Optimal pits and optimal transportation

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Introduction: Open Pit Mining

A Continuous Space Model

An Optimal Transportation Problem

The Kantorovich Dual

Elements of $c$-Convex Analysis

Solving the Dual Problem

Solving the Optimum Pit Problem

Perspectives
Open Pit Mining

To dig a hole in the ground and excavate valuable minerals
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- Diavik diamond mine, Canada
- Super Pit gold mine, Kalgoorli, Western Australia
- Chuquicamata copper mine, Chile
  \((4.3 \text{ km} \times 3 \text{ km} \times 900 \text{ m})\)
Mining Processes
Open Pit Mine Planning

1. Project evaluation: is it worth investing?
   ▶ Where to dig? How deep? What to process?
   Optimum open pit problem (determining ultimate pit limits)

2. Rough-cut planning: take time into account
   ▶ Where, when and what to excavate, to process subject to capacity and other resource constraints, and the time value of money (cash flows)
   ▶ Process choices, major equipment decisions
   Mine production planning problem (decisions over time)

3. Detailed operations planning
   ▶ Detailed mine design: benches, routes, facilities
   ▶ Operations scheduling, flows of materials, etc.

4. Execution...
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(25 million tons rock slide, 2006)

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- implemented in commercial software (Whittle, Geovia)
Discretized vs Continuous Models?

Discretized (block) models:

- are very large (100,000s to millions of blocks)
- production planning models even larger (× number of periods)
- the real problem is, to a large extent, continuous:
  - ore density and rock properties tend to vary continuously
  - their distributions are estimated ("smoothed") from sample (drill hole) data and other geological information
- block precedences only roughly model the slope constraints

Earlier continuous space models:

- Matheron (1975) (focus on "cutoff grade" parametrization)
- determine optimum depth \( \phi (y) \) under each surface point
- s.t. bounds on the derivative of \( \phi \) (wall slope constraints)

All these continuous space approaches suffer from lack of convexity:

- how to deal with local optima?
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Open Pit Problem: a Continuous Space Model

A general model

[Matheron 1975]: Given

\( E \subset \mathbb{R}^3 \): the domain to be mined
e.g.,

\[ E = A \times [h_1, h_2] \]

is the elevation or depth range

\( \Gamma : E \to E \): extracting \( x \) requires extracting all of \( \Gamma(x) \)

\( \Gamma \) is transitive:

\([\alpha \in \Gamma(x) \text{ and } \beta \in \Gamma(\alpha)] \Rightarrow \beta \in \Gamma(x)\)

\( \Gamma \) is reflexive:

\( x \in \Gamma(x) \)

\( \Gamma \) is closed:

\[ \{ (x, y) : x \in E, y \in \Gamma(x) \} \text{ is closed} \]

A pit \( F \) is a measurable subset of \( E \) closed under \( \Gamma \):

\( \Gamma(F) = F \)

where \( \Gamma(F) := \bigcup_{x \in F} \Gamma(x) \)

\( \Gamma \) is a continuous function

\[ g : E \to \mathbb{R} \]

\( g(x) \, dx \) net profit from volume element \( dx \)

\[ = dx_1 \, dx_2 \, dx_3 \]
at \( x \)

\[ g(F) := \int_F g(x) \, dx \]

total net profit from pit \( F \)

assume

\[ \int_E \max\left\{ 0, g(x) \right\} \, dx > 0 \]

(there is some profit to be made)

Optimum pit problem:

find \( F^* \in \arg \max \{ g(F) : F \text{ is a pit} \} \)
A general model [Matheron 1975]: Given

- compact $E \subset \mathbb{R}^3$: the domain to be mined
  e.g., $E = A \times [h_1, h_2]$, where $A \subset \mathbb{R}^2$ is the claim
  $[h_1, h_2]$ is the elevation or depth range
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  - \( g(F) := \int_F g(x)dx \) total net profit from pit \( F \)
Open Pit Problem: a Continuous Space Model

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A *pit* $F$ is a measurable subset of $E$ closed under $\Gamma$:

$$\Gamma(F) = F \quad \text{where } \Gamma(F) := \bigcup_{x \in F} \Gamma(x)$$

- continuous function $g : E \to \mathbb{R}$
  - $g(x)dx$ net profit from volume element $dx = dx_1 dx_2 dx_3$ at $x$
  - $g(F) := \int_F g(x)dx$ total net profit from pit $F$
  - assume $\int_E \max\{0, g(x)\} \, dx > 0$ (there is some profit to be made)
Open Pit Problem: a Continuous Space Model

**A general model** [Matheron 1975]: Given

- compact \( E \subset \mathbb{R}^3 \): the domain to be mined
  - e.g., \( E = A \times [h_1, h_2] \), where \( A \subset \mathbb{R}^2 \) is the *claim*
    - \([h_1, h_2]\) is the elevation or depth range

- map \( \Gamma : E \to E \): extracting \( x \) requires extracting all of \( \Gamma(x) \)
  - transitive: \([x' \in \Gamma(x) \text{ and } x'' \in \Gamma(x')] \implies x'' \in \Gamma(x)\)
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Optimum pit problem: find $F^* \in \arg\max\{g(F) : F \text{ is a pit}\}$
Introduction: Open Pit Mining

A Continuous Space Model

An Optimal Transportation Problem

The Kantorovich Dual

Elements of $c$-Convex Analysis

Solving the Dual Problem

Solving the Optimum Pit Problem

Perspectives
A Profit Allocation Model

Let $E^+ := \{ g(x) > 0 \}$ and $E^- := \{ g(x) \leq 0 \}$ (compact sets)

Add a sink $\omega$ unallocated profits from excavated points will be sent to $\omega$

and a source $\alpha$ unallocated costs of unexcavated points will be paid by $\alpha$

Let $X := E^+ \cup \{ \alpha \}$ and $Y := E^- \cup \{ \omega \}$ (also compact)

endowed with non-negative measures $\mu$ and $\nu$

\[
\mu(\{ \alpha \}) = \int_{E^-} |g(z)| \, dz, \quad \mu(E^+) = \int_{E^+} g(z) \, dz
\]

\[
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Profit allocations are allowed from every profitable $x \in E^+$ to every $y \in \Gamma(x) \cap E^-$

from source $\alpha$ to all $y \in E^-$ (unpaid costs)

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Allocation “Costs” and Optimum Profit Allocation

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- Minimizing total “costs” $\iff$ minimizing total unallocated profits

Lemma: $c$ is lower semi-continuous (l.s.c.)

Set $\Pi(\mu, \nu)$ of nonnegative Radon measures (profit allocations) $\pi$ with marginals $\pi_X = \mu$ and $\pi_Y = \nu$

Optimal transportation problem in Kantorovich form:

$$\min_{\pi} \mathbb{E}_\pi [c] := \int_{X \times Y} c(x, y) \, d\pi$$

s.t. $\pi \in \Pi(\mu, \nu)$

Proposition 1: Problem $(K)$ has a solution

Proof: The set of positive Radon measures on compact space $X \times Y$ is weak-* compact, and the map $\pi \mapsto \mathbb{E}_\pi [c]$ is weak-* l.s.c.
Minimizing total “costs” $\iff$ minimizing total unallocated profits

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Allocation “Costs” and Optimum Profit Allocation

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The Kantorovich Dual

\[ p \in L^1(X,\mu) \text{ associated with } \pi_X = \mu \]
\[ q \in L^1(Y,\nu) \text{ associated with } \pi_Y = \nu \]

Dual admissible set:
\[
A := \{ (p,q) : p(x) - q(y) \leq c(x,y) \, (\mu,\nu) \text{-a.s.} \}
\]

Dual objective:
\[
J(p,q) := \int_X p \, d\mu - \int_Y q \, d\nu = \int E^+ (p(z) - q(\omega)) \, d\mu - \int E^- (q(z) - p(\alpha)) \, d\nu
\]

Kantorovich dual:
\[
\sup_{J(p,q)} \text{ s.t. } (p,q) \in A(D)
\]

Theorem [Kantorovich, 1942]: When the cost function \( c \) is l.s.c.,
\[
\inf(K) = \sup(D)
\]

\( \therefore \) there is no duality gap (in continuous variables).
The Kantorovich Dual

**Potentials** (duals, Lagrange multipliers)

- \( p \in L^1(X, \mu) \) associated with \( \pi_X = \mu \)
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\[ J(p, q) := \int_X p \ d\mu - \int_Y q \ d\nu \]

\[ = \int_{E^+} (p(z) - q(\omega)) \ d\mu - \int_{E^-} (q(z) - p(\alpha)) \ d\nu \]

Kantorovich dual:

\[ \sup J(p, q) \ \text{s.t.} \ (p, q) \in \mathcal{A} \]  \hspace{1cm} (D)
The Kantorovich Dual

Potentials (duals, Lagrange multipliers)
- \( p \in L^1(X, \mu) \) associated with \( \pi_X = \mu \)
- \( q \in L^1(Y, \nu) \) associated with \( \pi_Y = \nu \)

Dual admissible set:
\[
\mathcal{A} := \{ (p, q) : p(x) - q(y) \leq c(x, y) \, (\mu, \nu)\text{-a.s.} \}
\]

Dual objective:
\[
J(p, q) := \int_X p \, d\mu - \int_Y q \, d\nu
\]
\[
= \int_{E^+} (p(z) - q(\omega)) \, d\mu - \int_{E^-} (q(z) - p(\alpha)) \, d\nu
\]

Kantorovich dual:
\[
\sup J(p, q) \quad \text{s.t.} \quad (p, q) \in \mathcal{A}
\]  \hspace{1cm} (D)

Theorem [Kantorovich, 1942]: When the cost function \( c \) is l.s.c.,
\[
\inf(K) = \sup(D)
\]
The Kantorovich Dual

**Potentials** (duals, Lagrange multipliers)
- \( p \in L^1(X, \mu) \) associated with \( \pi_X = \mu \)
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Dual admissible set:

\[
\mathcal{A} := \{ (p, q) : p(x) - q(y) \leq c(x, y) \ (\mu, \nu)\text{-a.s.} \}
\]

Dual objective:

\[
J(p, q) := \int_X p \, d\mu - \int_Y q \, d\nu
\]

\[
= \int_{E^+} (p(z) - q(\omega)) \, d\mu - \int_{E^-} (q(z) - p(\alpha)) \, d\nu
\]

Kantorovich dual:

\[
\sup J(p, q) \text{ s.t. } (p, q) \in \mathcal{A} \quad (D)
\]

**Theorem** [Kantorovich, 1942]: *When the cost function \( c \) is l.s.c.,*

\[
\inf(K) = \sup(D)
\]

- there is no *duality gap* (in continuous variables)
Connection to the Optimum Pit Problem

Let $F$ be a pit, $F^+ := F \cap E^+$ and $F^- := F \cap E^-$. Define $p_F : X \to \mathbb{R}$ and $q_F : Y \to \mathbb{R}$ by:

\[
p_F(\alpha) = 0, \quad p_F(x) = \begin{cases} 1 & \text{if } x \in F^+ \\ 0 & \text{otherwise} \end{cases}
\]

\[
q_F(\omega) = 0, \quad q_F(y) = \begin{cases} 1 & \text{if } y \in F^- \\ 0 & \text{otherwise} \end{cases}
\]

Then $(p_F, q_F)$ is admissible (i.e., in $A$) and $J(p_F, q_F) = g(F)$. 

Corollary: $\sup(P) \leq \inf(K)$, i.e., transportation problem (K) is a weak dual to the optimum pit problem (P).
Let $F$ be a pit, $F^+ := F \cap E^+$ and $F^- := F \cap E^-$
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Corollary: $\sup(P) \leq \inf(K)$ — i.e., transportation problem $(K)$ is a weak dual to the optimum pit problem $(P)$. 

Let $F$ be a pit, $F^+ := F \cap E^+$ and $F^- := F \cap E^-$

Define $p_F : X \to \mathbb{R}$ and $q_F : Y \to \mathbb{R}$ by:

\[
p_F(\alpha) = 0, \quad p_F(x) = \begin{cases} 
1 & \text{if } x \in F^+ \\
0 & \text{otherwise}
\end{cases}
\]

\[
q_F(\omega) = 0, \quad q_F(y) = \begin{cases} 
1 & \text{if } y \in F^- \\
0 & \text{otherwise}
\end{cases}
\]

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Let $F$ be a pit, $F^+ := F \cap E^+$ and $F^- := F \cap E^-

Define $p_F : X \to \mathbb{R}$ and $q_F : Y \to \mathbb{R}$ by:

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Then $(p_F, q_F)$ is admissible (i.e., in $\mathcal{A}$) and $J(p_F, q_F) = g(F)$

**Corollary:** $\sup(P) \leq \inf(K)$
Let $F$ be a pit, $F^+ := F \cap E^+$ and $F^- := F \cap E^-$

Define $p_F : X \rightarrow \mathbb{R}$ and $q_F : Y \rightarrow \mathbb{R}$ by:

\[
p_F(\alpha) = 0, \quad p_F(x) = \begin{cases} 
1 & \text{if } x \in F^+ \\
0 & \text{otherwise}
\end{cases}
\]

\[
q_F(\omega) = 0, \quad q_F(y) = \begin{cases} 
1 & \text{if } y \in F^- \\
0 & \text{otherwise}
\end{cases}
\]

Then $(p_F, q_F)$ is admissible (i.e., in $A$) and $J(p_F, q_F) = g(F)$

**Corollary:** $\sup(P) \leq \inf(K)$

- i.e., transportation problem $(K)$ is a *weak dual* to the optimum pit problem $(P)$
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Given $c: \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$, define the $c$-Fenchel conjugates (or $c$-Fenchel-Legendre transforms) $p^{\#}: \mathcal{Y} \to \mathbb{R}$ of any function $p \in L^1(\mathcal{X}, \mu)$ by

$$p^{\#}(y) := \inf_{x \in \mathcal{X}} (p(x) - c(x,y))$$

and $q^{\#}: \mathcal{X} \to \mathbb{R}$ of any function $q \in L^1(\mathcal{Y}, \nu)$ by

$$q^{\#}(x) := \inf_{y \in \mathcal{Y}} (q(y) + c(x,y))$$

where $\text{ess sup}_f(x) = \inf_{N \in \mathbb{N}} \sup_{x \in \mathcal{X} \setminus N} f(x)$, where $N$ is the set of measurable subsets $N \subset \mathcal{X}$ with $\mu(N) = 0$.

To simplify, we'll write $\sup$ and $\inf$ instead of $\text{ess sup}$ and $\text{ess inf}$.

Similarly, all equalities and inequalities will be $\mu$-a.e. in $\mathcal{X}$ and $\nu$-a.e. in $\mathcal{Y}$. 

---

**c-Fenchel Conjugates**
Given $c : X \times Y \to \mathbb{R}$, define the $c$-Fenchel conjugates (or $c$-Fenchel-Legendre transforms)

1. $p^\# : Y \to \mathbb{R}$ of any function $p \in L^1(X, \mu)$ by
   
   $$p^\#(y) := \text{ess sup}_{x \in X} (p(x) - c(x, y))$$

2. $q^\flat : X \to \mathbb{R}$ of any function $q \in L^1(Y, \nu)$ by
   
   $$q^\flat(x) := \text{ess inf}_{y \in Y} (q(y) + c(x, y))$$

where $\text{ess sup} f(x) = \inf_{N \in \mathcal{N}} \sup_{x \in X \setminus N} f(x)$, where $\mathcal{N}$ is the set of measurable subsets $N \subset X$ with $\mu(N) = 0$. 

To simplify, we'll write $\sup$ and $\inf$ instead of $\text{ess sup}$ and $\text{ess inf}$.
Given \( c : X \times Y \to \mathbb{R} \), define the \( c\)-Fenchel conjugates (or \( c\)-Fenchel-Legendre transforms)

- \( p^\# : Y \to \mathbb{R} \) of any function \( p \in L^1(X, \mu) \) by
  \[
  p^\#(y) := \esssup_{x \in X} (p(x) - c(x, y))
  \]

- \( q^\flat : X \to \mathbb{R} \) of any function \( q \in L^1(Y, \nu) \) by
  \[
  q^\flat(x) := \essinf_{y \in Y} (q(y) + c(x, y))
  \]

where \( \esssup f(x) = \inf_{N \in \mathcal{N}} \sup_{x \in X \setminus N} f(x) \), where \( \mathcal{N} \) is the set of measurable subsets \( N \subset X \) with \( \mu(N) = 0 \)

- To simplify, we’ll write \( \sup \) and \( \inf \) instead of \( \esssup \) and \( \essinf \)

- Similarly, all equalities and inequalities will be \( \mu\)-a.e. in \( X \) and \( \nu\)-a.e. in \( Y \)
Properties of $c$-Fenchel Conjugates

[Carlier, 2003; Ekeland, 2010]
Properties of $c$-Fenchel Conjugates

[Carlier, 2003; Ekeland, 2010]

For all $x \in X$, $y \in Y$,

\[ p(x) \leq c(x, y) + p^{\#}(y) \leq p^{\#\#}(x) \]
\[ q(y) \geq q^{\flat}(x) - c(x, y) \geq q^{\flat\#}(y) \]
Properties of $c$-Fenchel Conjugates

[Carlier, 2003; Ekeland, 2010]

For all $x \in X$, $y \in Y$,

$$p(x) \leq c(x, y) + p^\#(y) \leq p^{\#\#}(x)$$

$$q(y) \geq q^b(x) - c(x, y) \geq q^{\#\#}(y)$$

$c$-Fenchel duality:

$$p^{\#\#} = p^\# \quad \text{and} \quad q^{\#\#} = q^b$$
Properties of $c$-Fenchel Conjugates

[Carlier, 2003; Ekeland, 2010]

For all $x \in X$, $y \in Y$,

$$p(x) \leq c(x, y) + p^\#(y) \leq p^{\#\#}(x)$$
$$q(y) \geq q^b(x) - c(x, y) \geq q^{\#\#}(y)$$

$c$-Fenchel duality:

$$p^{\#\#} = p^\# \text{ and } q^{b\#\#} = q^b$$

Monotonicity:

$$p_1 \leq p_2 \implies p_1^\# \leq p_2^\#$$
$$q_1 \leq q_2 \implies q_1^b \leq q_2^b$$
c-Fenchel Transforms for the Open Pit Dual Problem

\[ p^\#(y) := \max \{ p(\alpha), \sup_{x \in \Gamma(x)} p(x) \} \quad \text{for} \quad y \in E^ - \]

\[ q^\♭(x) := \min \{ 1 + q(\omega), \inf_{y \in \Gamma(x)} q(y) \} \quad \text{for} \quad x \in E^ + \]

\[ p^\# \text{ and } q^\♭ \text{ are increasing with respect to } \Gamma : \]

\[ x' \in \Gamma(x) \implies q^\♭(x') \geq q^\♭(x) \]

\[ y' \in \Gamma(y) \implies p^\#(y') \geq p^\#(y) \]

For a pit \( F \),

\[ p^F = q^\♭ F \quad \text{and} \quad q^F = p^\# F \]
$p^\#(y) := \max \left\{ p(\alpha), \sup_{x : y \in \Gamma(x)} p(x) \right\}$

for $y \in E^-$

$p^\#(\omega) := \max \left\{ p(\alpha), \sup_{x \in E^+} p(x) - 1 \right\}$

$q^\flat(x) := \min \left\{ 1 + q(\omega), \inf_{y \in \Gamma(x)} q(y) \right\}$

for $x \in E^+$

$q^\flat(\alpha) := \min \left\{ q(\omega), \inf_{y \in E^-} q(y) \right\}$
$p^\#(y) := \max \left\{ p(\alpha), \sup_{x : y \in \Gamma(x)} p(x) \right\}$ for $y \in E^-$

$p^\#(\omega) := \max \left\{ p(\alpha), \sup_{x \in E^+} p(x) - 1 \right\}$

$q^\flat(x) := \min \left\{ 1 + q(\omega), \inf_{y \in \Gamma(x)} q(y) \right\}$ for $x \in E^+$

$q^\flat(\alpha) := \min \left\{ q(\omega), \inf_{y \in E^-} q(y) \right\}$

$p^\#$ and $q^\flat$ are increasing with respect to $\Gamma$:

$x' \in \Gamma(x) \implies q^\flat(x') \geq q^\flat(x)$

$y' \in \Gamma(y) \implies p^\#(y') \geq p^\#(y)$
\( p^\#(y) := \max \left\{ p(\alpha), \sup_{x : y \in \Gamma(x)} p(x) \right\} \) for \( y \in E^- \)

\( p^\#(\omega) := \max \left\{ p(\alpha), \sup_{x \in E^+} p(x) - 1 \right\} \)

\( q^\flat(x) := \min \left\{ 1 + q(\omega), \inf_{y \in \Gamma(x)} q(y) \right\} \) for \( x \in E^+ \)

\( q^\flat(\alpha) := \min \left\{ q(\omega), \inf_{y \in E^-} q(y) \right\} \)

\( p^\# \) and \( q^\flat \) are increasing with respect to \( \Gamma \):

\[ x' \in \Gamma(x) \implies q^\flat(x') \geq q^\flat(x) \]

\[ y' \in \Gamma(y) \implies p^\#(y') \geq p^\#(y) \]

For a pit \( F \), \( p_F = q^\flat_F \) and \( q_F = p^\#_F \).
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**Solving the Dual Problem**

Solving the Optimum Pit Problem

Perspectives
Translation Invariance

Given \((p, q) \in A\) and constants \(p_0, p_1, q_0, q_1\) satisfying:

\[
\mu \left( E + (q_0 - p_1) - \nu \left( E - (p_0 - q_1) \right) = 0 \right.
\]

define \(\tilde{p}\) and \(\tilde{q}\) by:

\[
\tilde{p}(\alpha) = p(\alpha) - p_0
\]

\[
\tilde{p}(x) = p(x) - p_1 \quad \text{for} \quad x \in E^+
\]

\[
\tilde{q}(\omega) = q(\omega) - q_0
\]

\[
\tilde{q}(y) = q(y) - q_1 \quad \text{for} \quad y \in E^-
\]

Then:

\[
J(\tilde{p}, \tilde{q}) = J(p, q)
\]
Translation Invariance

Given \((p, q) \in \mathcal{A}\) and constants \(p_0, p_1, q_0, q_1\) satisfying:

\[
\mu (E^+) (q_0 - p_1) - \nu (E^-) (p_0 - q_1) = 0
\]

define \(\tilde{p}\) and \(\tilde{q}\) by:

\[
\tilde{p}(\alpha) = p(\alpha) - p_0
\]
\[
\tilde{p}(x) = p(x) - p_1 \quad \text{for} \quad x \in E^+
\]
\[
\tilde{q}(\omega) = q(\omega) - q_0
\]
\[
\tilde{q}(y) = q(y) - q_1 \quad \text{for} \quad y \in E^-
\]

Then:

\[
J(\tilde{p}, \tilde{q}) = J(p, q)
\]
$c$-Fenchel Transforms Give Local Improvements
If \((p, q) \in A\), then \(p(x) - q(y) \leq c(x, y)\) for all \((x, y)\), so that:

\[
p(x) \leq \inf_{y} \{c(x, y) + q(y)\} = q^b(x)
\]

\[
q(y) \geq \sup_{x} \{p(x) - c(x, y)\} = p^\#(y)
\]
If \((p, q) \in A\), then \(p(x) - q(y) \leq c(x, y)\) for all \((x, y)\), so that:

\[
p(x) \leq \inf_y \{ c(x, y) + q(y) \} = q^b(x)
\]

\[
q(y) \geq \sup_x \{ p(x) - c(x, y) \} = p^\#(y)
\]

Therefore

\[
(p, p^\#) \in A \quad \text{and} \quad J(p, p^\#) \geq J(p, q)
\]

\[
(q^b, q) \in A \quad \text{and} \quad J(q^b, q) \geq J(p, q)
\]
If $(p, q) \in \mathcal{A}$, then $p(x) - q(y) \leq c(x, y)$ for all $(x, y)$, so that:

\[
p(x) \leq \inf_y \{ c(x, y) + q(y) \} = q^b(x)
\]

\[
q(y) \geq \sup_x \{ p(x) - c(x, y) \} = p^#(y)
\]

Therefore

\[
(p, p^#) \in \mathcal{A} \quad \text{and} \quad J(p, p^#) \geq J(p, q)
\]

\[
(q^b, q) \in \mathcal{A} \quad \text{and} \quad J(q^b, q) \geq J(p, q)
\]

This implies

\[
J(p, q) \leq J(p, p^#) \leq J(p^{#b}, p^#)
\]
If \((p, q) \in A\), then \(p(x) - q(y) \leq c(x, y)\) for all \((x, y)\), so that:

\[
p(x) \leq \inf_y \{c(x, y) + q(y)\} = q^b(x)
\]

\[
q(y) \geq \sup_x \{p(x) - c(x, y)\} = p^\#(y)
\]

Therefore

\[(p, p^\#) \in A \quad \text{and} \quad J(p, p^\#) \geq J(p, q)\]

\[(q^b, q) \in A \quad \text{and} \quad J(q^b, q) \geq J(p, q)\]

This implies

\[J(p, q) \leq J(p, p^\#) \leq J(p^b, p^\#)\]

Letting \(\bar{p} := p^b\) and \(\bar{q} := p^\#\), we get:

\[J(p, q) \leq J(\bar{p}, \bar{q})\]

\[\bar{p} = q^b \quad \text{and} \quad \bar{q} = p^\#\]
A Dual Solution

Proposition 2: Problem $\text{(D)}$ has a solution $(\bar{p}, \bar{q})$ with

$\bar{p} = \bar{q} \leq \bar{p} \leq 1$ $\alpha(p) = 0$,

$\bar{q} = \bar{p} \geq \bar{q} \leq 1$ $\omega(q) = 0$.

Proof: Take a maximizing sequence $(p_n, q_n) \in A$.

By preceding results, we may assume $p_n = q_n$ and $q_n = p_n$.

$p_n(\alpha) = 0$, $q_n(\omega) = 0$, and $\inf y \in E - q_n(y) = 0$.

Then, for all $x \in E^+$, $p_n(x) = \min \{1, \inf y \in \Gamma(x) \cap E - q_n(y)\}$.

This implies $0 \leq p_n(x) \leq 1$. Similarly, we get

$0 \leq q_n(x) \leq 1$.

So the family $(p_n, q_n)$ is equi-integrable in $L^1(\mu) \times L^1(\nu)$.

By the Dunford-Pettis Theorem, we can extract a subsequence which converges weakly to some $(\bar{p}, \bar{q})$.

A convex closed in $L^1(\mu) \times L^1(\nu)$ is weakly closed, so $(\bar{p}, \bar{q}) \in A$.

Since $J$ is linear and continuous on $L^1(\mu) \times L^1(\nu)$, we get:

$J(\bar{p}, \bar{q}) = \lim n J(p_n, q_n) = \sup(D)$. 
Proposition 2: **Problem (D) has a solution** \((\bar{p}, \bar{q})\) with

\[
\bar{p} = \bar{q}^\downarrow \quad 0 \leq \bar{p} \leq 1 \quad \bar{p}(\alpha) = 0 \\
\bar{q} = \bar{p}^\uparrow \quad 0 \leq \bar{q} \leq 1 \quad \bar{q}(\omega) = 0
\]
Proposition 2: Problem (D) has a solution \((\bar{p}, \bar{q})\) with
\[
\begin{align*}
\bar{p} &= \bar{q}^\flat \quad 0 \leq \bar{p} \leq 1 \quad \bar{p}(\alpha) = 0 \\
\bar{q} &= \bar{p}^\sharp \quad 0 \leq \bar{q} \leq 1 \quad \bar{q}(\omega) = 0
\end{align*}
\]

Proof: Take a maximizing sequence \((p_n, q_n) \in A\)
Proposition 2: Problem (D) has a solution \((\bar{p}, \bar{q})\) with
\[
\bar{p} = \bar{q}^\ast \quad 0 \leq \bar{p} \leq 1 \quad \bar{p}(\alpha) = 0
\]
\[
\bar{q} = \bar{p}^\dagger \quad 0 \leq \bar{q} \leq 1 \quad \bar{q}(\omega) = 0
\]

Proof: Take a maximizing sequence \((p_n, q_n) \in A\)

- By preceding results, we may assume \(p_n = q_n^\ast\) and \(q_n = p_n^\dagger\)
  
- \(p_n(\alpha) = 0\), \(q_n(\omega) = 0\), and \(\inf_{y \in E^-} q_n(y) = 0\)
Proposition 2: *Problem (D) has a solution* \((\bar{p}, \bar{q})\) *with*

\[
\begin{align*}
\bar{p} &= \bar{q}^\flat & 0 \leq \bar{p} \leq 1 & \bar{p}(\alpha) = 0 \\
\bar{q} &= \bar{p}^\sharp & 0 \leq \bar{q} \leq 1 & \bar{q}(\omega) = 0
\end{align*}
\]

*Proof:* Take a maximizing sequence \((p_n, q_n) \in A\)

- By preceding results, we may assume \(p_n = q_n^\flat\) and \(q_n = p_n^\sharp\)
  
  \[p_n(\alpha) = 0, \quad q_n(\omega) = 0, \quad \text{and} \quad \inf_{y \in E^-} q_n(y) = 0\]

- Then, for all \(x \in E^+\), \(p_n(x) = \min \{1, \inf_{y \in \Gamma(x) \cap E^-} q_n(y)\}\)
Proposition 2: Problem (D) has a solution \((\bar{p}, \bar{q})\) with
\[
\bar{p} = \bar{q}^\flat \quad 0 \leq \bar{p} \leq 1 \quad \bar{p}(\alpha) = 0 \\
\bar{q} = \bar{p}^\sharp \quad 0 \leq \bar{q} \leq 1 \quad \bar{q}(\omega) = 0
\]

Proof: Take a maximizing sequence \((p_n, q_n) \in A\)

- By preceding results, we may assume \(p_n = q_n^\flat\) and \(q_n = p_n^\sharp\)
  \(p_n(\alpha) = 0, \ q_n(\omega) = 0,\) and \(\inf_{y \in E^-} q_n(y) = 0\)

- Then, for all \(x \in E^+\), \(p_n(x) = \min\{1, \inf_{y \in \Gamma(x) \cap E^-} q_n(y)\}\)

- This implies \(0 \leq p_n(x) \leq 1\). Similarly, we get \(0 \leq q_n(x) \leq 1\)
Proposition 2: Problem (D) has a solution \((\bar{p}, \bar{q})\) with
\[
\begin{align*}
\bar{p} &= \bar{q}^\flat \\
0 &\leq \bar{p} \leq 1 \\
\bar{p}(\alpha) &= 0 \\
\bar{q} &= \bar{p}^\sharp \\
0 &\leq \bar{q} \leq 1 \\
\bar{q}(\omega) &= 0
\end{align*}
\]

Proof: Take a maximizing sequence \((p_n, q_n) \in A\)

- By preceding results, we may assume \(p_n = q_n^\flat\) and \(q_n = p_n^\sharp\)
  \(p_n(\alpha) = 0,\ q_n(\omega) = 0,\) and \(\inf_{y \in E^-} q_n(y) = 0\)
- Then, for all \(x \in E^+,\ p_n(x) = \min \{1, \inf_{y \in \Gamma(x) \cap E^-} q_n(y)\}\)
- This implies \(0 \leq p_n(x) \leq 1.\) Similarly, we get \(0 \leq q_n(x) \leq 1\)
- So the family \((p_n, q_n)\) is equi-integrable in \(L^1(\mu) \times L^1(\nu)\)
Proposition 2: Problem (D) has a solution $(\bar{p}, \bar{q})$ with

\[
\bar{p} = \bar{q}^♭ \quad 0 \leq \bar{p} \leq 1 \quad \bar{p}(\alpha) = 0
\]

\[
\bar{q} = \bar{p}^♯ \quad 0 \leq \bar{q} \leq 1 \quad \bar{q}(\omega) = 0
\]

Proof: Take a maximizing sequence $(p_n, q_n) \in A$

- By preceding results, we may assume $p_n = q_n^♭$ and $q_n = p_n^♯$

\[
p_n(\alpha) = 0, \quad q_n(\omega) = 0, \quad \text{and} \quad \inf_{y \in E^-} q_n(y) = 0
\]

- Then, for all $x \in E^+$, $p_n(x) = \min \{1, \inf_{y \in \Gamma(x) \cap E^-} q_n(y)\}$

- This implies $0 \leq p_n(x) \leq 1$. Similarly, we get $0 \leq q_n(x) \leq 1$

- So the family $(p_n, q_n)$ is equi-integrable in $L^1(\mu) \times L^1(\nu)$

- By the Dunford-Pettis Theorem, we can extract a subsequence which converges weakly to some $(\bar{p}, \bar{q})$
Proposition 2: Problem (D) has a solution \((\bar{p}, \bar{q})\) with
\[
\begin{align*}
\bar{p} &= \bar{q}^\sharp, & 0 \leq \bar{p} \leq 1 & \bar{p}(\alpha) = 0 \\
\bar{q} &= \bar{p}^\flat, & 0 \leq \bar{q} \leq 1 & \bar{q}(\omega) = 0
\end{align*}
\]

Proof: Take a maximizing sequence \((p_n, q_n) \in A\)

- By preceding results, we may assume \(p_n = q_n^\flat\) and \(q_n = p_n^\sharp\)
  \(p_n(\alpha) = 0, \ q_n(\omega) = 0, \) and \(\inf_{y \in E^-} q_n(y) = 0\)

- Then, for all \(x \in E^+, \ p_n(x) = \min\{1, \ inf_{y \in \Gamma(x) \cap E^-} q_n(y)\}\)

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- Since \(J\) is linear and continuous on \(L^1(\mu) \times L^1(\nu)\), we get:
  
  \[J(\bar{p}, \bar{q}) = \lim_n J(p_n, q_n) = \sup(D)\]
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If \( \pi \) is optimal to problem (K) and \((p,q)\) to its dual (D), then
\[
0 = J(p,q) - \int_{X \times Y} c(x,y) \, d\pi = \int_{X \times Y} (p(x) - q(y) - c(x,y)) \, d\pi
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implying the CS conditions:
\[
p(x) - q(y) - c(x,y) = 0, \quad \pi \text{-a.e.}
\]
Denote \( y \in \Gamma(x) \) by:
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y \succsim x \quad (\text{the preorder on } E \text{ defined by } \Gamma)
\]
Monotonicity Lemma:
If \((\bar{p}, \bar{q})\) is an optimal solution to (D) satisfying the properties in Proposition 2, then
\[
y'' \succsim y' \succsim x'' \succsim x' = \Rightarrow \bar{q}(y'') \geq \bar{q}(y') \geq \bar{p}(x'') \geq \bar{p}(x')
\]
Proof: The first and last inequalities follow from \( \bar{q} = \bar{p} ^\# \), \( \bar{p} = \bar{q} ^\flat \), and \( c\)-Fenchel conjugates increasing w.r.t. the middle inequality follows from \( \bar{p} ^\# (y) = \max \{ \bar{p}(\alpha), \sup_x y \in \Gamma(x) \bar{p}(x) \} \) for all \( y \in E \).
Complementary Slackness, and Monotonicity

If \( \pi \) is optimal to problem (K) and \((p, q)\) to its dual (D), then

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- the middle inequality follows from

$$\bar{p}^\#(y) = \max \left\{ \bar{p}(\alpha), \sup_{x : y \in \Gamma(x)} \bar{p}(x) \right\} \text{ for all } y \in E^-$$
Proposition 3: Let $(\bar{p}, \bar{q})$ be an optimal solution to problem (D) satisfying the properties in Proposition 2. Then $F := \{ x | \bar{p}(x) = 1 \} \cup \{ y | \bar{q}(y) = 1 \}$ defines an optimum pit.

Proof: $F$ is measurable, hence by the Monotonicity Lemma, a pit.

Letting $F^+ := F \cap E^+$ and $F^- := F \cap E^-$, we have $g(F) = \int F^+ d\mu - \int F^- d\nu \leq \sup P$.

Let $G^+ := E^+ \setminus F^+$ and $G^- := E^- \setminus F^-$ since $\bar{p} = 1$ on $F^+$, $\bar{q} = 1$ on $F^-$, and $\bar{p}(\alpha) = \bar{q}(\omega) = 0$,

$J(\bar{p}, \bar{q}) = \int F^+ d\mu - \int F^- d\nu + \int G^+ \bar{p} d\mu - \int G^- \bar{q} d\nu$. 
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- Let \(G^+ := E^+ \setminus F^+\) and \(G^- := E^- \setminus F^-\):
  since \(\bar{p} = 1\) on \(F^+\), \(\bar{q} = 1\) on \(F^-\), and \(\bar{p}(\alpha) = \bar{q}(\omega) = 0\),
  \[
  J(\bar{p}, \bar{q}) = \int_{F^+} d\mu - \int_{F^-} d\nu + \int_{G^+} \bar{p} d\mu - \int_{G^-} \bar{q} d\nu
  \]
Since $\nu$ is a marginal of $\pi$,
\[
\int G - \bar{q}(y) \, d\nu(y) = \int E + \times G - \bar{q}(y) \, d\pi(x,y)
\]
$c(x,y) = 0$ or $+\infty$ for $(x,y) \in E^+ \times E^-$,
CS conditions,
$0 \leq \bar{p} \leq 1$ and $0 \leq \bar{q} \leq 1$ imply that $\bar{p}(x) = \bar{q}(y)$ $\pi$-a.e. on $E^+ \times E^-$. Thus:
\[
\pi(F^+ \times G - \bar{q})(y) = 0 = \pi(G^+ \times F - \bar{p})(x)
\]
(zero allocations between excavated and unexcavated points), and
\[
\int E^+ \times G - \bar{q}(y) \, d\pi(x,y) = \int G^+ \times G - \bar{q}(y) \, d\pi(x,y) = \int G^+ \times \bar{p}(x) \, d\pi(x,y)
\]
$\Rightarrow J(\bar{p},\bar{q}) = g(F) = \sup(D) = \inf(K) \geq \sup(P) \geq g(F)$.
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\( \Rightarrow \)
\[
\int_{E^+ \times G^-} \bar{q}(y) d\pi(x, y) = \int_{E^+} \bar{p}(x) d\mu(x)
\]

\[\Rightarrow \]
\[
J(\bar{p}, \bar{q}) = \int F^+ d\mu - \int F^- d\nu = g(F)
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Hence
\[
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\(\text{c}(x, y) = 0 \text{ or } +\infty \) for \((x, y) \in E^+ \times E^-\), CS conditions, \(0 \leq \bar{p} \leq 1\) and \(0 \leq \bar{q} \leq 1\) imply that \(\bar{p}(x) = \bar{q}(y)\) \(\pi\)-a.e. on \(E^+ \times E^-\). Thus:
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Proof, continued

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$$\implies J(\bar{p}, \bar{q}) = \int_{F^+} d\mu - \int_{F^-} d\nu = g(F)$$

- Hence $g(F) = J(\bar{p}, \bar{q}) = \sup(D) = \inf(K) \geq \sup(P) \geq g(F)$
Main Result

**Theorem:** If

- $E$ is compact,
- $\Gamma$ is reflexive, transitive and has a closed graph, and
- $g(x)$ is continuous with $\int_E \max\{0, g(x)\} \, dx > 0$,

then:

1. Problem $(P)$ has an optimum solution, i.e., an optimal pit $F$
2. Its indicator functions $(p_F, q_F)$ define optimum potentials, i.e., optimal solutions to $(D)$
3. Problem $(K)$ has an optimum solution (profit allocation) and is a strong dual to $(P)$, i.e., $\min(K) = \max(P)$
4. A pit $F$ is optimal iff there exists a feasible solution $\pi$ to $(K)$ such that $(p_F, q_F)$ satisfies the CS conditions
Uniqueness?

**Theorem** [Matheron, 1975; also Topkis, 1976]:

1. The family $\mathcal{F}$ of all pits is closed under arbitrary unions and intersections:
\[
\bigcup_{F \in \mathcal{G}} F \in \mathcal{F} \quad \text{and} \quad \bigcap_{F \in \mathcal{G}} F \in \mathcal{F} \quad \text{for all } \mathcal{G} \subseteq \mathcal{F}
\]

2. The family of all optimum pits is also closed under arbitrary unions and intersections.

3. There exist a unique smallest optimum pit and a unique largest optimum pit.

The smallest optimum pit minimizes environmental impact without sacrificing total profit.
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  ▶ geological uncertainties on rock properties, amounts and location of ore, etc.
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Perspectives...

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  - Recall: production planning models include excavating and processing decisions over time, subject to capacity constraints, and with discounted cash flows
- **Taking uncertainties** into account:
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  - **operational** uncertainties (disruptions)
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- **Numerical implementation**
  - different from a blocks model...
That’s it, folks.

Any questions?