Asymmetry of information and risk sharing*

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1 The new paradigm

1.1 The old paradigm

Arrow and Debreu in 1954 [2] closed a line of investigation which had been open since Adam Smith by providing the mathematical foundations of general equilibrium theory. To quote Debreu [3]: "*A contract for the transfer of a commodity now specifies, in addition to its physical properties, its location and its date, an event on the occurrence of which the transfer is conditional*"

The mathematical model then consists of specifying, at the initial time $t = 0$

- a finite set of possible states of the world $\Omega = \{\omega_1, ..., \omega_K\}$; an event is a subset $A \subset \Omega$
- a finite set of commodities, indexed from $i = 1$ to I; each commodity is available in any non-negative quantity
- a finite set of traders, indexed from $j = 1$ to J; each trader is characterized by his preferences over goods bundles and his initial allocation

A goods bundle (also called a contingent claim) is a pair $(x \mid A)$, meaning that quantities $x = (x_1, .., x_I) \in R_+^I$ are to be delivered if the event A occurs. All trades occur at time $t = 0$, and traders are commited from then on. The market is complete if all contingent claims can be traded. An equilibrium price is a price system (one for each contingent claim) such that the market clears (demand equals supply)

If the market is complete, and if every trader has convex preferences over contingent claims, Arrow and Debreu proved the following:

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- there exists an (an possibly several) equilibrium[1]
- (first theorem of welfare economics) every equilibrium is Pareto optimal
- (second theorem of welfare economics) every Pareto optimum can be realized as an equilibrium for some initial allocation $\left(x^1, ..., x^J\right) \in \left(R_+^I\right)^J$.

This is the intellectual content of Adam Smith's famous idea of the "invisible hand", and has had tremendous influence in designing policies. Note that the invisible hand operates from an initial distribution of goods and can be seen as a "neutral" way of achieving an efficient redistribution - on the other hand, it has nothing to say about the initial distribution itself.

"*A model that I perceived is the critical functioning structure that defines how the world works*" Alan Greenspan, testimony to the US Congress, October 23, 2008

1.2 Information matters

To quote Debreu again, "*This new definition of a commodity allows one to obtain a theory of [risk] free from any probability concept*". This is nothing short of miraculous, and should arouse suspicion. In fact, the Arrow-Debreu model is not an accurate description of the way real economies or financial markets operate. The causes of so-called "market failures" have been classified under several headings, externalities, public goods, information. We will focus on the latter.

In the Arrow-Debreu model, everyone knows everything there is to know. In the real world, information is not readily available, one has to make a deliberate effort to get it, and even so one may never succeed, because people may lie to you. How important is that ? One may want to say: not very, there will always be enough information to get around, and anyway, markets are efficient, in the sense that prices will carry the relevant information. The major progress in economic theory in the past forty years has been to expose that as a fallacy ([8], [9])

1.2.1 No information, no market, no prices

Akerlof [?]

1.2.2 Markets with inefficient redistributions

Greenwald and Stiglitz ([4], [5])

[1] satisfying some technical conditions, such at the so-called Inada conditions: $u'(\lambda x) \to 0$ when $\lambda \to \infty$ and $u'(x) \to -\infty$ when x approaches the boundary of R_+^I

1.2.3 Prices do not reveal all relevant information

If all information was available through the prices (which are publicly known), then there would be no private reward to acquiring more information, and investors would have no incentive to dig into a company's record. .

Grossman and Stiglitz [6] provide a model where anyone can acquire information (at a price) and yet there are informed and uninformed traders in equilibrium (so that uninformed traders cannot learn the information through the prices).

It is a two-period model with two assets. There is a riskles asset, with deterministic return R, and a risky asset, with stochastic return $\tilde{u} = \tilde{\theta} + \tilde{\varepsilon}$ (we are putting $\tilde{}$ on random variable, so as to recognize them as random). The risky asset is in total supply $\tilde{x}$ (the realization of which is unobservable). The rv $\tilde{\theta}$ and $\tilde{\varepsilon}$ are Gaussian and independent, with $E[\tilde{\varepsilon}] = 0$ and $\mathrm{Var}\left(\tilde{u} \| \tilde{\theta}\right) = \mathrm{Var}(\tilde{\varepsilon}) = \sigma_\varepsilon^2$

All traders are identical. They all have exponential utility,(which, with Gaussian variables, make explicit computations possible)

$$E\left[-\exp\left(-aW_1\right)\right]$$

where W_1 is the value of their portfolio at time $t = 1$. The only difference is that some of them know $\tilde{\theta}$ and some don't. They both seek the optimal quantity X to invest in the risky asset

Let $\tilde{P}$ be the observed price of the risky asset at time $t = 0$. The expected utility of the informed trader is:

$$-\exp\left(-a\left[RW_0 + X\left(\tilde{\theta} - R\tilde{P}\right) - \frac{a}{2}X^2\sigma_\varepsilon^2\right]\right)$$

yielding

$$X_I(P, \theta) = \frac{\tilde{\theta} - R\tilde{P}}{a\sigma_\varepsilon^2}$$

Uninformed traders have figured out a relation between the price and the two sources of noise: $\tilde{P} = P^*(x, \theta)$. The whole idea of Grossman-Stiglitz is that they will not be able to disentangle the two noises, so that the observatio of the realizatiom P will not allow them to infer ε. Taking $P^*(x, \theta)$ as given, the expected utility of the uninformed trader is:

$$-\exp\left(-a\left[RW_0 + X\left(E\left[\tilde{u} \mid P^*\left(\tilde{x}, \tilde{\theta}\right)\right] - R\tilde{P}\right) - \frac{a}{2}X^2\mathrm{Var}\left[\tilde{u} \mid P^*\left(\tilde{x}, \tilde{\theta}\right)\right]\right]\right)$$

yielding

$$X_U(P, \theta) = \frac{E\left[\tilde{u} \mid P^*\left(\tilde{x}, \tilde{\theta}\right)\right] - R\tilde{P}}{a\mathrm{Var}\left[\tilde{u} \mid P^*\left(\tilde{x}, \tilde{\theta}\right)\right]}$$

If a proportion λ of the traders are informed, an equilibrium price is some $P_\lambda^*\left(\tilde{x}, \tilde{\theta}\right)$ such that, for $\tilde{P} = P_\lambda^*\left(\tilde{x}, \tilde{\theta}\right)$, the demand of risky assets of the informed and uninformed sum up to x for every value of $\tilde{\varepsilon}$. Grossman and

Stiglitz prove that there is such an equilibrium, and that it is a linear function of:

$$\tilde{w}_\lambda = \tilde{\theta} - \frac{a\sigma_\varepsilon^2}{\lambda}\left(\tilde{x} - E\tilde{x}\right)$$

Thus the price carries information about θ, but it does so imperfectly.

We now allow uninformed traders to become informed at a cost $c > 0$. The market then stabilizes at the value $\bar{\lambda}$ for which the ex-post utilities of informed and uninformed traders are the same. We get an overall equilibrium price $P^*(\tilde{x}, \tilde{\varepsilon})$. Define:

$$m = \left(\frac{a\sigma_\varepsilon^2}{\lambda}\right)\frac{\sigma_x^2}{\sigma_\theta^2}, \quad n = \frac{\sigma_\theta^2}{\sigma_x^2}$$

then the equilibrium value $\bar{\lambda}$ is the solution of:

$$m = \frac{e^{2ac} - 1}{1 + n - e^{2ac}}$$

The informativeness of the price system is then provided by the relations:

$$\begin{aligned} \mathrm{Cov}\left(P^*(\tilde{x}, \tilde{\varepsilon}), \tilde{\theta}\right) &= \frac{1}{1+m} \\ \mathrm{Cov}\left(\tilde{u}, \tilde{\theta}\right) &= \frac{n}{1+n} \end{aligned}$$

Note for instance that if $\sigma_x = 0$, the price system is fully informative, so there is no incentive to acquire information; an equilibrium exists only if $e^{ac} > \sqrt{1+n}$

1.2.4 Equilibria need not exist

Rothschild-Stiglitz [7]

1.3 Contract theory

Some characteristics: difficulty of appropriating the returns to creating information, markets do not have to clear, incentive for keeping information private. Information distorts actions: selling shares, offering guarantees, one-price vs quality increases with price. So one has developed instead a theory of monopoly. The *decision structure* is as follows:

- there is a principal and a set of agents
- the principal moves first and makes an offer to the agents
- each agent takes it or leaves it

There are two types of *information structures*

1.3.1 Adverse selection (hidden information)

- each agent has a type x
- each agent knows his type
- the principal knows the distribution of types $d\mu(x)$

The *incentive structure* is as follows:

- a contract consists of an action by the agent and a payment from (or to) the principal
- the principal offers a menu of contracts, and each agent picks the one he prefers, if any
- the agent then performs the action and gets (or gives) the payment
- each accepted contract brings some profit to the principal

The principal's problem consists of devising the contract menu so as to maximize his profit

1.3.2 Moral hazard (hidden action)

- the principal wants the agent to do something for him
- the actions $a \in A$ of the agent cannot be directly observed by the principal
- however, these actions will influence an outcome $z \in Z$ which can be observed by the principal and the agent

The *incentive structure* is as follows

- a contract consists a payment, contingent on the observed outcome
- the principal offers a contract to the agent
- the agent acts to maximize expected utility (which may lead him to turn down the contract, or to accept it and to shirk)

1.3.3 What to expect

- If the principal knows the agent's type, or can observe the agent's actions, the best she can expect is to get her reservation utility. This is the first-best situation.
- Asymmetry of information protects the agent. The principal then looks for a second-best outcome, which, from his point of view, will be inferior to the first-best
- So there is an informational rent to the agent, which is higher for high-quality agents than for low-quality ones (the poor and weak end up closer to their reservation utility than the rich and strong)

2 Adverse Selection

2.1 The Rochet-Choné problem

Let $\not\lesssim \subset \mathbb{R}^{d_1}$ be the space of types characterizing agents, and $X \subset \mathbb{R}^{d_2}$ be the space of actions which the principal wishes the agents to undertake. Let us think of the principal as a monopolist manufacturing cars; cars of quality x are priced at $p(x)$ and cost $c(x)$ to produce. Each agent buys 0 or 1 car, and the principal has to decide what qualities of cars to manufacture and at what prices to sell them in order to maximize his/her profit.

The distribution μ of types is known to the principal. An agent with type θ buying a car of quality x and paying p for it derives utility $u(\theta, x) - p$. If this is less than a certain quantity $\bar{u}(\theta)$ (his reservation utility), he will not undertake the action. A price menu is a map $\theta \to (\xi(\theta), p(\xi(\theta)))$. This menu will be *individually rational* if:

$$u(\theta, \xi(\theta)) - p(\xi(\theta)) \geq \bar{u}(\theta) \quad \forall \theta \tag{(IR)}$$

and *incentive-compatible* if:

$$u(\theta, \xi(\theta)) - p(\xi(\theta)) \geq u(\theta, \xi(\theta')) - p(\xi(\theta')) \quad \forall (\theta, \theta') \tag{(IC)}$$

The expected utility which the principal derives from this price menu is:

$$\int_A [p(\xi(\theta)) - c(\xi(\theta))] \, d\mu(\theta) \tag{1}$$

where $A \subset \not\lesssim$ is the set of agents which actually buy. The principal's problem consists in maximizing this integral over all individually rational and incentive-compatible contracts, that is, over all maps $\xi : \Theta \to X$ satisfying (IR) and (IC).

The key to solving this problem consists of introducing the function:

$$f(\theta) := \max_x \{u(\theta, x) - p(x)\}$$

From the point of view of mathematics, this is the *potential function* associated with an optimal transportation problem where the cost is $u(\theta, x)$. From the point of view of economics, this is the *indirect utility* which consumer θ derives from the contract p. We know that f is u-convex; if GSM holds, and μ is absolutely continous with respect to the Lebesgue measure, the u-subgradient map ξ, given by $\nabla f(\theta) = u(\theta, \xi(\theta))$, is well-defined a.e., and finding an incentive-compatible map $\xi : \Theta \to X$ is equivalent to finding its potential $f : \Theta \to \mathbb{R}$, which is a u-convex function. This is the basic simplification that connects optimal transportation and adverse selection.

Writing the integral (1) in terms of the potential f, we get:

$$\int_A [p(\xi(\theta)) - c(\xi(\theta))] \, d\mu(\theta) = \int_A [u(\theta, \xi(\theta)) - f(\theta) - c(\xi(\theta))] \, d\mu(\theta) \, .$$

Condition (IC) is equivalent to f being u-convex. Condition (IR) is equivalent to $f(\theta) \geq \bar{u}(\theta)$. If $f(\theta) > \bar{u}(\theta)$, type θ will buy. If $f(\theta) < \bar{u}(\theta)$, type θ will not buy from the principal. If $f(\theta) = \bar{u}(\theta)$, type θ is indifferent; if this occurs on a set of measure 0, it is unimportant, if it occurs on a set of positive measure, the modeller will break the tie. We end up with the following reformulation of *the principal-agent problem*:

$$\begin{array}{c} \sup_f \int_\Theta \left[u(\theta, \xi(\theta)) - f(\theta) - c(\xi(\theta))\right] d\mu \\ f(\theta) \geq \bar{u}(\theta), \quad f \quad u\text{-convex} \\ \nabla f(\theta) = u(\theta, \xi(\theta)) \quad \text{a.e} \end{array} \tag{(P)}$$

There is an existence theory for such problems, which was developed by Guillaume Carlier [?]. We will not give it here, and we will concentrate instead on the standard convex case, where u is linear with respect to θ. Note, however, the following general result, which is an economic version of the rearrangement theorem:

Proposition 1 *Let $\xi : \nleq \to X$ be an allocation such that $u(\theta, \xi(\theta)) - p(\xi(\theta)) \geq \bar{u}(\theta) \quad \forall \theta$. Assume μ is absolutely continuous with respect to the Lebesgue measure, and u satisfies GSM. Then there is an incentive-compatible allocation y with the same distribution.*

We will now take a linear specification for u in problem (P). Then u-convex functions are convex in the standard sense. This model was studied by Rochet and Chone in [?], and to this day is is the only truly multidimensional model of adverse selection which has been fully analysed and understood.

Set $\nleq = [a_, a+1]^2$, with $a > 0$. Assume the distribution of types is uniform so that μ is the Lebesgue measure. Set $X = \mathbb{R}_+^2$ and:

$$\begin{aligned} u(\theta, x) &= \theta_1 x_1 + \theta_2 x_2 \\ c(x) &= \frac{c}{2}\left(x_1^2 + x_2^2\right) \end{aligned}$$

GSM is satisfied. Note that, since x_1 and x_2 are positive, utility is increasing in the parameters θ. The higher θ_1 and θ_2, the more interested the agent is in the action θ. Let all agents have the same reservation utility 0. Problem (P) then becomes:

$$\sup \int_{\substack{a \leq \theta_1 \leq a+1 \\ a \leq \theta_2 \leq a+1}} \left[-\frac{c}{2}\left(\frac{\partial f}{\partial \theta_1}^2 + \frac{\partial f}{\partial \theta_2}^2\right) + \theta_1 \frac{\partial f}{\partial \theta_1} + \theta_2 \frac{\partial f}{\partial \theta_2} - f(\theta)\right] d\theta_1 d\theta_2$$

f convex, $f(\theta) \geq 0$ a.e.

The quality bought by agents of type θ is:

$$\xi(\theta) = \nabla f(\theta)$$

This problem was solved explicitly by Rochet and Chone. They find that the square Θ is partitioned into three separate regions $\Theta_i, i = 1, 2, 3$; the boundaries are parallel straight lines of slopes -1, and the three regions are ordered from the lower left corner Θ_1 to the upper right corner Θ_3, the middle region Θ_2 being sandwiched between them. More precisely:

$$\begin{aligned}\Theta_1 &= \{\theta \in \Theta \mid \theta_1 + \theta_2 \leq m_1\} \\ \Theta_2 &= \{\theta \in \Theta \mid m_1 \leq \theta_1 + \theta_2 \leq m_2\} \\ \Theta_1 &= \{\theta \in \Theta \mid m_2 \leq \theta_1 + \theta_2\}\end{aligned}$$

with:

$$\begin{aligned}m_1 &: = \frac{4a + \sqrt{4a^2 + 6}}{3}, \\ m_2 &: = 2a + \sqrt{\frac{2}{3}}\end{aligned}$$

Problem (P) has a unique solution f, which is described as follows. On Θ_1, we have $f = 0$: the individual rationality constraint is binding. On Θ_2, the incentive compatibility constraint is binding: f is constant along all lines with slope -1. In other words, there is a convex function $\varphi(t)$ of a single variable t such that, in the region Θ_2, we have $f(\theta_1, \theta_2) = \varphi(\theta_1 + \theta_2)$. In the third region, Θ_3, neither (IR) nor (IC) are binding, so that the function f is strictly convex and satisfies the Euler-Lagrange equation associated with the integral, namely:

$$c\left(\frac{\partial^2 f}{\partial \theta_1^2} + \frac{\partial^2 f}{\partial \theta_2^2}\right) = 3$$

From the economic point of view, Θ_1 is the no-buy region: all types $\theta \in \Theta_1$ stay out of the market. Θ_2 is the bunching region: types $\theta = (\theta_1, \theta_2)$ and $\theta' = (\theta_1', \theta_2')$ such that $\theta_1 + \theta_2 = \theta_1' + \theta_2'$ buy the same quality

$$\nabla f(\theta) = \left(\varphi'(\theta_1 + \theta_2), \varphi'(\theta_1 + \theta_2)\right)$$

Finally, Θ_3 is the screening region: in that region, individuals of different types buy different qualities, so that they reveal their type by buying (this is why it is called the screening region).

One can also figure out the set of qualities which are actually bought. It consists of the square $Q := \left[\frac{a}{c}, \ \frac{b}{c}\right]^2$, together with the straight segment L joining its lower left corner $\left(\frac{a}{c}, \frac{a}{c}\right)$ to $(0,0)$. Qualities in Q are bought by types in the screening region, qualities in L by individuals in the bunching region.

How robust is the Rochet-Chone solution ? Unfortunately, we do not know: their method of proof is heavily dependent on the particular form of the integral and the shape of the domain Θ. We refer to Carlier and Lachand-Robert [?] for more mathematics (they prove that the optimal f is C^1 in general situations), and to Carlier, Ekeland and Touzi [?] for more examples of adverse selection with multidimensional types. But it is fair to say that this area will be a topic of research for many year to come.

2.2 Risk transfer by OTC trading.

2.3 The agents

There is a principal and a set of agents. They will trade risk. The risk is represented by a random variable $X \in L^2(\Omega)$. If an agent has risk Y and acquires (or sells) risk X at a price π, her utility is

$$U(\theta; X) = \mathbb{E}[X+Y] - \lambda \mathbb{V}\text{ar}[X+Y] - \pi$$

Note that, for any constant a, we have $U(\theta; X+a) = U(\theta; X) + a$. So the agent is indifferent between buying (X, π) and $(X+a, \pi+a)$. In other words, if the price of X is π, then the price of $X - \mathbb{E}[X]$ is $\pi - \mathbb{E}[X]$. Without loss of generality, we can assume that all random variables are centered: $\mathbb{E}[X] = 0$.

The risk Y which the agent holds is an investment in securities $1, B_1, ..., B_k$:

$$Y = \sum_{k=1}^{K} \alpha_k B_k$$

with $\mathbb{E}[B_k] = 0, \quad \mathbb{C}\text{ov}(B, B) = I_K$. The type of the agent is then:

$$\theta = (\lambda, \beta_1, ..., \beta_K) \text{ with } \beta_k = \frac{\alpha_k}{2\lambda}$$

The utility of an agent of type θ is:

$$\begin{aligned} U(\lambda, \beta_1, ..., \beta_k; X) &= \mathbb{E}\left[X + \sum_{i=1}^{k} \beta_i B_i\right] - \lambda \mathbb{V}\text{ar}\left[X + \sum_{i=1}^{k} \beta_i B_i\right] \\ & \mathbb{E}[X] - \lambda \mathbb{V}\text{ar}[X] - \beta \mathbb{C}\text{ov}(X, B) - 4\lambda^3 \|\beta\|^2 \end{aligned}$$

The constant term at the end plays no role in the optimisation. We are left with:

$$U(\theta; X) = \theta G(X) + g(X) \tag{2}$$

$$G(\theta) = (-\mathbb{V}\text{ar}[X], -\mathbb{C}\text{ov}(X, B)) \tag{3}$$

$$g(X) = \mathbb{E}[X] \tag{4}$$

Note that $G(0) = 0$ and $g(0) = 0$. The reservation utility (no incentive to trade) is $U(\theta; 0) = 0$. Hence the definition

Definition 2 *A* contract *is a pair (X, π) of maps $\theta \longmapsto (X_\theta, \pi_\theta)$ from Θ to $L^2 \times R$. A contract (X, π) satisfies the* individual rationality condition *(IR) if*

$$U(\theta, X_\theta) - \pi_\theta \geq 0$$

It satisfies the incentive-compatibility condition *(IC) if:*

$$U(\theta, X_\theta) - \pi_\theta \geq U(\theta, X_{\theta'}) - \pi_{\theta'} \quad \forall \theta'$$

An allocation *$\theta \to X_\theta$ is incentive-compatible if there exists some $\theta \to \pi_\theta$ such that the contract (X, π) is incentive-compatible*

The individual rationality condition states that the agent is allowed not to participate if it gives him/her less than his/her reservation utility. The incentive-compatibility condition states that agents of type θ will choose the contract X_θ, π_θ which the principal designed for them, that is they will buy (or sell) the derivative X_θ and pay (or get) π_θ.

We introduce the indirect utility of agent θ :

$$v(\theta) = \max_{\theta'} \{U(\theta, X_{\theta'}) - \pi_{\theta'}\} = \max_{\theta'} \{\theta G(X(\theta')) + g(X(\theta')) - \pi(\theta')\} \quad (5)$$

Proposition 3 *v is a convex function of θ, and an allocation $\theta \to X_\theta$ is incentive-compatible if and only if*

$$\forall \theta, \quad G(X(\theta)) \in \partial v(\theta) \quad (6)$$

Conversely, if v is a convex function and an allocation $\theta \to X_\theta$ satisfies (6), then it is incentive-compatible

Proof. The formula (5) defines v as the pointwise supremum of a family of affine functions. So $v(\theta)$ is a convex function. If $\theta \to X_\theta$ is IC, then there exists some $\theta \to \pi_\theta$ such that (X_θ, π_θ) is IC, so that the maximum on the right-hand side of (5) is attained for $\theta' = \theta$. This means precisely that $G(X(\theta)) \in \partial v(\theta)$

Conversely, suppose v is convex and $G(X(\theta)) \in \partial v(\theta)$. Set

$$\pi_\theta = \theta G(X(\theta)) + g(X(\theta)) - v(\theta)$$

I claim that (X_θ, π_θ) is an incentive-compatible contract. From the definition of the subgradient, we have:

$$\begin{aligned} v(\theta) - v(\theta') &\geq G(X(\theta'))(\theta - \theta') \\ v(\theta) - G(X(\theta'))\theta &\geq v(\theta') - G(X(\theta'))\theta' \end{aligned}$$

Plugging in the value for $\pi(\theta)$, this yields:

$$\theta(G(X(\theta)) - G(X(\theta'))) + g(X(\theta)) - \pi(\theta) \geq g(X(\theta')) - \pi(\theta')$$

Rearranging, this becomes:

$$\theta G(X(\theta)) + g(X(\theta)) - \pi(\theta) \geq \theta G(X(\theta'))\, g(X(\theta')) - \pi(\theta')$$

which is the definition incentive-compatibility. ∎

In the case at hand, using (2), (3). (4), we obtain:

$$\begin{aligned} -\mathbb{V}\mathrm{ar}[X_\theta] &= \frac{\partial v}{\partial \lambda} \\ -\mathbb{C}\mathrm{ov}(X_\theta, B_i) &= \frac{\partial v}{\partial \beta_i} \end{aligned}$$

2.4 Profit-maximizing principal.

The principal can produce any random variable X at a cost $C(X)$. For instance, if he has access to a financial market, he has $C(X) = \mathbb{E}[ZX]$ with $Z \geq 0$ and $\mathbb{E}[X] = 1$. If he sells X_θ to type θ, he makes π_θ. He knows the density μ of types:

$$\mu(\theta) \geq 0 \text{ and } \int \mu(\theta)\, d\theta < \infty$$

He is risk-neutral, so he is maximizing his expected profit:

$$\Phi(X, \pi) = \sup \int (\pi_\theta - C(X_\theta))\, \mu(\theta)\, d\theta$$

over of all (IR) and (IC) contracts. Define:

$$H(p) = \inf \{C(X) - g(X) \mid G(X) = p\}$$

Proposition 4 *The principal's problem consists of:*

$$\sup \int (\theta \nabla v - v - H(\nabla v))\, d\mu(\theta) \tag{7}$$

$$v \text{ convex, } v \geq 0 \tag{8}$$

Proof.

$$\begin{aligned}
\int (\pi_\theta - C(X_\theta))\, \mu(\theta)\, d\theta &= \int (U(\theta, X_\theta) - v_\theta - C(X_\theta))\, \mu(\theta)\, d\theta \\
&= \int (\theta G(\nabla v(\theta)) + g(X) - v(\theta) - C(X))\, \mu(\theta)\, d\theta
\end{aligned}$$

■

Substituting the values from (2), (3). (4), we obtain:

$$H(x, y_1, ..., y_K) = \min \{\mathbb{E}[(Z-1)X] \mid -\mathbb{V}\mathrm{ar}[X] = x,\ -\mathbb{C}\mathrm{ov}(X, B_i) = y_i,\ \mathbb{E}[X] = 0\}$$

This can be rewritten as an optimization problem in L^2

$$\begin{aligned}
& \min (Z - 1, X) \\
(X, B_k) &= y_k,\ 1 \leq k \leq K \\
X^2 &= x
\end{aligned}$$

The solution is given by:

$$H(x, y) = -\sum y_k \mathbb{C}\mathrm{ov}(X, B_k) - \sqrt{-x - \sum y_k^2} \sqrt{\mathbb{V}\mathrm{ar}[Z] - \sum \mathbb{C}\mathrm{ov}(X, B_k)^2}$$

Setting $\mathbb{C}\mathrm{ov}(X, B_k) = \xi_k$ and $\mathbb{V}\mathrm{ar}[Z] = \zeta$, the principal's problem is:

$$\max \int \left[\lambda \frac{\partial v}{\partial \lambda} + \sum (\beta_k + \xi_k) \frac{\partial v}{\partial \beta_k} - v + \sqrt{\xi - \sum \zeta_k^2} \sqrt{-x - \sum y_k^2}\right]$$

v convex, $v \geq 0$

We now have to prove that the principal's problem has a solution. The following result will be useful in the proof; it is a rare instance of economic intuition contributing to a mathematical proof.

Proposition 5 *Suppose X is such that*

$$\theta \nabla v(\theta) - v(\theta) - H(\nabla v(\theta)) < 0$$

on a subset $\Omega \subset \Theta$ with $\mu(\Omega) > 0$. Then there is some w, convex and non-negative, such that:

$$\begin{aligned} 0 &\leq w \leq v \ , \\ \theta \nabla w(\theta) - w(\theta) - H(\nabla w(\theta)) &\geq 0 \\ \int (\theta \nabla w - w - H(\nabla w))\, d\mu &\geq \int (\theta \nabla v - v - H(\nabla v))\, d\mu \end{aligned}$$

Proof. Take $\theta \in \Omega$, and consider the contract (X_θ, π_θ) offered to type θ. We have:

$$\pi_\theta - C(X_\theta) = \theta \nabla v(\theta) - v(\theta) - H(\nabla v(\theta)) < 0$$

In other words, the principal is losing money on this contract. He will be better off offering type θ the contract $(0, 0)$. Let him do precisely that, i.e. cancel all the contracts . Types $\theta \in \Omega$ then have the choice of taking $(0, 0)$ or any one of the remaining contracts (X_θ, π_θ) for $\theta \notin \Omega$. The corresponding indirect utility is:

$$w(\theta) = \max\{0, \sup \bar{v}(\theta)\}$$

with:

$$\begin{aligned} \bar{v}(\theta) &= \max\{\theta G(X(\theta')) + g(X(\theta')) - \pi(\theta') \mid \theta' \notin \Omega\} \\ &= \max\{\theta \nabla v(\theta') + v(\theta') - \theta' \nabla v(\theta') \mid \theta' \notin \Omega\} \\ &= \max\{(\theta - \theta') \nabla v(\theta') + v(\theta') \mid \theta' \notin \Omega\} \end{aligned}$$

Clearly $\bar{v} \leq v$, $\bar{v}$ coincides with v on $\Theta \backslash \Omega$, and $\bar{v}$ is convex. So the principal is not changing his profit on all the types $\theta \notin \Omega$. On the other hand, if $\theta \in \Omega$, then type θ will either choose the contract of a type $\theta' \notin \Omega$, in which case the principal is making money, or she will choose the contract $(0, 0)$, in which case the principal is not losing money. In either case,

$$\theta \nabla w(\theta) - w(\theta) - H(\nabla w(\theta)) \geq 0$$

and the result follows. ■

Introduce the Fenchel transform of H :

$$H^*(\theta) = \sup_p \{\theta p - H(p)\}$$

H^* is convex, although H is not.

Proposition 6 *Suppose that H^* is integrable:*

$$\int_\Theta H^*(\theta)\, d\mu(\theta) < \infty$$

Then the principal's problem has a solution

Proof. Let v_n be a maximising sequence:

$$\int (\theta \nabla v_n - v_n - H(\nabla v_n))\, d\mu \longrightarrow \sup \tag{9}$$

By the preceding Proposition, we may assume that:

$$\begin{aligned} 0 &\leq \theta \nabla v_n(\theta) - v_n(\theta) - H(\nabla v_n(\theta)) \\ 0 &\leq v_n(\theta) \leq \theta \nabla v_n(\theta) - H(\nabla v_n(\theta)) \leq H^*(\nabla v_n(\theta)) \end{aligned}$$

It follows from the last inequality and the convexity of the v_n that the v_n are locally uniformly Lipschitz: for any compact subset Ω of Θ, there is a constant $C(\Omega)$ such that, at every point $\theta \in \Omega$ where v_n is differentiable, we have:

$$|\nabla v_n(\theta)| \leq C(\Omega)$$

It follows that the v_n are uniformly continous on Ω, so that we can extract a subsequence (still denoted by v_n) which converges uniformly to $\bar{v}$. It is easily seen that $\bar{v}$ is convex.

Let θ be a point where the v_n and $\bar{v}$ are differentiable. Consider the sequence $\nabla v_n(\theta)$. It is bounded, so that we can extract a convergent subsequence: $\nabla v_n(\theta) \to p$. Since $\nabla v_n(\theta)$ belongs to the subdifferential of v_n at θ, we have:

$$v_n(\theta') - v_n(\theta) \geq (\nabla v_n(\theta), \theta' - \theta)$$

and going to the limit:

$$\bar{v}(\theta') - \bar{v}(\theta) \geq (p, \theta' - \theta)$$

But this means precisely that $p \in \partial \bar{v}(\theta)$. Since θ has been chosen to be a point of differentiability, we have $p = \nabla v(\theta)$, and since all subsequences of $\nabla v_n(\theta)$ converge to $\nabla v(\theta)$, so does the sequence $\nabla v_n(\theta)$ itself.

We conclude by applying Lebesgue's dominated convergence theorem to (9):

$$\int (\theta \nabla \bar{v} - \bar{v} - H(\nabla \bar{v}))\, d\mu = \lim \int (\theta \nabla v_n - v_n - H(\nabla v_n))\, d\mu = \sup$$

■

For:

$$H(x, y) = -\sum_{k=1}^{K} \zeta_k y_k + \sqrt{\xi - \sum \zeta_k^2} \sqrt{-x - \sum y_k^2}$$

we find:

$$H^*(\lambda, \beta) = \frac{1}{4\lambda}\left(\xi - \sum \zeta_k^2 + \sum (\beta_k + \zeta_k)^2\right)$$

So the principal's problem has a solution provided:

$$\int \frac{1}{4\lambda}\left(\xi - \sum \zeta_k^2 + \sum (\beta_k + \zeta_k)^2\right) \mu(\lambda, \beta)\, d\lambda d\beta_1 ... d\beta_K < \infty$$

3 Moral hazard

3.1 The problem of limited liability

- Shareholders vs management
- The public vs the firm (BP)
- The government vs the banks (too big to fail)

3.2 Moral hazard in continuous time: the Toulouse model

The agent is in charge of a project which generates a stream of revenue, which accrue to the principal

Accidents occur, generating large losses, the cost of which will be borne by the principal

The risk (probability of losses) can be reduced by due diligence from the agent

Due diligence is costly to the agent, and not directly observable by the principal

The principal seeks to ensure due diligence from the agent by offering her a performance-based contract

Contracts must be based on observables, ie the stream of revenue and the occurence of accidents

3.2.1 Modeling assumptions

The revenue is a constant fraction μ of the size X_t of the project

Accidents occur according to a jump process N_t The corresponding losses are a constant fraction C of the size X_t of the project at the time of the accident.

Profits between t and $t + dt$ are:

$$X_t(\mu dt - C dN_t)$$

The size of the project is commanded by the principal, who can either

- downsize it ,all the way to 0 if necessary (in which case the project is shut down)

- upsize it by investing, at a maximum rate γ

With these assumptions, the only source of noise in the system is the point process N_t.

3.2.2 The agent

Between t and $t + dt$, the agent has two possibilities:

- either exterting effort, in which case the probability of an accident is $\Lambda_t = \lambda$ and her cost is 0
- or shirking, in which case the probability raises to $\Lambda_t = \lambda + \Delta\lambda$ and her private benefit is B

In other words, the accidents are random events, which constitute a Poisson process with intensity λ if the agent performs effort, and $(\lambda + \Delta\lambda)$ if she does not.

She seeks to maximize:

$$E\left[\int_0^\tau e^{-\rho t}\left(L_t + \mathbf{1}_{\{\Lambda_t = \lambda + \Delta\lambda\}} B dt\right)\right]$$

A contract $\Gamma = (X_t, \tau, \Lambda_t, L_t)$ will specify the rules for down/upsizing the project, the rules for terminating it (so τ is a stopping time), as well as the agent's effort Λ_t, and her salary, which is a continuous stream $L_t dt$.

3.2.3 Mathematics of Poisson processes

Let N_t^λ be a Poisson process with intensity λ and $\mathcal{F}_t^\lambda$ the corresponding filtration. By definition, N_t^λ is right continuous and:

$$\int_0^t \left(\lambda ds - dN_s^\lambda\right) = \int_0^t \lambda ds - N_t^\lambda(\omega) = M_t$$

is a right-continuous $\mathcal{F}_t^\lambda$-martingale, with zero expectation:

$$\begin{aligned}
E\left[M_T \mid \mathcal{F}_t^\lambda\right] &= \int_0^t \lambda ds - N_t^\lambda(\omega) + E\left[\int_0^T \lambda ds - N_T^\lambda(\omega) \mid \mathcal{F}_t^\lambda\right] \\
&= \int_0^t \lambda ds - N_t^\lambda(\omega) + \int_t^T \lambda ds - E\left[N_T^\lambda(\omega) \mid \mathcal{F}_t^\lambda\right] \\
&= M_t + \left(\int_0^{T-t} \lambda ds - N_{T-t}^\lambda(\omega)\right) = M_t
\end{aligned}$$

A process X_t is $\mathcal{F}_t^\lambda$-*adapted* if, X_t is $\mathcal{F}_t^\lambda$-measurable for every t. It is *predictable* if its trajectories are left continuous at the random jumpes, i.e., for every t where $dN_t(\omega) = 1$, we have:

$$\lim_{\substack{s \to t \\ s \le t}} X(s) = X(t)$$

Whenever Z_t is a predictable (left-continuous) process, then

$$I_T := \int_0^T Z_t dM_t = \int_0^T Z_t(\omega)\lambda dt + \sum \lim_{\substack{s\to t\\ s<t}} Z_s$$

where the sum in the last term is taken over all the jumps, is a centered$\mathcal{F}_t^\lambda$-martingale:

$$\begin{aligned} E\left[I_T \mid \mathcal{F}_t^\lambda\right] &= I_t \text{ for } t \le T \\ E\left[I_t\right] &= 0 \text{ for all } t \end{aligned}$$

Conversely, any centered $\mathcal{F}_t^\lambda$-martingale, i.e. any $\mathcal{F}_t^\lambda$-adapted process is of that form: this is the martingale representation theorem

3.2.4 Contracts inducing maximum effort

A contract $\Gamma = (X_t, \tau, \Lambda_t, L_t)$ is specified. It asks the agent to perform maximum effort

$$\Lambda_t = \lambda$$

so that the resulting accident rate is λ.

Let us assume first that the agent conforms to the contract, so that the accidents constitute a Poisson process with intensity λ. The salary is a $\mathcal{F}_t^\lambda$-predictable process, and the expected utility of the agent at time t (taking into account the fact that he is supposed to exert maximum effort, and gets no private rewards for shirking) is:

$$\begin{aligned} U_t(\Gamma) &= E\left[\int_0^\tau e^{-\rho s} L_s ds \mid \mathcal{F}_t^\lambda\right] && (10) \\ &= \int_0^{\tau\wedge t} e^{-\rho s} L_s ds + e^{-\rho t} W_t(\Gamma) && (11) \\ dU_t &= e^{-\rho t} L_t dt - \rho e^{-\rho t} W_t(\Gamma) + e^{-\rho t} dW_t && (12) \end{aligned}$$

where $W_t(\Gamma)$ (the continuation utility of the agent), is given by:

$$W_t(\Gamma) = E\left[\int_t^\tau e^{-\rho s} L_s ds \mid \mathcal{F}_t^\lambda\right]$$

On the other hand, $U_t(\Gamma) - U_0(\Gamma)$ is a centered $\mathcal{F}_t^\lambda$-martingale, so by the martingale representation theorem, we have:

$$\begin{aligned} U_t(\Gamma) &= U_0(\Gamma) + \int_0^{t\wedge\tau} e^{-\rho s} H_s\left(\lambda ds - dN_s^\lambda\right) && (13) \\ dU_t &= e^{-\rho t} H_t\left(\lambda dt - dN_t^\lambda\right) && (14) \end{aligned}$$

for some $\mathcal{F}_t^\lambda$-predictable process H_t. Note that the jump in U_t is due, not to a fine paid by the agent, but simply to the change in her prospects (as specified by the contract) due to the accident. Setting $dN_t^\lambda = 1$, we see that $e^{-\rho t} H_t$ is the amount (discounted at time 0) by which the utility of the agent is reduced in case of an accident. In other words, H_t is the cost of an accident to the agent, in terms of forgone future utilities, if she sticks to the contract, i.e. if she exerts maximum effort throughout.

Comparing this with (12), we get:

$$dW_t = \rho W_t dt + H_t \left(\lambda dt - dN_t^\lambda\right) - L_t dt \tag{15}$$

Taking expectations, we get

$$E\left[dW_t\right] = \left(\rho W_t - L_t\right) dt \tag{16}$$

Let us now check whether the contract is incentive-compatible (IC), that is, that the agent will indeed exert maximum effort, even though the principal cannot observe her. If the agent shirks, she changes dN_t^λ to $dN_t^{\lambda+\Delta\lambda}$ (the frequency of accidents increases to$(\lambda + \Delta\lambda)\, dt$) and gets private returns $BXdt$. The expected return becomes:

$$E\left[dW_t\right] = \left(\rho W_t - H_t \Delta\lambda + BX_t - L_t\right) dt \tag{17}$$

Comparing (16) with (17), we find that, the expected benefit from shirking between t and $t + dt$ are $\left(BX_t - H_t \Delta\lambda\right) dt$. For the contract to be IC, we need:

$$BX_t \leq H_t \Delta\lambda$$

Setting $b = \frac{B}{\Delta\lambda}$, this becomes:

$$H_t \geq bX_t \tag{(IC)}$$

What about the IR constraint ? The agent can get constant utility 0 at any time by resigning. So the IR condition is:

$$W_t \geq 0 \tag{(IR)}$$

3.2.5 The principal

Consider the continuation value of the principal:

$$F\left(X, W\right) = \max_\Gamma E\left[\int_0^\tau e^{-rt}\left\{X_t\left[\mu - cg_t\right]dt - CX_t dN_t^\lambda - dL_t\right\} \parallel X_0 = X,\ W_0 = W\right]$$

over all effort-inducing contracts Γ. It is defined for $X \geq 0$ and $W \geq bX$. Recall that:

$$\begin{aligned} X_t &= X_0 + X_t^i + X_t^d \\ dX_t^i &= g_t X_t dt, \quad 0 \leq g_t \leq \gamma \\ dX_t^d &= \left(x_t - 1\right) X_t, \quad 0 \leq x_t \leq 1 \\ dW_t &= \rho W_t dt - L_t dt + H_t \left(\lambda dt - dN_t^\lambda\right) \end{aligned}$$

The controls are

$$g_t, l_t = \frac{L_t}{X_t}, h_t = \frac{H_t}{X_t}, x_t$$

The corresponding HJB equation is:

$$\text{(HJB)} \quad \begin{cases} rF = \max\{X_t\left[\mu - \lambda C - cg_t - \ell_t\right] + \\ (\rho W_t + h_t X_t \lambda - \ell_t X_t) \frac{\partial F}{\partial W} + g_t X_t \frac{\partial F}{\partial X} \\ -\lambda\left[F - F\left(x_t X_t, W_t - h_t X_t\right)\right]\} \end{cases}$$

We do not have a general method for solving such equations. We will proceed by finding an (almost) explicit solution. This solution will have two properties:

- It will be homogeneous: $F(X, W) = Xf\left(\frac{W}{X}\right) = f(w)$.
- The size-adjusted value function $f(w)$ is concave

We will also need to observations on the controls

- Comparing (IR) with (IC), we derive a more stringent condition. Indeed, (IC) stipulates that if there is an accident, and the size of the project is X_t, the utility of the agent must be instantaneously reduced by bX at least. If $W_t \leq bX_t$, the new utility of the agent becomes $W_t - H_t \leq 0$, so the agent resigns. If the principal wants to keep the agent active, he will need to keep W_t above bX_t, namely:

$$W_t \geq bX_t$$

- In particular, the principal will have to take care that the forbidden region $W_t < bX_t$ is not entered as the result of an accident. In other words, if the current state is (X_t, W_t), with $W_t \geq bX_t$, and an accident occurs, we want the new state (X_t', W_t') to satisfy $W_t^{'} \geq bX_t'$. We have $W_t' \leq W_t - H_t$, so this requires that:

$$X_t' \leq \frac{W_t - H_t}{b}$$

  If there is no downsizing, $X_t' = X_t$, this in turn requires that $X_t \leq \frac{W_t - H_t}{b}$. If $X_t > \frac{W_t - H_t}{b}$, downsizing will be required, that is $X_t' = xX_t$, with:

$$x \leq \frac{W_t - H_t}{bX_t}$$

The system now becomes:

$$\begin{aligned} 0 &\leq g_t \leq \gamma \\ b &\leq h_t \\ 0 &\leq \ell_t \\ 0 &\leq x_t \leq \frac{w - h}{b} \end{aligned}$$

and we are looking for a function $f(w)$, which is concave and satisfies the following delay-differential equation:

$$\text{(HJB reduced)} \quad \begin{cases} rf = \{\mu - \lambda(C + f(w)) + f'(w) \\ +\max_{g,h,l,x}\{g[f(w) - wf'(w) - c] - \ell[1 + f'(w)] \\ +h\lambda f'(w) + \lambda x f\left(\frac{w-h}{x}\right)\} \end{cases}$$

We perform the maximization wrt to each variable separately:

1: wrt ℓ_t

$$\begin{aligned} \ell &= 0 \text{ for } f'(w) > -1 \\ 0 &\leq w \leq w^p \text{ with } f'(w^p) = -1 \end{aligned} \tag{18}$$

2: wrt x Since the map $x \to xf\left(\frac{w-h}{x}\right)$ is increasing, we get:

$$x = \min\left\{\frac{w-h}{b}, 1\right\}$$

It is the smallest possible jump compatible with keeping the agent responsive.

3: wrt h Substituting, we find:

$$hf'(w) + \left(\frac{w-h}{b}\right) f(b) = \frac{w}{b} + h\left(f'(w) - \frac{f(b)}{b}\right)$$

and the coefficient of h is ≤ 0 since f is concave, $f(0) = 0$ and $w \geq b$. It follows that we should choose the smallest possible h namely:

$$h = b$$

4: wrt g Clearly:

$$\begin{aligned} g &= \gamma \text{ if } f(w) - wf'(w) > c \\ g &= 0 \text{ otherwise} \end{aligned}$$

Using again the fact that f is concave and $f(0) = 0$, we find that there is some w^i such that:

$$f(w) - wf'(w) > c \Longleftrightarrow w \geq w^i \tag{19}$$

If $f(w^p) + w^p = f(w^p) - w^p f'(w^p) > c$, we find that $w^i < w^p$

5: determining ℓ at $w = w^p$ When w_t reaches w^p, the agent starts receiving compensation. It is determined by writing:

$$\begin{aligned} dW_t &= \rho W_t dt - X_t \ell_t dt + b\lambda X_t dt \\ \gamma w^p X_t dt &= \rho X_t w^p dt - X_t \ell_t dt + b\lambda X_t dt \\ \ell_t &= (\rho - \gamma) w^p + b\lambda \end{aligned} \tag{20}$$

So she receives a salary $L_t = ((\rho - \gamma) w^p + b\lambda) X_t$, proportional to the size of the project (which increases exponentially) as long as $w_t = w^p$ and there is no accident.

6: summing up

a As long as $\frac{W}{X} = w_p$, and there is no accident, the principal invests at the maximum rate γ and the agent gets ℓ defined by (20). Her utility $W_t = w^p X_t$ increases proportionally to X_t

b When an accident occurs, the principal stops paying the agent. Her utility drops because of forgone salary. She will no longer be paid until her utility raises to $w^p X_t$ again. The principal continues to invest at maximum rate γ.

c When further accidents occur, the principal continues to invest at maximal rate until w_t is brought into the region $w < w^i$. He then stops investing until the w_t is brought back in the region $w_t > w^i$

d The principal does not downside after an accident, as long as $w_t > 2b$. If $w_t < 2b$ and an accident occurs, the principal downsizes the project to bring w_t back to level b.

e As a consequence, w_t never enters the regions $w > w^p$ nor $w < b$

Note that, because of (15), we have:

$$\begin{aligned} dw_t &= \frac{dW_t}{X_t} - W_t \frac{dX_t}{X_t^2} \\ &= \frac{\rho W_t dt + H_t \left(\lambda dt - dN_t^\lambda\right) - L_t dt}{X_t} - w_t \frac{dX_t}{X_t} \\ &= \left(\rho dt - \frac{dX_t}{X_t}\right) w_t + (h_t \lambda - \ell_t) dt - dN_t^\lambda \end{aligned}$$

so that w_t increases in all the regions: the only decreases are due to accidents.

We now need two theorems, one of which will state that the HJB system has a solution $f(w)$, the other one stating that, with this solution, the optimal contract for the principal is found

Theorem 7 *Suppose:*

$$\mu - \lambda C > (\rho - r)\, b \left(2 + \frac{r}{\lambda}\right)$$

Then there is a constant $\bar{c} > 0$ *such that, if* $c < \bar{c}$, *the delay-differential equation:*

$$\begin{gathered} f(w) = \frac{f(b)}{b} w \text{ for } 0 \leq w \leq b \\ r f(w) = \mu - \lambda C + \mathcal{L} f(w) \text{ for } b \leq w \leq w^i \\ (r - \gamma) f(w) = \mu - \lambda C - \gamma c + \mathcal{L}_\gamma f(w) \text{ for } w^i \leq w \leq w^p \\ f(w) = f(w^p) + w^p - w \text{ for } w \geq w^p \end{gathered}$$

has a maximal solution $f(w)$. *The thresholds* w^i *and* w^p *are determined endogeneously by (18) and (19). We have* $w^i < w^p$ *and the operators* $\mathcal{L}$ *and* $\mathcal{L}_\gamma$ *are given by:*

$$\begin{aligned} \mathcal{L} f(w) &= (\rho w + \lambda b) f'(w) - \lambda \left[f(w) - f(w - b)\right] \\ \mathcal{L}_\gamma f(w) &= ((\rho - \gamma) w + \lambda b) f'(w) - \lambda \left[f(w) - f(w - b)\right] \end{aligned}$$

The solution f *is globally concave, and differentiable except at* $w = b$.

Theorem 8 *If* f *is a maximal solution, then the optimal contract* Γ *is the one described above, and* $H_t = h_t X_t$ *satisfies (14), where* U_t *is given by (10). The value of this contract is* w *for the agent and* $F(w)$ *for the principal, where* $W_0 = w$ *is the initial value of* W_t

The interpretation is as follows. At time $t = 0$, the principal decides how much the contract should be worth for the agent. That gives him W_0, while X_0 is the initial size of the project. Applying the contract from (W_0, X_0) then defines subsequent values of (W_t, X_t).

3.3 Moral hazard in continuous time: the Sannikov model

3.3.1 The model:

The agent is in charge of a project which generates a stream of revenue, which accrue to the principal:

$$dX_t = A_t dt + \sigma dZ_t \tag{21}$$

Here Z_t is a standard Brownian motion, $\sigma > 0$ is a constant, and $A_t \geq 0$ is the agent's effort, which is adapted to the filtration $\mathcal{F}_t$ generated by Z_t.

The agent's intertemporal utility is:

$$rE\left[\int_0^\infty e^{-rt} \left(u(C_t) - h(A_t)\right) dt\right]$$

where C_t is the compensation paid by the principal to the agent, u is her utility function, and h the cost of effort.

Effort is non-negative, $0 \leq a$, and we have

$$\begin{aligned} u(c) &\geq 0, \quad u(0)=0, \quad u\prime(c) \to 0 \text{ when } c \to \infty \\ & u \text{ increasing and concave} \end{aligned}$$

$$\begin{aligned} h(0) &= 0, \quad h(a) \geq \gamma_0 a \text{ with } \gamma_0 > 0 \\ & h \text{ increasing and convex} \end{aligned}$$

If $C_t = A_t = 0$, the agent gets no salary and exerts no effort, she gets utility 0.

A *contract* is a specification of some $\mathcal{F}_t$-adapted processes (A_t, C_t).

3.3.2 Incentive-compatible contracts

Assume a contract (C_t, A_t) has been specified. The continuation value of the agent is defined as;

$$W_t(C,A) = rE^A \left[\int_t^\infty e^{-r(s-t)} (u(C_s) - h(A_s))\, ds \mid \mathcal{F}_t \right]$$

Proposition 9 *There is an adapted process Y_t (depending on A and C) such that, for all $t \geq 0$*

$$W_t(C,A) = W_0(C,A) + \int_0^t (W_s(C,A) - u(C_s) + h(A_s))\, ds + r\int_t^T Y_t \sigma dZ_t$$

Proof. Consider the total discounted payoff of the agent:

$$V_t(C,A) = rE^A \left[\int_0^\infty e^{-rt} (u(C_t) - h(A_t))\, dt \mid \mathcal{F}_t \right]$$

By the Martingale Representation Theorem, we have:

$$\begin{aligned} V_t(C,A) &= V_0(C,A) + \int_t^\infty re^{-rs} Y_s \sigma dZ \\ dV_t &= re^{-rt} Y_t \sigma dZ \end{aligned}$$

On the other hand, we have also:

$$\begin{aligned} V_t(C,A) &= r\int_0^t e^{-rs} (u(C_s) - h(A_s))\, ds + e^{-rt} W_t(C,A) \\ dV_t &= e^{-rt} (u(C_t) - h(A_t))\, dt + e^{-rt} dW_t - re^{-rt} W_t dt \end{aligned}$$

Comparing the two expressions for dV_t, we get the result ■

We rewrite this formula in differential form

$$dW_t = r(W_t - u(C_t) + h(A_t))\, dt + rY_t \sigma dZ$$

The Brownian motion dZ_t is not observable by the principal: so he bases his payoff on the assumption that the agent conforms to the contract, i.e.

$$dW_t = r\left(\left(W_t - u\left(C_t\right) + h\left(A_t\right) - A_t\right)dt + Y_t dX_t\right) \tag{22}$$

so that rY_t is the sensitivity of the agent's utility to observed output.

If the agent exerts effort a, while the principal believes her to be exerting effort $a\left(W_t\right)$ she changes $h\left(A_t\right)$ to $h\left(a\right)$ and $E\left[X_t\right]$ to adt. Her expected benefit is:

- $-rh\left(a\right)dt$ between t and $t+dt$
- $rY_t adt$ between $t+dt$ and ∞

The contract will be incentive-compatible if and only if:

$$A_t = \arg\max_a \left\{aY_t - h\left(a\right)\right\}$$

For instance, if h is differentiable and the maximum is attained at a point a with $0 < a < \bar{a}$, the level of Y_t which induces effort $a\left(W_t\right)$ is given by:

$$Y_t = h'\left(A_t\right)$$

We now give a formal proof:

Proposition 10 *(A, C) is IC if:*

$$Y_t A_t - h\left(A_t\right) = \max_{0 \leq a \leq \bar{a}} \left\{Y_t a - h\left(a\right)\right\} \text{ for a.e. } (t, \omega)$$

Proof. Suppose this does not hold. Then there is an alternative strategy A_t^* such that $Y_t A_t^* - h\left(A_t\right) \geq Y_t A_t - h\left(A_t\right)$ everywhere, and $Y_t A_t^* - h\left(A_t h^*\right) > Y_t A_t - h\left(A_t\right)$ on a set of $(t\,\omega)$ of positive measure.

The agent picks a time $t > 0$ and plans to apply the strategy A_s^* for $s \leq t$ and A_s for $s \geq t$. Consider his expected total utility at time t, conditional on available information:

$$\hat{V}_t = r\int_0^t e^{-rs}\left(u\left(C_s\right) - h\left(A_s^*\right)\right)ds + e^{-rt}W_t\left(C, A\right)$$

Note that the last term is $W_t\left(C, A\right)$ and not $W_t\left(C, A^*\right)$ since $A_t = A_t^*$ for $t \geq t$. Hence:

$$\begin{aligned}
e^{rt}d\hat{V}_t &= r\left(u\left(C_t\right) - h\left(A_t^*\right)\right)dt - rW_t dt + dW_t \\
&= r\left(u\left(C_t\right) - h\left(A_t^*\right) - W_t\right)dt + r\left(W_t - u\left(C_t\right) + h\left(A_t\right)\right)dt + rY_t\left(dX - A_t dt\right) \\
&= r\left(h\left(A_t\right) - h\left(A_t^*\right) - A_t + A_t^*\right)dt + rY_t\left(dX - A_t^* dt\right)
\end{aligned}$$

So:

$$\hat{V}_t = W_0\left(C, A\right) + \int_0^t r\left(h\left(A_s\right) - h\left(A_s^*\right) - A_s + A_s^*\right)ds + r\int_0^t Y_s dM_s$$

where $dM_s = dX_s - A_s^* ds$ is a martingale. Hence:

$$E\left[\hat{V}_t\right] = W_0(C, A) + E\left[\int_0^t r\left(h(A_s) - h(A_s^*) - A_s + A_s^*\right) ds\right]$$

The integrand is non-negative, and positive on a set of positive measure. It follows that there is some t such that:

$$E\left[\hat{V}_t\right] > W_0(C, A)$$

But the right-hand side is the expected utility of the agent at time 0 if she applies strategy A from the beginning, while the left-hand side is her expected utility if she applies A^* initially and switches to A at time t. The second strategy is clearly superior, so the contract is not IC. ■

3.3.3 The principal's problem

If the principal does not retire the agent, his intertemporal utility is:

$$rE\left[\int_0^\infty e^{-rt}(dX_t - C_t)\, dt\right] = rE\left[\int_0^\infty e^{-rt}(A_t - C_t)\, dt\right]$$

The principal can retire the agent at any time τ, provided he offers her continuation utility: if the latter is W_τ, the principal must offer the agent a constant stream of consumption c such that:

$$r\int_t^\infty e^{-r(s-t)} u(c)\, dt = u(c) = W_\tau$$

and the cost to the principal is:

$$r\int_t^\infty e^{-r(s-t)} c dt = c$$

So the principal's intertemporal utility is given by:

$$\begin{aligned} &rE\left[\int_0^\tau e^{-rt}(A_t - C_t)\, dt + F_0(W_\tau)\right] \\ &F_0(u(c)) = -c \end{aligned}$$

Note that the principal is risk-neutral, while the agent is risk-averse.

The principal's problem consists of maximizing his utility, subject to:

- the contract being (IC), so that $Y_t = h'(A_t)$
- the contract being (IR), that is, the continuation value W_t being ≥ 0 at all times

We set:

$$F(w) = \sup rE\left[\int_0^\tau e^{-rt}(A_t - C_t)\,dt \mid w = rE\left[\int_0^\tau e^{-rt}(u(C_t) - h(A_t))\,dt\right]\right]$$

The HJB equation is given by:

$$\text{(QVI)}\quad \left\{ \begin{array}{c} F : [0,\ \infty) \to R \text{ is continuous, and } F(w) \geq F_0(w) \text{ everywhere} \\ \max_{a,c}\left\{a - c + F'(w)(w - u(c) + h(a))\,dt + \frac{1}{2}F''(w)\,rh'(a)^2\sigma^2\right\} - rF(w) = 0 \text{ when } F(w) > \\ \max_{a,c}\left\{a - c + F'(w)(w - u(c) + h(a))\,dt + \frac{1}{2}F''(w)\,rh'(a)^2\sigma^2\right\} - rF(w) \leq 0 \text{ when } F(w) = \end{array} \right.$$

This is a quasi-variational inequality, a fertile source of mathematical problems. We will not solve it.

Note that in the region where $F(w) \geq F_0(w)$ it can be rewritten as follows:

$$F''(w) = \min_{a,c} \frac{F(w) - a + c - F'(w)(w - u(c) + h(a))}{rh'(a)^2\sigma^2}$$

Denoting by $a(w, F, F')$ and $c(F')$ the points where the min on the right-hand side is attained (and assuming that they are uniquely defined, otherwise the matter is slightly more complicated), this becomes an equation:

$$rh'(a(w,F,F'))^2\sigma^2 F'' - F + F'(w - u(c(F')) + h(a(w,F,F'))) - a(w,F,F') + c = 0 \tag{23}$$

valid in the region $F(w) \geq F_0(w)$

Proposition 11 *Suppose F solves (QVI) with $F(0) = 0$. Then, for any starting condition $w \geq 0$ such that $F(W_0) > F_0(W_0)$, consider the process W_t given by:*

$$dW_t = r\left((W_t - u(C(W_t)) + h(a(W_t)) - a(W_t))\,dt + h'(a(W_t))\,dX_t\right), \quad W_0 = w \tag{24}$$

and define τ as the first time $F(W_t) = F_0(W_t)$ unexpected''inmath Then $\tau < \infty$ and the contract:

- *$C_t = C(W_t)$ and $A_t = a(W_t)$ for $t \leq \tau$*
- *$C_t = -F_0(W_\tau)$ and $A_t = 0$ for $t > \tau$*

is (IC), (IR), and has value w_0 for the agent and $F(w_0)$ for the principal.

Proof. Denote by $W_t(C, A)$ the continuation value of that contract. We claim that $W_t(C, A) = W_t$. Indeed, equations (22) and (24) coincide, so that two solutions starting from the same point have the same law, and hence:

$$W_0(C, A) = W_0 = w_0$$

so the contract has value w_0 for the agent. Its continuation value at time t is $W_t \geq 0$, so it is (IR). It is (IC) by construction.

Consider the random variable G_t given by:

$$G_t = r \int_0^t e^{-rs} \left(A_s - C_s\right) ds + e^{-rt} F\left(W_t\right)$$

This is a diffusion, and for $t < \tau$, equation (23) and Ito's Lemma tell us that its drift vanishes. So G_t is a martingale until it reaches τ, and by the optional stopping theorem, we have:

$$E\left[r \int_0^\tau e^{-rs} \left(A_s - C_s\right) ds + e^{-r\tau} F\left(W_\tau\right)\right] = E\left[G_\tau\right] = G_0 = F\left(W_0\right)$$

But the value of the contract for the principal is given by:

$$E\left[r \int_0^\tau e^{-rs} \left(A_s - C_s\right) ds + e^{-r\tau} F_0\left(W_\tau\right)\right]$$

and since $F\left(W_\tau\right) = F_0\left(W_\tau\right)$, the result follows. ■ This can happen in two ways, either $W_t = 0$, since $F_0\left(0\right) = F\left(0\right) = 0$, or W_t reaching the lowest value $\bar{w} > 0$ such that $F_0\left(\bar{w}\right) = F\left(\bar{w}\right)$

The fact that we have found the optimal contract for the principal now follows from the next result

Proposition 12 *Any (IC) contract* $\left(A^*, C^*\right)$ *starting from* $W_0 = w$ *achieves profit at most* $F\left(w\right)$

Proof. Consider the agent's continuation value $V\left(A^*, C^*\right)$. Consider again the random variable G_t given by:

$$G_t = r \int_0^t e^{-rs} \left(A_s^* - C_s^*\right) ds + e^{-rt} F\left(W_t\right)$$

■

It follows from equation (23) and Ito's Lemma that its drift is always non-positive for $t < \tau$. So G_t is a supermartingale until the stopping time. It can be shown that if he does not retire the agent when $F\left(W_t\right) = F_0\left(W_t\right)$, his continuation utility is at most $F_0\left(W_t\right)$. So the expected profit of the principal at time 0 is less than:

$$E\left[r \int_0^\tau e^{-rs} \left(A_s - C_s\right) ds + e^{-r\tau} F_0\left(W_\tau\right)\right] \leq E\left[G_\tau\right] = G_0 = F\left(W_0\right)$$

and the result is proved.

To sum up, we have proved the following: among all incentive-compatible contracts yielding value w to the agent, the best one for the principal is the one described in Proposition 11. At time $t = 0$, the principal tells te agen how much expected utility w he is willing to give her, and then implements the contract starting with $W_0 = w$. His expected utility is then $F\left(w\right)$

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