) - c(\xi(\theta))] \, d\mu(\theta).$$

Condition (IC) is equivalent to f being u-convex. Condition (IR) is equivalent to $f(\theta) \geq \bar{u}(\theta)$. If $f(\theta) > \bar{u}(\theta)$, type θ will buy. If $f(\theta) < \bar{u}(\theta)$, type θ will not buy from the principal. If $f(\theta) = \bar{u}(\theta)$, type θ is indifferent; if this occurs on a set of measure 0, it is unimportant, if it occurs on a set of positive measure, the modeler will break the tie. We end up with the following reformulation of *the principal-agent problem*:

$$\begin{aligned} &\sup_f \int_\Theta [u(\theta, \xi(\theta)) - f(\theta) - c(\xi(\theta))] \, d\mu \\ &f(\theta) \geq \bar{u}(\theta), \quad f \ \ u\text{-convex} \\ &\nabla f(\theta) = u(\theta, \xi(\theta)) \quad \text{a.e.} \end{aligned} \tag{P}$$

There is an existence theory for such problems, which was developed by Guillaume Carlier (2001). We will not give it here, and we will concentrate instead on the standard convex case, where u is linear with respect to θ. Note, however, the following general result, which is an economic version of the rearrangement theorem.

Proposition 27 *Let $\xi : \Theta \to X$ be an allocation such that $u(\theta, \xi(\theta)) - p(\xi(\theta)) \geq \bar{u}(\theta) \quad \forall \theta$. Assume μ is absolutely continuous with respect to the Lebesgue measure, and u satisfies GSM. Then there is an incentive-compatible allocation y with the same distribution.*

Proof Just take for ν the image of μ by ξ, so that $\nu := \xi(\mu)$, and apply the results in the preceding subsection. □

We will now take a linear specification for u in problem (P). Then u-convex functions are convex in the standard sense. This model was studied by Rochet and Chone (1998), and to this day is the only truly multidimensional model of adverse selection which has been fully analyzed and understood.

Set $\Theta = [a, a+1]^2$, with $a > 0$. Assume the distribution of types is uniform so that μ is the Lebesgue measure. Set $X = \mathbb{R}_+^2$ and:

$$\begin{aligned} u(\theta, x) &= \theta_1 x_1 + \theta_2 x_2 \\ c(x) &= \frac{c}{2}\left(x_1^2 + x_2^2\right). \end{aligned}$$

GSM is satisfied. Note that, since x_1 and x_2 are positive, utility is increasing in the parameters θ. The higher θ_1 and θ_2, the more interested the agent is in the action θ. Let all agents have the same reservation utility 0. Problem (P) then becomes:

$$\sup \int_{\substack{a\leq\theta_1\leq a+1\\ a\leq\theta_2\leq a+1}} \left[-\frac{c}{2}\left(\frac{\partial f}{\partial\theta_1}^2+\frac{\partial f}{\partial\theta_2}^2\right)+\theta_1\frac{\partial f}{\partial\theta_1}+\theta_2\frac{\partial f}{\partial\theta_2}-f(\theta)\right]\mathrm{d}\theta_1\mathrm{d}\theta_2$$
$$f \text{ convex},\quad f(\theta)\geq 0 \text{ a.e.}$$

The quality bought by agents of type θ is:

$$\xi(\theta)=\nabla f(\theta).$$

This problem was solved explicitly by Rochet and Chone. They find that the square Θ is partitioned into three separate regions Θ_i, $i=1,2,3$; the boundaries are parallel straight lines of slopes -1, and the three regions are ordered from the lower left corner Θ_1 to the upper right corner Θ_3, the middle region Θ_2 being sandwiched between them. More precisely:

$$\begin{aligned}\Theta_1&=\{\theta\in\Theta\mid\theta_1+\theta_2\leq m_1\}\\ \Theta_2&=\{\theta\in\Theta\mid m_1\leq\theta_1+\theta_2\leq m_2\}\\ \Theta_1&=\{\theta\in\Theta\mid m_2\leq\theta_1+\theta_2\}\end{aligned}$$

with:

$$\begin{aligned}m_1&:=\frac{4a+\sqrt{4a^2+6}}{3},\\ m_2&:=2a+\sqrt{\frac{2}{3}}.\end{aligned}$$

Problem (P) has a unique solution f, which is described as follows. On Θ_1, we have $f=0$: the individual rationality constraint is binding. On Θ_2, the incentive compatibility constraint is binding: f is constant along all lines with slope -1. In other words, there is a convex function $\varphi(t)$ of a single variable t such that, in the region Θ_2, we have $f(\theta_1,\theta_2)=\varphi(\theta_1+\theta_2)$. In the third region, Θ_3, neither (IR) nor (IC) are binding, so that the function f is strictly convex and satisfies the Euler-Lagrange equation associated with the integral, namely:

$$c\left(\frac{\partial^2 f}{\partial\theta_1^2}+\frac{\partial^2 f}{\partial\theta_2^2}\right)=3.$$

From the economic point of view, Θ_1 is the no-buy region: all types $\theta\in\Theta_1$ stay out of the market. Θ_2 is the bunching region: types $\theta=(\theta_1,\theta_2)$ and $\theta'=(\theta_1',\theta_2')$ such that $\theta_1+\theta_2=\theta_1'+\theta_2'$ buy the same quality

$$\nabla f(\theta) = (\varphi'(\theta_1 + \theta_2), \varphi'(\theta_1 + \theta_2)).$$

Finally, Θ_3 is the screening region: in that region, individuals of different types buy different qualities, so that they reveal their type by buying (this is why it is called the screening region).

One can also figure out the set of qualities which are actually bought. It consists of the square $Q := \left[\frac{a}{c}, \frac{b}{c}\right]^2$, together with the straight segment L joining its lower left corner $\left(\frac{a}{c}, \frac{a}{c}\right)$ to $(0, 0)$. Qualities in Q are bought by types in the screening region, qualities in L by individuals in the bunching region.

How robust is the Rochet-Chone solution? Unfortunately, we do not know: their method of proof is heavily dependent on the particular form of the integral and the shape of the domain Θ. We refer to Carlier and Lachand-Robert (2001) for more mathematics (they prove that the optimal f is C^1 in general situations), and to Carlier et al. (2007) for more examples of adverse selection with multidimensional types. But it is fair to say that this area will be a topic of research for many year to come.

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