Adverse selection and optimal transportation

A personal history

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In memory of Jean-Jacques Laffont

- The University of Paris-Dauphine was founded in 1969, with the mission of teaching management in a university context. I was appointed a professor there in 1970.
- The UER Mathématiques de la Décision (MD) and the Centre de Recherches de Mathématiques de la Décision (CEREMADE) were founded in 1970 by Jean-Pierre Aubin. We were the first in the world to create a full mathematical program (BSc and Masters) where the *natural sciences* (physics and chemistry) were replaced by *social sciences* (economics, finance), plus statistics and computer science
- Our main interests at the time (Aubin, Bensoussan, Ekeland) were in *optimization and control*. This is when I wrote my book with Temam on convex analysis. Bensoussan was the first person to apply stochastic optimal control to problems of inventory management and real options

- In economics, our contacts where Pierre-Marie Larnac and Jean-Jacques Laffont . He would often come to my office and explain to me the mathematical problems arising from asymmetry of information (moral hazard and adverse selection). I concluded at the time that they were intractable. He left in 1977, after the agrégation, but the seeds had been sown.
- Outside France, our main contact was José Scheinkman, who was a frequent visitor in Dauphine, as I was in Chicago

Dauphine in the eighties

- Jean-Michel Lasry and Pierre-Louis Lions were appointed as professors, and young students arrived from École Normale: Hervé Moulin, Bernard Cornet, Georges Haddad, and Jean-Charles Rochet
- Rochet's thesis (1986), under the supervision of Lasry, was for me the first step towards understanding problems in adverse selection (moral hazard was still out of reach). He gave a mathematical framework, defined incentive-compatible contracts , and showed (relying on previous work by Terry Rockafellar) how to characterize them by a potential function.
- Rochet then went to join Laffont in Toulouse, and published a famous paper (Rochet-Choné, 1998) where he treated completely a simple two-dimensional example

Rochet's thesis

An individual of type t performing a task x gets a reward w. The function $t \longrightarrow (x(t), w(t))$ is the contract. The individual has a separable (dis)utility $u(t, x) + w$. A contract is incentive-compatible (IC) if:

$$u(t, x(t)) + w(t) \geq u(t, x(t')) + w(t') \text{ for all } (t, t')$$

Theorem

A contract $t \longrightarrow (x(t), w(t))$ is IC iff, for any closed chain $(t_0, t_1, \ldots, t_N = t_0)$, we have

$$\sum_{n=0}^{N-1} u(t_n, x(t_n)) - u(t_n, x(t_{n+1})) \geq 0$$

Rochet's thesis (cont'd)

Theorem

Let $t \to (x(t), w(t))$ satisfy this condition. Pick a type $\bar{t}$. Then the function:

$$V(t) := \inf \left\{ \sum_{n=0}^{N-1} u(t_n, x(t_n)) - u(t_n, x(t_{n+1})) \mid t_0 = \bar{t},\ t_N = t \right\}$$

satisfies:

$$V(t) = \sup_{t'} \left\{ u(t, x(t')) + V(t') - u(t', x(t')) \right\} \tag{E}$$

The function V is the potential associated with the problem. It is the key to solving it: instead of seeking a contract, that is a map from T to $X \times \mathbb{R}$, we are seeking a single function on T

Connexions with convex analysis

Suppose first $u(t,x) = t \cdot x$.Then the equation becomes:

$$V(t) = \sup_{t'} \left\{ (t - t') \cdot x(t') + V(t') \right\}$$

So V is convex , and:

$$x(t) \in \partial V(t)$$

If u is not convex, we can generalize these formulas. V is the supremum of functions of the type $t \rightarrow u(t,x) + c$. Such functions will be called u-affine , and we have a corresponding u-convex analysis. Equation (E) will be written:

$$x(t) \in \partial_u V(t)$$

The impact of this is that if you know the function $V(t)$, you know the whole contract

The Rochet-Choné example

- The principal can produce any quality $x \in \mathbb{R}^2_+$ at a cost $\frac{c}{2}|x|^2$ and sets the price $w(x)$
- Customer types are uniformly distributed in $T = [a,\ 1+a]^2$
- If customer t buys quality x at price w, utility is $t \cdot x - w$
- The principal's problem is to maximize

$$\int_T \left[w(x(t)) - \frac{1}{2}|x(t)|^2 \right] dt$$

among all contracts $(x(t), w(t))$ which are IC and IR

The Rochet-Choné problem (cont'd)

- Introduce the indirect utility which customer t derives from the contract:t'

$$V(t) = \sup_{t'} \left\{ t' \cdot x - w(t') \right\}$$

- It is convex if the contract is IC, non-negative if it is IR, and $\nabla V(t) = x(t)$ by the envelope theorem.The problem can now be restated in terms of $V(t)$. Introduce:

$$J(V) := \int_T \left[-\frac{1}{2} |\nabla V|^2 + t \cdot \nabla V - V \right] dt$$

Then the principal 's problem becomes:

$$\sup \left\{ J(V) \mid V \in L^2(T), \quad V \geq 0, \quad V \text{ convex} \right\}$$

The Rochet-Choné solution

Introduce the cone $\mathcal{C}$ of convex non-negative functions in $L^2(T)$. The problem consists of maximizing a positive definite quadratic form on L^2 (it may take the value $-\infty$, but it is upper semi-continuous) on a closed convex cone. So the solution exists and is unique.

Question: what does it look like ? For this, we need a characterization

First set of optimality conditions: one would expect $\partial J(\bar{V}) \in N_{\mathcal{C}}(\bar{V})$. At the optimum $\bar{V}$, the subgradient of J should belong to the normal cone of $\mathcal{C}$. The problem is that no one knows what this normal cone looks like (except for Pierre-Louis Lions in the case of H_0^1)

The Rochet-Choné solution

Second set of optimality conditions: change $\bar{V}$ to $\bar{V} + \varepsilon V$, let $\varepsilon \to 0$ and integrate by parts. This is the standard method in the calculus of variations, going back to La Vallée-Poussin. One gets:

$$J'(\bar{V}) \cdot V = \int_T [-\nabla \bar{V} \cdot \nabla V + t \cdot \nabla V - V]\, dt$$
$$= \int_{\partial T} (t - \nabla \bar{V})_n V ds + \int_T (\Delta \bar{V} - 3) V dt \geq 0 \quad \forall V \in C$$

A probability μ is a balayée of a probabilité ν if $\int f d\mu \geq \int f d\nu$ for every convex function f. This is an (incomplete) order relation.It is known in potential theory as sweeping (μ est une balayee de ν), or Choquet ordering. There is a very interesting representation theory connected with it in terms of Markov kernels and Strassen's theorem:

$$\mu = \int \alpha_x d\nu(x) \quad \text{where the barycenter of } \alpha_x \text{ is x}$$

The Rochet-Choné solution

Dauphine in the nineties

I went in Paris to the seminar of a young Canadian researcher, named Robert McCann. He was investigating the optimal transportation problem: given two discrete sets T and X, with $|T| = |X|$, and a transportation cost $c(t, x)$, find a bijection $t \to x(t)$ which will minimize $\sum_t u(t, x(t))$. He wrote on the blackboard the following:

Theorem

The transportation plan x is optimal if, for any closed chain $(t_0, t_1, ..., t_N = t_0)$:

$$\sum_{n=0}^{N-1} u(t_n, x(t_n)) - u(t_n, x(t_{n+1})) \geq 0$$

I recognized immediately equation (E) from Rochet's thesis ten years earlier

Dauphine in the nineties (cont'd)

I discussed with McCann in Paris, and introduced to the subject a PhD student of mine, named Guillaume Carlier. It was the beginning of the use of optimal transportation methods in economics:

- the structure of cities (IE-Carlier)
- equilibrium in hedonic models (IE, Chiappori-McCann-Nesheim)

This line of research is very active today, and happily represented at TSE (Adrien Blanchet). Going back to the Rochet-Chone model, we find:

Theorem

Given the optimal distribution ν of qualities, the optimal contract can be found by solving an optimal transportation problem from μ (the distribution of types) to ν

So, it would be enough to find ν to solve the whole problem.

Carlier and Lachand-Robert

In his thesis, Carlier, together with Thomas Lachand-Robert, proved an optimal regularity result (smooth pasting)

Theorem

Consider the problem:

$$\min \int_\Omega L\left(t, V\left(t\right), \nabla V\left(t\right)\right) dt$$

$$V \geq 0, \ \ convex, \ V\left(t\right) = V_0\left(t\right) \ \ for\ t \in \partial\Omega$$

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where the integrand L is smooth, convex w.r.t $\left(V, \nabla V\right)$ and coercive, the boundary Ω and the boundary condition V_0 are smooth. Then the solution $\bar{V}$ is C^1

Situations are known where $\bar{V}$ is not C^2

The numerics

Problems with global convexity constraints had appeared in mathematics from quite another angle (Newton's problem), and the several numerical methods had been suggested. They were all of finite elements type (fixed grid) Some of them were wrong, all were slow. Together with Maury, Carlier and Lachand-Robert devised more efficient methods, which were still fixed-grid. They did not apply them to the Rochet-Choné problem. At this point, there had been several papers introducing adverse selection into portfolio management (IE-Carlier-Touzi, Horst-Moreno) and I became convinced that we needed efficient numerical methods if we were to penetrate the economic mainstream, which was confined to one-dimensional types, along the lines of Laffont Tirole and Martimort

Numerics: the Ekeland-Moreno method

With my student Santiago Moreno, we sought a method adapted to the convexity of the problem.

$$\max \int_{\Omega} L\left(t, V\left(t\right), \nabla V\left(t\right)\right) dt$$
$$V \geq 0, \text{ convex}$$

Choose a set of points t^n, $1 \leq n \leq N$ in T. Consider the discretized problem

$$\max \sum_{n=1}^{N} L\left(t^n, v^n, w^n\right)$$
$$v^n \geq 0 \ \forall n \text{ and } v^n \geq v^m + \left(t^n - t^m\right) \cdot w^m \quad \forall \left(n, m\right)$$

If the solution is $\left(\bar{t}^n, \bar{v}^n, \bar{w}^n\right)$, the approximate solution is:

$$\bar{V}^n\left(t\right) = \max_m \left\{v^m + \left(t - t^m\right) \cdot w^m\right\}$$

Note that the grid is not fixed: it is a result of the procedure

Numerics: the Mirebeau method

In 2012 I discussed the problem with Jean-Marie Mirebeau, from Dauphine, who came up with an entirely different method

Conclusion

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- The situation is now reversed: the theory lags behind the numerics. We cannot justify mathematically the solution of the Rochet-Choné problem
- The economics is ahead of the mathematics: what we need here is a qualitative theory of partial differential equaitions. We want to move away from the Rochet-Choné model. Generally speaking, do we always find three regions: no-buy, bunching and screening ?
- There are many open questions: what about competing principals ? what about IC constraints for coalitions, not only individuals ? Laffont had started studying some of them. We will need much time to catch up. We miss him.