If not us, who ? And if not now, when ?

Growth theory and sustainable development

Ivar Ekeland, Yiming Long, Qinlong Zhou
with special thanks to Gerd Asheim and Larry Karp

CEREMADE, Universite Paris-Dauphine and Chern Institute. TianJin

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The Chichilnisky model

There is one good in the economy. All members of society have the same preferences and discount the future at the same rate $\delta > 0$, which reflects mostly their probability of dying. To model sustainable development, Chichilinisky has introduced the following criterion:

$$I_\alpha(c) := (1-\alpha)\int_0^\infty \delta u(c(t))\, e^{-\delta t} dt + \alpha u\left(\lim_{t\to\infty} c(t)\right)$$

The first term represents what the present generation will experience directly, and the second what it will not experience, but still cares about, for ethical reasons. The Pareto weight $0 < \alpha < 1$ strikes the balance.

- $\alpha = 0$. This is the Business as Usual (BAU) situation: dictatorship of the present
- $\alpha = 1$. This is the Extreme Ecological situation: dictatorship of the future

The technology

There is a set $\mathcal{A}(k_0)$ of admissible paths:

$$c \in \mathcal{A}(k_0) \iff \begin{array}{c} \frac{dk}{dt} = f(k) - c(t), \quad k(0) = k_0, \\ k(t) \geq 0, \quad c(t) \geq 0 \end{array}$$

The functions $f(k)$ (production) and $u(c)$ (consumption) are C^2 on $]0, \infty[$, with $f'' < 0$ and $u'' < 0$. We assume $u' > 0$ and $f(0) = 0$, $\lim_{k\to 0} f'(k) = \infty$, $\lim_{k\to\infty} f'(k) \leq 0$. There are two separate cases:

- if $\lim_{k\to\infty} f'(k) = 0$, then k is manufactured capital. Typically k_0 is low and will be built up
- if $\lim_{k\to\infty} f'(k) < 0$, the k is a renewable natural resource. Typically k_0 is high and will be depleted

Optimal growth in BAU

For $\alpha = 0$, we get the classical Ramsey-Cass-Koopmans model. A benevolent planner sets herself the optimal control problem:

$$\sup_{c \in \mathcal{A}(k_0)} \int_0^{\infty} \delta u\left(c\left(t\right)\right) e^{-\delta t} dt$$

There is a single stationary state $\underline{k}$ which is hyperbolic, defined by:

$$f'\left(\underline{k}\right) = \delta,\ f\left(\underline{k}\right) = \underline{c}$$

Theorem

There is a unique optimal trajectory $k\left(t\right)$. It converges to $\underline{k}$, when $t \to \infty$

Note that $\underline{k}$ depends only on the technology $f\left(k\right)$ and the discount rate δ, not on the initial capital k_0 nor the utility function $u\left(c\right)$

Optimal sustainable growth ?

$$I_\alpha(c) := (1-\alpha)\int_0^\infty \delta u(c(t))\, e^{-\delta t} dt + \alpha u\left(\lim_{t\to\infty} c(t)\right) \quad \text{with} \quad \alpha > 0$$

Lemma (Asheim, 1996)

If $\alpha > 0$, and $u(f(k))$ is unbounded, then $\sup I_\alpha(c) = +\infty$. In particular, there is no optimal solution

- Good news: by tightening its belt, and expecting others to do so, the present generation can achieve any level of utility, no matter how large.

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- Good news: by tightening its belt, and expecting others to do so, the present generation can achieve any level of utility, no matter how large.
- Bad news: by not tightening its belt, and expecting others to do so, the present generation can do even better
- Conclusion: strategies of this kind are useless. because, in the absence of a commitment mechanism, they are not implementable

Time-inconsistency

Since there is no intertemporal commitment mechanism, we have a leader-follower game between successive planners.
We consider Markov strategies $c = \sigma(k)$, resulting in a flow:

$$\frac{dk}{dt} = f(k) - \sigma(k)$$

A strategy $c = \sigma(k)$ is implementable if it is a subgame-perfect Nash equilibriumof that game. This has been formalized by Phelps (1968), Harris and Laibson (2001), Krusell and Smith (2003), in the discrete case. In the continuous case, there are two equivalent approaches, by Karp (2003), and by Ekeland and Lazrak (2006).

Equilibrium strategies

Let $c = \sigma(k)$ be a strategy (consumption plan) converging to some k_∞.
At time t:

- denote by $k_0(t)$ the corresponding trajectory starting from k_0
- denote by $I_t(\sigma, k)$ the corresponding value of the criterion, when we start from k at time t
- denote $\sigma^c_{t,\varepsilon}$ the strategy which consists of consuming c between t and $t + \varepsilon$ and reverting to σ after $t + \varepsilon$

Definition

The strategy σ is an equilibrium if:

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[I_t\left(\sigma^c_{t,\varepsilon}, k_0(t)\right) - I_t\left(\sigma, k_0(t)\right) \right] \leq 0 \text{ for all } c$$

Each generation is always better off applying $\sigma(k)$ than deviating unilaterally.

Equilibrium strategies in the Chichilnisky model

For each $k_\infty > 0$, we consider the optimization problem:

$$\sup \left\{ \int_0^T u\left(c\left(t\right)\right) e^{-\delta t} dt \mid c \in A\left(k_0\right),\ T > 0,\ k\left(T\right) = k_\infty \right\}$$

and we denote by $c = \sigma_{k_\infty}\left(k\right)$ the corresponding optimal strategy.

Theorem (No turning back)

All equilibrium strategies in the Chichilnisky model are of the type σ_{k_∞}, for some $k_\infty > 0$, and conversely. More precisely, for any $k_0 > 0$, and any k_∞ between k_0 and $\underline{k}$ (the BAU stationary level), σ_{k_∞} is an equilibrium strategy on the interval $[0, k_\infty]$ or $[k_\infty, \infty]$.

- for $k_\infty = \underline{k}$, we get the BAU strategy
- if $k_\infty \neq \underline{k}$, we get one-sided strategies
- say $k_0 > \underline{k}$ (depleting a natural resource). Then we can stop the depletion at any level $k_\infty \in [\underline{k}, k_\infty]$, but we can never aim for a level $k_\infty > \underline{k}$ (restore past levels) or $k_\infty < \underline{k}$ (deplete beyond BAU)

Some remarks

- It is much more difficult to rebuild a natural resource than to preserve it. Legislating a minimal level of natural capital is implementable if one starts from higher levels, not if one starts from lower levels. In addition to the well-known physical inertia in climate change, there is a "rational inertia "

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- Whatever the level of concern of individuals or governments for the environment, time inconsistency will be a strong factor in deferring decisions which have no immediate effect, but would strongly affect the future, which is the standard situation in climate policy (Stern Review)

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- Whatever the level of concern of individuals or governments for the environment, time inconsistency will be a strong factor in deferring decisions which have no immediate effect, but would strongly affect the future, which is the standard situation in climate policy (Stern Review)
- Economic theory gives the wrong answer to the famous question of Rabbi Hillel: "If not us, who ? If not now, when ?". This is a failure, not of rationality, but of utilitarianism: Rabbi Hillel's is fundamentally a question about ethics, and ethics are part of a wider rationality

The HJB equation

Let us go back to the Ramsey model. Consider the optimal value V as a function of the initial point:

$$V(k_0) := \sup \{ I_0(c, k) \mid (k, c) \in \mathcal{A}(k_0) \}$$

If $V(k)$ is C^1, it must satisfy the HJB equation, namely:

$$\delta V(k) = u^* \left(V'(k) \right) + f(k) V'(k)$$

where u^* is the conjugate of u, defined by:

$$u^*(x) = \max_c \{ u(c) - cx \}$$
$$u(c) = \min_y \{ u^*(x) + xc \}$$

For instance, if $u(c) = -\frac{1}{p} c^p$, then $u^*(x) = \frac{1}{q} x^q$ with $p^{-1} + q^{-1} = 1$

Equilibrium strategies in the Chichilnisky model:

Theorem (Characterization)

If V is a C^2 solution of:

$$u^* \left(V'(k)\right) + f(k)V'(k) = \delta V \quad \text{(HJBE)}$$

$$V(k_\infty) = \frac{1}{\delta} u(f(k_\infty)) \quad \text{(BC)}$$

and $\sigma(k) := -\nabla u^* (V'(k))$ converges to k_∞, then $\sigma(k)$ is an equilibrium strategy.

HJB as an implicit ODE

HJB is an implicit ODE for the unknown function $V(k)$. It turns out that it cannot always be solved wrt V':

$$\delta V(k) = u^*\left(V'(k)\right) + f(k)V'(k) \geq \min_x \{u^*(x) + f(k)x\} = u(f(k))$$

The curve $\delta V = \delta u(f(k))$ divides the positive orthant (k, V) in three regions:

- in the upper region, $\delta V > u(f(k))$, the equation $\delta V(k) = u^*(x) + f(k)x$ has two solutions x_1 and x_2
- in the lower region, $\delta V < u(f(k))$, the equation $\delta V(k) = u^*(x) + f(k)x$ has no solution
- on the frontier, $\delta V = u(f(k))$, the equation $\delta V(k) = u^*(x) + f(k)x$ has one solution x

Note that the boundary condition $\delta V(k_\infty) = u(f(k_\infty))$ lies on the frontier ! At this point, HJB cannot be solved wrt V', so this is not an initial-value problem

The phase space

- in the upper region, $\delta V > u(f(k))$, there are two solutions through every point $(\bar{V}, \bar{k})$, corresponding to $V'(\bar{k}) = x_1$ and $V'(\bar{k}) = x_2$.
- in the lower region, $\delta V < u(f(k))$, there is no solution
- on the frontierr, $\delta V = u(f(k))$, if $f'(k) \neq \delta$, there are two half-solutions with the same tangent. More precisely, if $\bar{V} = u(f(\bar{k}))$, and $\bar{x}$ is the solution of $\delta \bar{V} = u^*(\bar{V}) + f(\bar{k})\bar{V}$, then there are two functions $V_1(k)$ and $V_2(k)$, defined on the same interval $[\bar{k} - \varepsilon, \ \bar{k}]$ or $[\bar{k}, \ \bar{k} + \varepsilon]$, and satisfying $V_1'(\bar{k}) = V_2'(\bar{k}) = \bar{x}$.

Theorem

For $k_\infty = \underline{k}$, there is a C^2 solution going through the point $(\underline{k}, \underline{V})$ where $f'(\underline{k}) = \delta$, $\delta \underline{V} = u(f(\underline{k}))$.

Figure 1

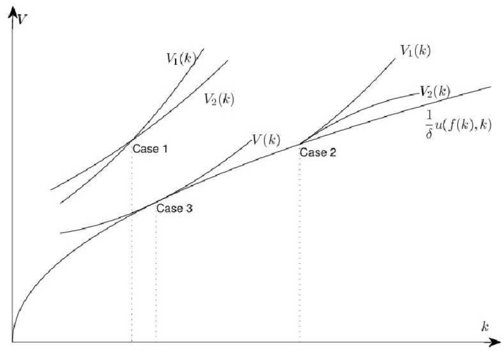

Figure 1. The illustration of the solutions

Figure 2

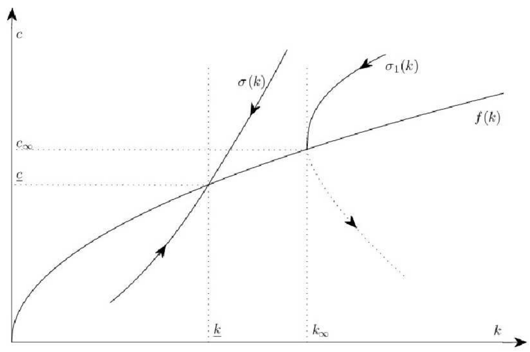

Figure 2. The phase diagram for the Euler equation

Proof of the "no turning back" Theorem

Setting $V := V_0 - \frac{\alpha}{(1-\alpha)\delta} u\left(f\left(k_\infty\right)\right)$, we see that V is a solution f the boundary-value problem:

$$u^*\left(V'\left(k\right)\right) + f(k)V'(k) = \delta V$$
$$V\left(k_\infty\right) = \frac{1}{\delta} u\left(f\left(k_\infty\right)\right)$$

This recognize the HJB equation of the Ramsey problem but with a different boundary condition, and we apply our analysis of the phase space: since $\delta V_\infty = u\left(f\left(k_\infty\right)\right)$, the point $\left(V_\infty, k_\infty\right)$ is on the frontier separating the two regions, and there are two half-solutions with the same tangent.