The Micro Economics of Group Behavior: Identification*

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Abstract

Consider a group consisting of S members facing a common budget constraint $p'\xi = 1$; any demand vector belonging to the budget set can be (privately or publicly) consumed by the members. Although the intra-group decision process is not known, it is assumed to generate

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Pareto efficient outcomes; neither individual consumptions nor intra-group transfers are observable. The paper analyzes when, to what extent and under which conditions it is possible to recover the underlying structure - individual preferences and the decision process - from the group's aggregate behavior. We show that although the general version of the model is not identified, specific restrictions such as private or public consumptions and exclusivity can be used to obtain identification. We also show how the presence of distribution factors, defined as variables that can influence behavior only through their impact on the decision process, can be used in the estimation process. We conclude that in all the cases we consider (private consumption only, public consumption only, private and public consumption), while the detailed structure may or may not be fully identifiable, a simple exclusivity assumption (each member is the exclusive consumer of at least one good) is almost always sufficient to formulate unambiguous welfare judgments.

1 Introduction

Group behavior: beyond the 'black box' Consider a group consisting of S members. The group has limited resources; specifically, its global consumption vector ξ must satisfy a standard market budget constraint of the form $p'\xi = 1$ (where p is a vector of prices, and where total group income is normalized to one). Any demand vector belonging to the global budget set thus defined can be consumed by the members. Some of the goods can be privately consumed, while others may be publicly used. The decision process within the group is not known, and is only assumed to generate Pareto efficient outcomes[1]. Finally, neither individual consumptions nor intra-group transfers are observable. In other words, the group is perceived as a 'black box'; only its aggregate behavior, summarized by the demand function $\xi(p)$, is recorded. The goal of the present paper is to provide answers to the

[1]We view efficiency as a natural assumption in many contexts, and as a natural benchmark in all cases. For instance, the analysis of household behavior often takes the 'collective' point of view, where efficiency is the basic postulate. Other models, in particular in the literature on firm behavior, are based on cooperative game theory in a symmetric information context, where efficiency is paramount (see for instance the 'insider-outsider' literature, and more generally the models involving bargaining between management and workers or unions). The analysis of intra group risk sharing, starting with Townsend's seminal paper (1994), provides other interesting examples. Finally, even in the presence of asymmetric information, first best efficiency is a natural benchmark. For instance, a large part of the empirical literature on contract theory tests models involving asymmetric information against the null of symmetric information and first best efficiency (see Chiappori and Salanie (2000) for a recent survey).

general question: when is it possible to recover the underlying structure - namely, individual preferences, the decision process and the resulting intra-group transfers - from the group's aggregate behavior?

In the (very) particular case where the group consists of only one member, the answer is well known: individual demand uniquely defines the underlying preferences. Not much is known in the case of a larger group. However, recent results in the literature on household behavior suggest that, surprisingly enough, when the group is 'small', the structure can be recovered under reasonably mild assumptions. For instance, in the model of household labor supply proposed by Chiappori (1988, 1992), two individuals privately consume leisure and some Hicksian composite good. The main conclusion is that the two individual preferences and the decision process can generically be recovered (up to an additive constant) from the two labor supply functions. This result has been empirically applied (among others) by Fortin and Lacroix (1997) and Chiappori Fortin and Lacroix (2002), and extended by Chiappori (1997) to household production and by Blundell et al. (2000) to discrete participation decisions. Fong and Zhang (2001) consider a more general model where leisure can be consumed both privately and publicly. Although the two alternative uses are not independently observed, they can in general be identified under a separability restriction, provided that the consumption of another exclusive good (e.g. clothing) is observed.

Altogether, these results suggest that there is information to be gained on the contents of the 'black box'. In a companion paper (Chiappori and Ekeland 2004), we investigate the properties of aggregate behavior stemming from the efficiency assumption. We conclude that when the group is small enough, a lot of structure is imposed on collective demand by this basic assumption: there exist strong, testable, restrictions on the way the black box may operate. The main point of the present paper is complementary. We investigate to what extent, and under which conditions, it is possible to recover much (or all) of the interior structure of the black box without opening it. We first show that in the most general case, there exists a continuum of observationally equivalent models - i.e. a continuum of different structural settings generating identical observable behavior. This negative result implies that additional assumptions are required.

We then provide examples of such assumptions, and show that they are surprisingly mild. Essentially, each agent in the group must be the exclusive consumer of (at least) one commodity; moreover, in some case the availability of a 'distribution factor' is required (see below). Under these conditions, the structure that is relevant to formulate welfare judgments is non parametric identified in general (in a sense that is made clear below), *irrespective of the total number of commodities.* We conclude that even when decision processes or intra group transfers are not known, much can be learned about them from the sole observation of the group's aggregate behavior. This conclusion generalizes the earlier intuition of Chiappori (1988, 1992); it shows that the results obtained in these early contributions, far from being specific to the particular settings under consideration, were in fact general.

Identifiability and identification From a methodological perspective, it may be useful to define more precisely what is meant by 'recovering the underlying structure'. The structure, in our case, is defined by the (strictly convex) preferences of individuals in the group and the decision process. Because of the efficiency assumption, for any particular cardinalization of individual utilities the decision process is fully summarized by the Pareto weights corresponding to the outcome at stake. The structure, thus, consists in a set of individual preferences (with a particular cardinalization) and Pareto weights (with some normalization - e.g., the sum of Pareto weights is one).

This structure is not observable; what can be recorded is the group's aggregate demand function $\xi(p)$. In practice, the 'observation' of $\xi(p)$ is a complex process, that entails specific difficulties. For instance, one never observes a (continuous) function, but only a finite number of values on the function's graph. These values arte measured with some errors, which raises problems of statistical inference. In some cases, the data are cross-sectional, in the sense that different groups are observed in different situations; specific assumptions have to be made on the nature and the form of (observed and unobserved) heterogeneity between the groups. Even when the same group is observed in different contexts (say, from panel data), other assumptions are needed on the dynamics of the situation, e.g. on the way past behavior influences present choices. All these issues, which lay at the core of what is usually called the *identification* problem, are outside the scope of the paper.

Our interest, here, is in what has been called the *identifiability* problem, which can be defined as follows: when is it the case that the (hypothetically) perfect knowledge of a smooth demand *function* $\xi(p)$ uniquely defines the underlying structure within a given class? A more formal version of the definition is the following. For any given structure, the maximization of the (Pareto) weighted sum of utilities generates a unique demand function. This defines a mapping from the set of structures to the set of demand functions. Identifiability obtains if this mapping is *injective*, in the sense that two different structures can never generate the same demand function. For instance, in the case of *individual* behavior, a standard result in consumer theory states that identifiability always obtains, meaning that an individual demand function uniquely identifies the underlying preferences. Usual as this property may have become, it arguably remains one of the strongest results in microeconomic theory. The present work can be seen as an attempt at generalizing this classical identifiability property to efficient groups of arbitrary sizes.[2]

Identifiability is a necessary condition for identification. If different structures are observationally equivalent, there is no hope that observed behavior will help to distinguish between them - only ad hoc functional form restrictions can do that. Clearly, observationally equivalent models may have very different welfare implications. Without uniqueness, any normative recommendation based on one particular structural model is thus unreliable, since it will typically be based on the purely arbitrary choice of one underlying

[2]The distinction between identification and identifiability can be traced back to Koopmans's (1949) seminal paper (we thank Martin Browning for suggesting this reference). A difference is that Koopmans's defines a 'structure' as 'a combination of a specific set of structural equations and a specific distribution function of the latent variables' - a 'model' being defined as a 'set of structures'. Koopmans clearly distinguishes two types of identification problems, namely those linked with 'statistical inference' and those due to 'identifiablity'.

structural model among the various possible selections. Clearly, identifiability is only a necessary first step for identification (in the standard, econometric sense). Whether an *identifiable* model is econometrically *identified* depends on the stochastic structure representing the various statistical issues (measurement errors, unobserved heterogeneity,...) discussed above. After all, the abundant empirical literature on consumer behavior, while dealing with a model that is always identifiable, has convinced us that identification crucially depends on the nature of available data.

Parametric versus non-parametric identifiability The identifiability problem may be approached from a parametric or a non parametric perspective. In the parametric approach, a particular functional form is chosen for the structural model, and a reduced form for the demand function is derived. In particular, the derivation highlights the links between the parameters of the structural model and the coefficient of the demand function that will be taken to data. Identification, in this context, is equivalent to the uniqueness of the set of parameters of the structural model corresponding to any specified values for the (estimated) coefficients of the reduced form. Note that in such a context, uniqueness or identifiability are *conditional on the functional* *form*; i.e. it obtains (at best) within a specific and narrow set, defined by the functional form chosen at the outset.

Throughout this paper, our approach, on the contrary, is explicitly *non-parametric*. That is, we try to find conditions that guarantee uniqueness within the general class of smooth, strictly convex preferences and differentiable Pareto weights. We give below an example in which identifiability obtains in a purely parametric sense, while it does not obtain in the non-parametric setting, and we argue that welfare recommendations made in this case are unreliable.

Distribution factors An important tool to achieve identification is the presence of *distribution factors*; see Bourguignon, Browning and Chiappori (1995). These are defined as variables that can affect group behavior only through their impact on the decision process. Think, for instance, of the choices as resulting from a bargaining process. Typically, the outcomes will depend on the members' respective bargaining positions; hence, any factor of the group's environment that may influence these positions (EEPs in McElroy's (1990) terminology) potentially affects the outcome. Such effects are of course paramount, and their relevance is not restricted to bargaining in any particular sense. In general, group behavior depends not only on preferences and budget constraint, but also on the members' respective 'power' in the decision process. Any variable that changes the powers may have an impact on observed collective behavior.

In many cases, distribution factors are readily observable. An example is provided by the literature on household behavior. In their study of household labor supply, Chiappori, Fortin and Lacroix (2002) use the state of the marriage market, as proxied by the sex ratio by age, race and state, and the legislation on divorce, as particular distribution factors affecting the intrahousehold decision process, and therebt its outcome, i.e. labor supplies. They find, indeed, that factors more favorable to women significantly decrease (resp. increase) female (resp. male) labor supply. Using similar tools, Oreffice (2005) concludes that the legalization of abortion had a significant impact on intrahousehold allocation of power. In a similar context, Rubalcava and Thomas (2000) use the generosity of single parent benefits and reach identical conclusions. Thomas, Contreras, and Frankenberg(1997), using an Indonesian survey, show that the distribution of wealth by gender at marriage - another candidate distribution factor - has a significant impact on children health in those areas where wealth remains under the contributor's control[3]. Duflo (2000) has derived related conclusions from a careful analysis of a reform of the South African social pension program that extended the benefits to a large, previously not covered black population. She finds that the recipient's gender - a typical distribution factor - is of considerable importance for the consequences of the transfers on children's health.

Whenever the aggregate group demand is observable as a function of prices *and* distribution factors, one can expect that identification may be easier to obtain. This is actually known to be the case in particular situations. For instance, Chiappori, Fortin and Lacroix (2002) show how the use of distribution factors allows a simpler and more robust estimation of a collective model of labor supply. In the present paper, we generalize these results by providing a general analysis of the estimation of collective models in different contexts, with and without distribution factors.

The results Our main conclusions can be summarized as follows:

- In its most general formulation, the model is not identifiable. Any given aggregate demand that is compatible with efficiency can be derived either from a model with private consumption only, or from a model with

[3]See also Galasso (1999) for a similar investigation.

public consumption only. Even when it is assumed that all consumptions are private (or that they are all public, or that some commodities are privately and other publicly consumed), in the absence of exclusive consumptions there exists a continuum of different structural models that generate the same aggregate demand.

- In the public goods case, a simple exclusivity assumption is in general sufficient to guarantee full, non-parametric identifiability. Specifically, if, for each agent of the group, there exists a commodity which is exclusively consumed by that agent, then, in general, individual preferences and the corresponding Pareto weights can be uniquely recovered, irrespectively of the total number of commodities.

- In the private consumption case, efficiency has a simple interpretation; namely, the decision process can be viewed as involving two stages, one in which the agents agree on some sharing rule for aggregate income, and one in which each agent independently chooses his/her optimal consumption bundle. In this context, the exclusivity assumption allows to recover the sharing rule, up to some additive function ϕ of the prices of the non-exclusive commodities; for each choice of ϕ, preferences are uniquely determined.

A very important remark, however, is that the additive function ϕ is welfare irrelevant. Specifically, we define the *collective indirect utility* of agent s as the utility he/she gets for any values of prices, aggregate income and distribution factors, taking into account the sharing rule within the group: we show that the collective indirect utility of each agent is exactly identifiable. It follows, in particular, that whenever some given change in prices, income and distribution factors is found to benefit agent s for one particular choice of ϕ, then the conclusion holds for all possible choices of ϕ.

- Finally, the same results obtain in the case of public and private consumptions.

Our general conclusion, hence, is that in all three cases we consider (private consumption only, public consumption only, private and public consumption), *exclusivity is sufficient to identify all welfare-relevant aspects of the collective model.*

Section 2 describes the model. A precise statement of the identifiability problem, as well as a negative result for the general case are stated in Section 3. We then consider three specific cases - public consumptions, private consumptions, public and private consumptions - in the next three sections.

2 The model

2.1 Preferences

We consider a S person group. Purchases are denoted by the vector $x \in \mathbb{R}^N$ where $N = n + K$[4]. The group demand can in principle be divided between two uses : private consumption by each person[5], $x_1, ...x_S$, and public consumption X. Here, $x_s \in \mathbb{R}^n$, $X \in \mathbb{R}^K$ and corresponding prices are $(p, P) \in \mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^K$, giving the budget constraint[6]:

$$p' (x_1 + ... + x_S) + P'X = 1$$

Each member has her/his own preferences over the goods consumed in

[4]Formally purchases could include leisure; then the price vector includes the wages - or virtual wages for non-participants.

[5]Throughout the paper, x_s^i denotes the private consumption of commodity i by agent s, and x_s is the vector of private consumption for agent s.

[6]In most of what follows, the group's total income (or total expenditure for empirical purposes) is normalized to one. In particular, we implicitly assume that all functions at stake are homogeneous in prices and income. In some cases, however, we abandon the normalization for explanatory convenience. Then y denotes the group's aggregate income.

the group. In the most general case, each member's preferences can depend on other members' private and public consumptions; this allows for altruism, but also for externalities or any other preference interaction. Then preferences of member s are then of the form $U^s(x_1, ...x_S, X)$, where U^s is strongly concave, twice differentiable in $(x_1, ...x_S, X)$, and strictly increasing in (x_s, X). However, we shall see that identification does not obtain in this general setting, and we shall concentrate on more specific preferences. We will analyze in detail three cases:

1. all goods are publicly consumed: then $n = 0, N = K$ and preferences are $U^s(X)$ for $1 \leq s \leq S$.

2. all goods are privately consumed, with no externalities (except for altruism); then $K = 0, N = n$ and preferences are egoistic $U^s(x_s)$

3. some goods are publicly consumed while others are privately consumed with no externalities; then $K \geq 0,\ n \geq 0,\ N = K + n$. While this case is in a sense 'general', it should be noted that we do not allow for externalities of private consumptions, and that we assume that any given good is known to be either public or private.

Each setting can in principle be extended to allow for preferences of the

’caring’ type (i.e., agent s maximizes an index of the form $W^s\left(U^1,...,U^S\right)$; however, we do not discuss the identifiability of the W^s.[7]

Finally, we shall denote by z the vector of distribution factors.

2.2 The decision process.

We now consider the mechanism that the group uses to decide on what to buy. Note, first, that if the functions $U^1,...,U^S$ represent the same preferences then we are in a ’unitary’ model where the common utility is maximized under the budget constraint. The same conclusion obtains if one of the partners can act as a dictator and impose her (or his) preferences as the group’s maximand. Clearly, these are very particular cases. In general, the ’process’ that takes place within the group is more complex.

Following the ’collective’ approach, we shall throughout the paper postulate efficiency, as expressed in the following axiom :

Axiom 1 (Efficiency) *The outcome of the group decision process is Pareto efficient; that is, for any prices* (p,P) *and distribution factors* z*, the consumption* $(x_1,...x_S,X)$ *chosen by the group is such that no other vector*

[7] Each allocation that is efficient with respect to the W^s must also be efficient with respect to the U^s. The converse is not true (e.g., an allocation which is too unequal may fail to be efficient for the W^s), a property that has sometimes been used to achieve identification.

$\left(\bar{x}_1, ...\bar{x}_S, \bar{X}\right)$ *in the budget set could make all members better off, one of them strictly so.*

Denote the vector of Pareto weights by $\mu = (\mu_1, ..., \mu_S)$, with the normalization $\sum_s \mu_s = 1$. The axiom can be restated as follows: there exists S scalar functions $\mu_s(p, P, z) \geq 0,\ 1 \leq s \leq S$, with $\sum \mu_s = 1$, such that $(x_1, ...x_S, X)$ is a solution of:

$$(\mathrm{P}) \quad \left\{ \begin{array}{c} \max_{x_1,...x_S,X} \sum \mu_s\left(p, P, z\right) U^s(x_1, ...x_S, X) \\ p'\left(x_1 + ... + x_S\right) + P'X = 1 \end{array} \right.$$

For any given utility functions U^1,...,U^S and any price-income bundle, the budget constraint defines a Pareto frontier for the group. From the Efficiency Axiom, the final outcome will be located on this frontier. It is well-known that, for every (p, P, z), any point on the Pareto frontier can be obtained as a solution to problem (P): the vector $\mu\left(p, P, z\right)$, which belongs to the $(S-1)$-dimensional simplex, summarizes the decision process because it determines the final location of the demand vector on this frontier. The map μ describes the distribution of power. If one of the weights, μ_s, is equal to one for every (p, P, z), then the group behaves as though s is the effective dictator. For intermediate values, the group behaves as though each person s has some decision power, and the person's weight μ_s can be seen as an indicator of this power[8].

It is important to note that the weights μ_s will in general depend on prices p and distribution factors z, since these variables may in principle influence the distribution of 'power' within the group, hence the location of the final choice over the Pareto frontier. However, while prices enter both Pareto weights and the budget constraint, distribution factors matter only (if at all) through their impact on μ.

Following Browning and Chiappori (1998), we add some structure by assuming that the functions $\mu_s(\pi, z)$ are continuously differentiable for $s = 1, ..., S$

[8] This interpretation must be used with care, since the Pareto coefficient μ_s obviously depend on the particular cardinalization adopted for individual preferences; in particular, $\mu_s > \mu_t$ does *not* necessarily mean that 's has more power than t'. However, the *variations of* μ_s are significant, in the sense that for any given cardinalization, a change in parameters that increases μ_s while leaving μ_t constant unambiguously ameliorates the relative position of s.

2.3 Characterization of aggregate demand.

Set:

$$\begin{aligned}\pi &:= (p, P) \\ \xi(\pi, z) &:= (x_1(\pi, z) + ... + x_s(\pi, z),\ X(\pi, z)) \in \mathbb{R}^N\end{aligned}$$

The map ξ is the *aggregate demand* of the group, i.e. the solution of program (P). Note that, by the Walras law:

$$\pi' \xi(\pi, z) = 1$$

We define a demand function $\xi(\pi, z)$ to be S*-admissible* if one can find a group of size S such that $\xi(\pi, z)$ is Pareto efficient for the group. Formally:

Definition 2 *Assume that prices π vary on some open subset $\mathcal{P}$ of the positive orthant $\mathbb{R}_+^N$, while distribution factors z vary within some open subset $\mathcal{D}$ of $\mathbb{R}^d$*

- *A demand function $\xi(\pi, z)$ is $S-$*admissible *if there exists S utility functions $U^1, ..., U^S$, strictly increasing, with negative definite Hessian, and a differentiable map μ from $\mathcal{P} \times \mathcal{D}$ into the $(S-1)$-dimensional* *simplex such that* $\xi(\pi, z)$ *solves program (P) for all* (π, z)*.*

- *It is* locally $S-$admissible *near* $\xi(\bar{\pi}, \bar{z})$ *if there exists an open neighborhood* $\mathcal{P}$ *of* $\bar{\pi}$*, an open neighborhood* $\mathcal{D}$ *of* $\bar{z}$*, an open neighborhood* $\mathcal{N}$ *of* $\xi(\bar{\pi}, \bar{z})$*,* S *utility functions* $U^1, ..., U^S$ *defined on* $\mathcal{N}$*, strictly increasing, with negative definite Hessian, and a differentiable map* μ *from* $\mathcal{P} \times \mathcal{D}$ *into the* $(S-1)$*-dimensional simplex, such that* $\xi(\pi, z)$ *solves program (P) for all* (π, z) *in* $\mathcal{P} \times \mathcal{D}$ *.*

In a companion paper, Chiappori and Ekeland (2005), we derive necessary and sufficient conditions for a function $\xi(\pi, z)$ to be S-admissible. For the sake of completeness, we briefly restate these conditions below. Let us first omit the distribution factors:

Proposition 3 *If* $\xi(\pi)$ *is* S*-admissible, then the Slutsky matrix* $S(\pi) = (D_\pi \xi)(I - \pi\xi')$ *can be decomposed as:*

$$S(\pi) = \Sigma(\pi) + R(\pi) \tag{1}$$

where the matrix Σ *is symmetric, negative definite and the matrix* R *is of rank at most* $S-1$*. Equivalently, there exists a subspace* $\mathcal{R}$ *of dimension at least* $N-(S-1)$ *such that the restriction of* $S(\pi)$ *to* $\mathcal{R}$ *is symmetric and* *negative definite. Conversely, if a map* $\xi(\pi)$ *satisfies the Walras law* $\pi'\xi = 1$ *and condition* $SR(S-1)$ *in some neighborhood of* $\bar{\pi}$*, and if the Jacobian* $D_\pi\xi(\bar{\pi})$ *is invertible, then* ξ *is locally* S*-admissible.*

Relation (1) is known as the SR($S-1$) condition. According to Proposition 3, if a map ξ satisfies Walras and SR($S-1$), then one can recover S utility functions of the general form $U^s(x_1, ...x_S, X)$ and S Pareto weights $\mu_s(p, P) \geq 0$ such that $\xi\,(p, P)$ is the collective demand associated with problem (P). A natural question is whether more knowledge about intra-group consumption will generate stronger restrictions. Assume, for instance, that commodities are known to be privately consumed, so that the utility functions are of the form $U_s(x_s)$, or, alternatively, that consumption is exclusively public, so that the preferences are $U_s\,(X)$. Does the integration result still hold when utilities are constrained to belong to these specific classes?

Interestingly enough, the answer is positive. In fact, it is impossible to distinguish the two cases by looking at the aggregate demand only. In the paper mentioned above, we prove the following result:

Proposition 4 *For any given function* $\xi(\pi)$*, with* $\pi'\xi(\pi) = 1$*, satisfying* $SR(S-1)$ *in some neighborhood of* $\bar{\pi}$*, and such that the Jacobian* $D_\pi\xi(\bar{\pi})$ *is* *invertible:*

- *there exist S strictly increasing, strictly concave functions $U^1(X), ..., U^S(X)$, defined in some neighborhood of $\bar{\xi} = \xi(\bar{\pi})$, and S Pareto weights $\mu_1(\pi), ..., \mu_S(\pi)$, defined in some neighborhood $\mathcal{N}$ of $\bar{\pi}$ such that, for all $\pi \in \mathcal{N}$:*

$$\xi(\pi) = \text{ArgMax}\left\{\sum_{s=1}^{S} \mu_s(\pi) U^s(X) \mid \pi' X = 1\right\}$$

- *there exist S maps $x_s(\pi)$, and S Pareto weights $\mu_1(\pi), ..., \mu_S(\pi)$, all defined in some neighborhood $\mathcal{N}$ of $\bar{\pi}$, and S strictly increasing, strictly quasi-concave functions $U^s(x)$ defined in some neighborhood of $\bar{\xi} = \xi(\bar{\pi})$, such that, for all $\pi \in \mathcal{N}$:*

$$\begin{aligned} \xi(\pi) &= \sum x_s(\pi) \\ (x_1(\pi), ..., x_S(\pi)) &= \text{ArgMax}\left\{\sum_{s=1}^{S} \mu_s(\pi) U^s(x_s) \mid \pi'(x_1(\pi) + ... + x_S(\pi)) = 1\right\} \end{aligned}$$

Finally, the same paper provides necessary condition on the effect of distribution factors. Denoting by d the number of distribution factors, so that $z = (z_1, ..., z_d))$ and by Z the Jacobian matrix $D_z\xi$, with general term

$\partial\xi_i/\partial z_k$, one has the following result:

Proposition 5 *If $d \geq S-1$, then $\text{rank } Z \leq S-1$. Denote by $\mathcal{Z}$ the space of vectors $v \in \mathbb{R}^N$ such that $v'Z = 0$. If $\text{rank } Z = S-1$, then the restriction of the Slutsky matrix $S = (D_\pi \xi)(I - \pi\xi')$ to $\mathcal{Z}$ is symmetric and negative definite.*

3 Identifiability: the general problem

3.1 Statement of the problem

Following the discussion above, we now consider the following, general question:

Question I (Identifiability): *Take an arbitrary, S-admissible demand $\xi(\pi, z)$. Is there a* unique *family of differentiable, strictly increasing, strictly convex preference relations on $\mathbb{R}^N$, represented by (non-unique) utility functions $U^s(x_1, ..., x_S, X)$, and, for each cardinalization of preferences, a* unique *family of differentiable Pareto weights $\mu_s(\pi, z)$, $1 \leq s \leq S$, such that $\xi(\pi, z)$ is the aggregate demand associated with problem (P) ?*

Question I refers to what could be called a *non-parametric* definition of identifiability, because uniqueness is required within the general set of well-behaved functions, rather than within the set of functions sharing a specific parametric form in which a finite number of parameters can be varied. We now examine a specific example that illustrates the nature and the scope of this important distinction

3.2 Parametric versus non-parametric identifiability.

The example is borrowed from Blundell, Chiappori and Meghir (2004). Consider a 2-person household in a collective model of labor supply. There are three commodities: two individual leisure L^1, L^2 and a Hicksian composite good C. Wages are denoted w_1 and w_2, non-labor income by y, while the price of the Hicksian good is normalized to one. The Hicksian good is used for private expenditures and some public consumption:

$$C = C^1 + C^2 + K \tag{2}$$

where K denotes the level of expenditures for public consumption. Finally, assume that preferences are Cobb-Douglas:

$$U^s(C^s, L^s, K) = \alpha_s \log L^s + (1 - \alpha_s) \log C^s + \delta_s \log K \tag{3}$$

and that the Pareto weights are related to wages by:

$$\mu_1 = \frac{lw^1}{lw^1 + w^2}, \ \mu_2 = \frac{w^2}{lw^1 + w^2}$$

Here, the α_s, δ_s and l are parameters to be estimated.

Solving (P) leads to the following demand functions:

$$\begin{aligned} L^1 &= \frac{\alpha_1 l}{(1+\delta_1)\, lw^1 + (1+\delta_2)\, w^2} \left(w^1 + w^2 + y\right) \\ L^2 &= \frac{\alpha_2}{(1+\delta_1)\, lw^1 + (1+\delta_2)\, w^2} \left(w^1 + w^2 + y\right) \\ K &= \frac{\delta_1 lw^1 + \delta_2 w^2}{(1+\delta_1)\, lw^1 + (1+\delta_2)\, w^2} \left(w^1 + w^2 + y\right) \end{aligned}$$

Finally, let us introduce, for expositional convenience, a very simple stochastic structure; i.e., the only shock is an additive measurement error, so that the econometric model is

$$
\begin{aligned}
L^1 &= \frac{\alpha_1 l}{(1+\delta_1)\, l w^1 + (1+\delta_2)\, w^2} \left(w^1 + w^2 + y\right) + \varepsilon_1 \\
L^2 &= \frac{\alpha_2}{(1+\delta_1)\, l w^1 + (1+\delta_2)\, w^2} \left(w^1 + w^2 + y\right) + \varepsilon_2 \\
K &= \frac{\delta_1 l w^1 + \delta_2 w^2}{(1+\delta_1)\, l w^1 + (1+\delta_2)\, w^2} \left(w^1 + w^2 + y\right) + \eta
\end{aligned}
\tag{4}
$$

where $(\varepsilon_1, \varepsilon_2, \eta)$ follows some known stochastic structure.

If adequate data are available, this model can be consistently estimated. Then the third equation allows to recover δ_1, δ_2 and l; and the two first give α_1 and α_2 with additional, overidentifying restrictions. One might conclude that the model is fully (actually, over-) identified, that individual preferences and the Pareto weights can be estimated, and that these estimates can be used to formulate normative recommendations.

However, such a conclusion would be very fragile, because it is entirely driven by the choice of the functional form. Indeed, a consequence of the results derived below is that the model at stake is *not* identifiable. There exists a continuum of different structural models (i.e., functional forms for preferences and Pareto weights) that generate the same collective demand

(4)[9].

Non identifiability, here, does not result from the econometrician's inability to exactly recover the form of demand functions - say, because only noisy estimates of the parameters in (4) can be obtained, or even because the functional form itself (and the stochastic structure added to it) have been arbitrarily chosen. These econometric questions have, at least to some extent, econometric or statistical answers. For instance, confidence intervals can be computed for the parameters (and become negligible when the sample size grows); the relevance of the functional form can be checked using specification tests; etc. The non identifiability problem has a different nature: even if a *perfect* fit to *ideal* data was feasible, it would still be impossible to recover the underlying structure from observed behavior, hence to emit reliable normative judgments.

In practice, parametric models are often convenient; the discussion above should by no means be taken to imply that parametric estimations should not be used, or even that it should be resorted to with some reluctance. Postulating a specific functional form is a standard, well established and often

[9]More precisely, uniqueness obtains only within the class of separable preferences (to which our Cobb-Douglas example obviously belongs). General identifiability requires in addition the availability of a distribution factor (see below).

extremely fruitful methodology. We do however submit that the status of the conclusions drawn from parametric estimations crucially depend on whether or not the underlying model is *non-parametrically identifiable*. If it is, then the reliability of the parametric estimates (and, consequently, of the conclusions drawn from it) is directly related to the quality of the empirical fit. If the econometrician can convince himself (and the scientific community) that the model provides a pretty faithful representation of the real phenomenon, then the same level of trust could in principle be put into the conclusions derived from it. The case is however much weaker in the absence of non parametric identifiability. A good empirical fit is no longer sufficient: by definition, many different structural models, with potentially divergent normative implications, have exactly the same fit (since they generate the same reduced forms), hence are exactly as well supported by the data as the initial one.

Of course, this discussion should not be interpreted too strictly. In the end, identifying assumptions are (almost) always needed. The absence of non parametrically identifiability, thus, should not necessarily be viewed as a major weakness. We believe, however, that it justifies a more cautious interpretation of the estimates. More importantly, we submit, as a basic, methodological rule, that an explicit analysis of non parametric identifiability is a necessary first step in any consistent empirical strategy - if only to suggest the most adequate identifying assumptions. Applying this approach to collective models is indeed the main purpose of this paper.

3.3 A general negative result

It should be clear that in the most general version of the model we consider, identifiability cannot obtain. To see why, take a demand function that satisfies SR($S-1$). From Proposition 4 above, it is compatible with at least two different structural models: one where all commodities are privately consumed, and one in which all consumption is public. Quite obviously, these models have very different welfare implications, although they generate the same aggregate demand. This suggests that more specific assumptions are needed. We explore below assumptions regarding the private or public nature of consumptions; i.e. we shall consider models in which either all commodities are publicly consumed, or all consumptions are private, or more generally some commodities are exclusively private and other exclusively public, each consumption type being known ex ante. We shall actually see that the nature of the indeterminacy is deeper than suggested by the previous remark. Even when all consumptions are assumed to be public (or, alternatively, private), it is still the case that a continuum of different structural models generate the same group demand function. In other words, identifying restrictions are needed, that go beyond the publicness (or privateness) of individual consumptions. However, such restrictions may be far from stringent, as it will be documented below.

4 Identifiability with purely public consumptions.

We first consider the benchmark case where all commodities are publicly consumed; then $n = 0$ and $N = K$, and problem (P) becomes:

$$(P1) \begin{cases} \max_X \sum_{s=1}^{S} \mu_s(P)\, U^s(X) \\ X \in \mathbb{R}^N, \quad P'X = 1 \end{cases}$$

4.1 A negative result

A first finding is that, even in the pure public goods context, identifiability does not obtain without additional assumptions. We state this formally.

Definition 6 *Take any C^2 function $X(P)$, with $P'X(P) = 1$, defined on an* *open subset* $\mathcal{P}$ *of* $\mathbb{R}^N_+$*. We shall say that* $X(P)$ *is* S-admissible on $\mathcal{P}$ with public consumption *if there exist* S *utility functions* $U^1, ..., U^S$*, defined in a neighborhood of* $X(\mathcal{P})$*, strictly increasing, with negative definite Hessian, and a* C^1 *map* μ *from* $\mathcal{P}$ *into the* $(S-1)$*-dimensional simplex such that* $X(P)$ *solves problem (P1) for every* $P \in \mathcal{P}$ *. In that case, we shall say that* $(U^1, ..., U^S)$ support *the demand function* $X(P)$*, and that the underlying preferences* $(\preceq_{U^1}, ..., \preceq_{U^S})$ support *the demand function* $X(P)$*.*

If a given family of utility function $(U^1, ..., U^S)$ supports $X(P)$ then so does any other family $(V^1, ..., V^S)$ which has the same underlying preferences. So identifiability can only hold at the level of preferences. In fact, it does not without further assumptions:

Proposition 7 *Assume that all goods are publicly consumed. Let* $X(P)$ *be a demand function defined on an open subset* $\mathcal{P}$ *of* $\mathbb{R}^N_+$*, and supported by a family* $(U^1, ..., U^S)$ *of preferences. Let* $\mathcal{K} \subset \mathcal{P}$ *be a compact subset, set* $\mathcal{X} = X(\mathcal{K})$*, and assume that* $\partial U^s / \partial X_i > 0$ *over* $\mathcal{X}$ *for all* s, i*. Then for every* ε *small enough, there exists a family of utility functions* $(U^1_\varepsilon, ..., U^S_\varepsilon)$*, defined in a neighborhood of* $\mathcal{X}$ *, strictly increasing, with negative definite Hessian, defined in a neighborhood of* $\mathcal{X}$ *and supporting* $X(P)$*. In addition,* *if there is a point $\bar{x} \in \mathcal{X}$ where $\nabla U^s(\bar{x})$ and $\sum_{t \neq s} \nabla U^t(\bar{x})$ are not collinear, then the preferences associated with the $U^s_{\varepsilon_1}$ and $U^s_{\varepsilon_2}$ are different for $\varepsilon_1 \neq \varepsilon_2$*

In other words, if some function $X(P)$ solves problem (P) on some compact set for a particular choice of preferences and Pareto weights, then it solves problem (P) on the same compact set for many other different choices of preferences and Pareto weights - in fact, for a whole continuum of them. Clearly, all the structural models thus defined are observationally equivalent, since they generate the same demand $X(P)$. The condition that $\partial U^i / \partial X_j > 0$ over $\mathcal{X}$ for all i, j is crucial; as we shall see below, exclusive consumptions, that is, commodities such that $\partial U^i / \partial X_j = 0$, are a typical identifying assumption, leading to uniqueness, hence to full identifiability.

Proof. Starting from a particular choice $\{U^s(X), \mu_s(P)\}$, $1 \leq s \leq S$, we shall show that there exists a continuous family $F^\varepsilon = (F^\varepsilon_1, ..., F^\varepsilon_S)$ of linear invertible maps of $\mathbb{R}^S$ into itself, such that if U^s_ε is defined by $U^s_\varepsilon(X) = F^\varepsilon_s\left(U^1(X), ..., U^S(X)\right)$ then $X(P)$ is supported by $\left(U^1_\varepsilon(X), ..., U^S_\varepsilon(X)\right)$ over $\mathcal{P}$.

Set:

$$F^{\varepsilon} = \begin{pmatrix} 1 & -\varepsilon & \cdots & -\varepsilon \\ -\varepsilon & 1 & \cdots & -\varepsilon \\ \vdots & \vdots & \ddots & \vdots \\ -\varepsilon & -\varepsilon & \cdots & 1 \end{pmatrix}$$

The inverse of F^{ε} is:

$$\left[F^{\varepsilon}\right]^{-1} = \begin{pmatrix} A & b & \cdots & b \\ b & A & \cdots & b \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \cdots & A \end{pmatrix}$$

with:

$$\begin{aligned} A &= 1 - \frac{(K-1)\,\varepsilon^2}{1-(K-2)\,\varepsilon-(K-1)\,\varepsilon^2} \\ b &= \frac{\varepsilon}{1-(K-2)\,\varepsilon-(K-1)\,\varepsilon^2} \end{aligned}$$

Both A and b are positive if $\varepsilon > 0$ is small enough. Setting $U_{\varepsilon}^{s}(X) = F_{\varepsilon}^{s}\left(U^{1}(X), ..., U^{S}(X)\right)$, we have:

$$\frac{\partial U_\varepsilon^s}{\partial X_j} = \frac{\partial U^s}{\partial X_j} - \sum_{t \neq s} \varepsilon \frac{\partial U^t}{\partial X_j}$$

$$\frac{\partial^2 U_\varepsilon^s}{\partial X_i \partial X_j} = \frac{\partial^2 U^s}{\partial X_i \partial X_j} - \sum_{t \neq s} \varepsilon \frac{\partial^2 U^t}{\partial X_i \partial X_j}$$

so that U_ε^s behaves like U (that is, it is increasing and has negative definite Hessian) on the compact set $\mathcal{X}$, provided ε is small enough.

Let us look at the indifference curve $\mathcal{C}_\varepsilon = \{x \mid U_\varepsilon^s(x) = U_\varepsilon^s(x)\}$. The normal to $\mathcal{C}_\varepsilon$ at $\bar{x}$ is the unitary vector in the direction of the gradient:

$$\nabla U_\varepsilon^s(\bar{x}) = \nabla U^s(\bar{x}) - \varepsilon \sum_{t \neq s} \nabla U^t(\bar{x})$$

and it changes with ε since $\nabla U^s(\bar{x})$ and $\sum_{t \neq s} \nabla U^t(\bar{x})$ are not collinear.

We conclude by showing that any allocation that is Pareto efficient for the U^s must be Pareto efficient for the U_ε^s. Assume that $X(P)$ is not efficient for the U_ε^s, for some $P \in \mathcal{P}$. Then we know that there exists some ξ such that $P'\xi = 1$ and ξ is preferred to $X(P)$ by all the U_ε^s. and strictly preferred by at least one of them. But then

$$U^s(\xi) = AU^s_\varepsilon(\xi) + \sum_{t\neq s} bU^t_\varepsilon(\xi) > AU^s_\varepsilon(X(P)) + \sum_{t\neq s} bU^t_\varepsilon(X(P)) = U^s(X(P))$$

for all s, so $X(P)$ could not be efficient for U^s either.

Finally, it should be stressed that the F^ε need not be linear in U; by continuity, since $\mathcal{P}$ is compact, any function that is 'close enough' to the U^s in the C^2 topology will lead to the same conclusion. Thus the set of possible choices of preferences and Pareto weights is very large. ■

The intuition behind Proposition 7 is easy to get: it is possible to replace each U^s with a function of all the U^t without decreasing the set of Pareto efficient allocations.

The next result is a by-product of our approach to existence, based on exterior differential calculus. It states essentially that the type of indeterminacy described in Proposition 7 is (almost) the only one. For simplicity, we formulate the result for the case $S = 2$, although it is valid for any S.

Lemma 8 *Assume that all goods are publicly consumed. Let* $X(P)$*, with* $P'X(P) = 1$*, be a map satisfying SR(1) but not SR(0) in some neighborhood of* $\bar{P}$ *and assume the Jacobian* $D_P X(\bar{P})$ *is invertible, so that the inverse map*

$P(X)$ is locally well-defined. Then $X(P)$ is locally S-admissible with public consumption, and the sets

$$\mathcal{H}_{h_1,h_2,h_3} = \left\{ X \mid U^1(X) = h_1,\ U^2(X) = h_2,\ \frac{\mu_1(P(X))}{\mu_2(P(X))} = h_3 \right\}$$

are uniquely identified near $X\left(\bar{P}\right)$

Proof. See Appendix 1. ■

This strong result lies at the core of the identifiability findings below. It states that U^1, U^2 and μ are identified up to one mapping of $\mathbb{R}^3$: if U^1, U^2, μ_1, μ_2, with $\mu_1 + \mu_2 = 1$, support $X(P)$, then for any other family $\bar{U}^1, \bar{U}^2, \bar{\mu}_1, \bar{\mu}_2$, with $\bar{\mu}_1 + \bar{\mu}_2 = 1$, which also support $X(P)$, there must exist functions F, G and H such that

$$\begin{aligned} \bar{U}^1 &= F\left(U^1, U^2, \mu_1\right) \\ \bar{U}^2 &= G\left(U^1, U^2, \mu_1\right) \\ \bar{\mu}_1 &= H\left(U^1, U^2, \mu_1\right) \end{aligned} \tag{5}$$

In other words, given the collective demand function, one can identify, in the space of goods, subsets of codimension 3 along which both members of the household are indifferent and the Pareto weight μ is constant. But one could not, for instance, identify subsets along which both members of the household are indifferent, as defined by the equations $U^2(X) = h_1,\ U^2(X) = h_2$. So the preference relations are not identified.

4.2 Generic identifiability with exclusive goods.

The next task is to work out additional conditions under which full identifiability obtains. A natural assumption is the existence of *exclusive goods*, i.e. goods that are consumed by one member only. We first consider the simple case of two agents ($S = 2$). The following result states that, in that case, one exclusive good per member is sufficient for identifiability, except in particular, non-robust, situations.

Assume henceforth that $S = 2 \leq K$, and that prices vary within some compact set $\mathcal{K}$. Assume that we are observing a demand function $\bar{X}(P),\ P \in \mathcal{K}$, and that it is invertible, leading to an inverse demand function $\bar{P}(X), X \in \bar{X}(P)$. Both are supposed to be C^1.

Assume, furthermore, that all goods are publicly consumed, except for good 1 (resp. 2) that is consumed exclusively by member 1 (resp. 2), so that the two utility functions have the form $U^1(X^1, X^3, ..., X^K)$ and $U^2\left(X^2, X^3, ..., X^K\right)$.

Given such utility functions U^1 and U^2, and $\mu \in C^2(\mathcal{K})$ a distribution function, so that $\mu_1(P) = \mu(P) > 0$ and $\mu_2(P) = 1 - \mu_1(P) > 0$ are Pareto weights, we shall denote by $X(U^1, U^2, \mu, P)$ be the corresponding demand function, that is, the solution problem (P1) for $S = 2$.

Proposition 9 *Assume that $\partial\bar{P}/\partial X^1$ and $\partial\bar{P}/\partial X^2$ are nonzero everywhere. Suppose that (U^1, U^2, μ) and $\left(\tilde{U}^1, \tilde{U}^2, \tilde{\mu}\right)$ both support $\bar{X}(P)$:*

$$X\left(U^1, U^2, \mu, P\right) = \bar{X}(P) = X(\tilde{U}^1, \tilde{U}^2, \tilde{\mu}, P) \ \ \forall P \in \mathcal{P}$$

Then we have the following alternative:

- *either there are two functions φ_1 and φ_2 such that $\widetilde{U}^1 = \varphi_1(U^1)$ and $\widetilde{U}^2 = \varphi_2(U^2)$*
- *or $\mu(P)$ satisfies the following partial differential equation:*

$$\frac{\partial^2}{\partial X^1 \partial X^2}\left[\log \mu\left(P\left(\mu, X\right)\right) - \log(1 - \mu\left(P\left(\mu, X\right)\right)\right] = 0$$

Proof. See Appendix 2. ■

As noted earlier, the function $\mu(P)$ represents the distribution of power within the group. It is extremely unlikely that it would turn out to verify that particular partial differential equation; if it did, in any given case, some rationale would have to be provided why it should. Note that if $\mu(P)$ did satisfy that equation, one could find another distribution function $\nu(P)$, arbitrarily close to $\mu(P)$ in the C^2 topology, which did not; hence any model which would have $\mu(P)$ verify that equation would have to be non-robust.

So Proposition 9 really means that if each individual is the exclusive consumer of at least one commodity, then the preferences are identifiable, except in very particular and non-robust situations. In addition, the location of the final choice $X(P)$ on the Pareto frontier is the same, even though the parametrization may change from (U^1, U^2, μ) to $\left(\tilde{U}^1, \tilde{U}^2, \tilde{\mu}\right)$.

4.3 Collective models of labor supply with public consumptions

An immediate application is to the collective model of household labor supply, initially introduced by Chiappori (1988, 1992). The idea is to consider the household as a two-person group making Pareto efficient decisions on consumption and labor supply; let L^s denote the leisure of member s, and

w_s the corresponding wage. Various versions of the model can be considered. In each of them Proposition 9 applies, leading to full identifiability of the model.

1. **Leisure as an exclusive good**

   In the first model, each member's leisure is exclusive and there is no household production. Labor and non labor incomes are used to purchase commodities $X^1, ..., X^K$ that are publicly consumed within the household; utilities are thus of the form $U^s\left(L^s, X^1, ..., X^K\right)$.

2. **Leisures are public, one exclusive good per member**

   In the second model, leisure of one member is also consumed by the other member; again, there is no household production. The identifying assumption is that there exists two commodities (say, 1 and 2) such that commodity i is exclusively consumed by member s.[10] One can think, for instance, of clothing as the exclusive commodity (as in Browning et al 1994), but many other examples can be considered. Utilities are then of the form $U^s\left(L^1, L^2, X^s, X^3, ..., X^K\right)$. Again, Proposition 9 applies: from the observation of the two labor supplies and the K

[10] This framework is close to (but less general than) that of Fong and Zhang (2000)

consumptions as functions of prices, wages and non labor income, it is possible to uniquely recover preferences and Pareto weights. This is a strong result indeed, since it states that one can, from the sole observation of household labor supply and consumption, identify the partials $\partial U^i / \partial L^j$, $i \neq j$, that is, deduce to what extent individual leisures are publicly consumed.

3. **Leisure as exclusive goods with household production**

As a third example, assume that individual time can be devoted to three different uses: leisure, market work and household production. The domestic good Y is produced from domestic labor under some constant return to scale technology, say $Y = f(t^1, t^2)$,[11] and publicly consumed within the household. Preferences are of the form $U^s\left(L^s, Y, X^1, ..., X^K\right)$ and one can define

$$\tilde{U}^s\left(L^s, t^1, t^2, X^1, ..., X^K\right) = U^s\left(L^s, f\left(t^1, t^2\right), X^1, ..., X^K\right) \qquad (6)$$

Here, Y is not observable in general, but t^1 and t^2 are observed, which typically requires data over time use (obviously, there is little chance to

[11] Other inputs can be introduced at no cost, provided they are observable.

identify household production if neither the output nor the input are observable).

From Proposition 9, the $\tilde{U}^s$ are identified. Then the production technology can be recovered up to a scaling factor, using the assumption of constant return to scale, from the relation:

$$\frac{\partial \tilde{U}^1/\partial t^1}{\partial \tilde{U}^1/\partial t^2} = \frac{\partial \tilde{U}^2/\partial t^1}{\partial \tilde{U}^2/\partial t^2} = \frac{\partial f/\partial t^1}{\partial f/\partial t^2}$$

which in addition generates an overidentifying restriction. Finally, (6) allows to recover the U^s; again, the separability property in (6) generates additional, testable restrictions.

4. **Leisures are public, one exclusive good per member and household production**

   Finally, one can combine models 2 and 3 by assuming that leisure is a public good, but the demand for two other exclusive goods can be observed. Again, identifiability generically obtains in this context.

4.4 Identifiability with $S > 2$ members

The identifiability result extends to larger groups. Say $2 < S < K$ and prices vary within some compact set $\mathcal{P}$. Given a function $X(P)$, with $P'X(P) = 1$, recall that $X(P)$ is supported by $(\mu_1, ..., \mu_S,\ U^1, ..., U^S)$ if $X(P)$ is the collective demand function associated with $(\mu_1, ..., \mu_S,\ U^1, ..., U^S)$, that is, the solution of problem (P1).

Given S and K, we show in the appendix how to define an integer $\bar{R}$ and family of polynomials $\Pi_m, 1 \leq m \leq M$ in $S(1+\sum_{r=1}^{\bar{R}} C_K^{K+r-1})$ variables. The polynomials Π_m depend only on S and K, and can be computed explicitly in each particular case.

Proposition 10 *Assume that $S < K$ and that prices vary within some compact set $\mathcal{P}$. Let $X(P)$, $P \in \mathcal{P}$, be the observed demand of the group, and assume that the inverse demand $P(X)$ is well-defined and C^1. Assume moreover that each group member i is the exclusive consumer of good i, with $1 \leq i \leq S$, while commodities $S+1$ to K are consumed by all agents. If $X(P)$ is supported by two families $(\mu_1, ..., \mu_S,\ U^1, ..., U^S)$ and $(\tilde{\mu}_1, ..., \tilde{\mu}_S,\ \tilde{U}^1, ..., \tilde{U}^S)$, then one of the following alternatives holds:*

(a) either there are functions $\varphi_s, 1 \leq s \leq S$ such that $\tilde{U}^s = \varphi_s(U^s)\ \ \forall s$.

(b) or the functions $\mu_1, ..., \mu_S$ satisfy a system of nonlinear partial differ-

ential equations:

$$\Pi_m\left(\mu_s, \frac{\partial \mu_s}{\partial X^k}, ..., \frac{\partial^r \mu_s}{(\partial X^1)^{r_1} ... (\partial X^K)^{r_K}}\right) = 0,\ 1 \leq m \leq M, \tag{7}$$

Proof. See Appendix 3. ■

The functions $\mu_1, ..., \mu_S$ with $0 < \mu_i < 1$ and $\sum \mu_i = 1$ represent the division of power within the group. As noted earlier, in the case $S = 2$, it is extremely unlikely that they would turn out to verify that particular partial differential equation, and if they did, it would be a non-robust property that would need some supporting rationale to be credible. So the preceding Proposition really means that if each individual is the exclusive consumer of at least one commodity, then the preferences are identifiable, except in pathological situations.

It is important to note that in the public good case with exclusive commodities, identifiability obtains without the help of distribution factors. These can be used to increase the robustness of the estimation and to generate overidentifying restrictions (see Chiappori and Ekeland 2001).

5 Identifiability with purely private consumptions

We now consider the second benchmark case where all commodities are privately consumed; then $K = 0$ and $N = n$.

5.1 The sharing rule interpretation

In the private good context, a key concept is that of a sharing rule. The basic remark is that the group can be formally seen as a small Arrow-Debreu economy, endowed with a linear 'production' technology characterized by the production constraint $p' \sum_s x_s \leq 1$. From the second welfare theorem, any efficient allocation is an equilibrium of this economy. Also, because of the linear nature of the production technology, the equilibrium price vector must equal p (up to normalization). The decision process, thus, can be summarized by the transfers that are needed to implement the selected equilibrium. This motivates the following result:

Proposition 11 *Assume that all goods are privately consumed and there is no consumption externality. The the efficiency axiom is equivalent to the following: there exists a* sharing rule $\rho(p) = (\rho_1(p), ..., \rho_S(p))$ *, with* $\rho_s(p) \geq$

0 *and* $\sum \rho_s(p) = 1$, *such that the consumption of member* $s = 1, ..., S$ *solves*

$$x_s(p) = \arg\max \{U^s(x) \mid p'x \leq \rho_s(p)\} \tag{8}$$

Efficiency, in the private consumption case, is equivalent to a two-stage process. In stage one, agents agree on (or bargain over) a sharing rule that defines the transfers between members. At stage two, each agent chooses her consumption subject to the budget constraint defined by the sharing rule. Such a process is always efficient, whatever the particular sharing rule at stake. There is a one-to-one, increasing correspondence between a member's share $\rho_s(p)$ and her Pareto weight $\mu_s(p)$: a member who has more weight in the decision process will be able to attract a larger fraction of the group income. From a welfare viewpoint, shares are more convenient conceptual tools than Pareto weights, because they are expressed in monetary units and are independent of the cardinal representation of preferences. These advantages come however at a price: the sharing rule approach can only be adopted in the pure private goods case.[12]

[12] However, one can, in a model with private and public consumption, define a *conditional* sharing rule (see below).

In the sequel, we shall need some more insight into the optimization problem (8). The necessary condition for optimality is:

$$DU^s(x) = \lambda_s(x)\,p$$

for some Lagrange multiplier $\lambda_s(p) \geq 0$, and the second-order condition is that the restriction of the Hessian $D^2U^s(x)$ to the hyperplane $[DU^s(x)]^\perp$ is positive semi-definite.

Definition 12 *Assume U^s is C^2. We shall say that $x_s(p)$ is a strong maximizer for (U^s, ρ_s) at p if:*

- *there is no other maximizer of problem (8)*
- *the Lagrange multiplier $\lambda_s(p)$ is positive*
- *the restriction of $D^2U^s(x)$ to $[DU^s(x)]^\perp$ is positive definite.*

Now consider a C^2 family of utility functions $U^s_\varepsilon(x)$ and sharing rules $\rho^\varepsilon_s(p)$ with $U^s_0 = U^s$ and $\rho^\varepsilon_s = \rho_s$. Consider the optimization problem::

$$\max\left\{U^s_\varepsilon(x)\right\} \mid p'x \leq \rho^\varepsilon_s(p) \tag{9}$$

If $x_s(p)$ is a strong maximizer for (U^s, ρ_s) at $\bar{p}$, then the implicit function theorem can be applied near $\bar{p}$ to show that, for ε close enough to zero, the following system, considered as $(N+1)$ equations for the $(N+1)$ variables (x, λ) :

$$\begin{aligned} DU_\varepsilon^s(x) &= \lambda p \\ p'x &= \rho_s^\varepsilon(p) \end{aligned}$$

has a unique solution $(x_s^\varepsilon(p), \lambda_s^\varepsilon(p))$, defined in a neighborhood of $\bar{p}$ and C^1 in that neighborhood. This solution must then be the global maximizer. In other words, the necessary conditions for optimality are also sufficient for small perturbations of the initial problem. Another consequence of the implicit function theorem is that the individual demand function x_s is locally invertible near p, and so of course are the x_s^ε for ε small enough.

5.2 Welfare-relevant identifiability

Before investigating the identifiability properties of the private good model, a remark is in order. In general, the main reason why identifiability matters is the ability to formulate normative judgments. When considering a possible reform that would affect the group's budget constraint and/or its decision process, one would like to assess who in the group is likely to gain or lose from the reform, and how much. For instance, if a tax reform or the targetting of a family benefit to a specific member may have an impact on the intrahousehold allocation of welfare, as argued by numerous studies, then one should ideally be able to assess this impact.

Clearly, complete identifiability of preferences and the decision process (as summarized by the Pareto weights) is a sufficient condition for such welfare judgments to be formulated. In the public good case, the condition is also necessary; i.e., if different structural models arer compatible with the same observed behavior, these models will in general generate different welfare evaluations. However, in the case of private commodities, this equivalence does not hold. Indeed, we shall see that there typically exist a class of *different* structural models that (i) are compatible with the same observed behavior, and (ii) always generate *identical* welfare assessments - so that the differences are welfare-irrelevant.

To see why this is the case, consider the following model of household labor supply, directly borrowed from Chiappori (1992). The economy has three commodities: two leisures and some private, Hicksian composite consumption good, the price of which is normalized to one. Preferences are of the form $U^s(L_s, C_s)$, where L_s (resp. C_s) stands for s's leisure (resp. consumption). Individual labor supply functions $L^s(w_1, w_2, y)$ are observed; individual consumptions are not.

In this context, one can prove that the sharing rule is identifiable, but only up to an additive constant. Namely, if some function $\rho(w_1, w_2, y)$ is compatible with observed behavior, so is any function of the form $\bar{\rho}(w_1, w_2, y) = \rho(w_1, w_2, y) + K$, where K is a constant. The intuition for this result is straightforward. If U^1, U^2 are the individual utilities corresponding to ρ, define two alternative utilities $\bar{U}^1, \bar{U}^2$ by

$$\begin{aligned}\bar{U}^1(L_1, C_1) &= U^1(L_1, C_1 - K) \\ \bar{U}^2(L_2, C_2) &= U^2(L_2, C_2 + K)\end{aligned}$$

Graphically, in a standard two good diagram with leisure on the horizontal axis and consumption on the vertical one, the indifference curves of $\bar{U}^1$ (resp. $\bar{U}^2$) are those of U^1 (resp. U^2), only shifted downwards (resp. upwards). Clearly, for any price-income bundle the individual labor supply generated by $\bar{U}^1$ and the sharing rule $\bar{\rho}$ is equal to that generated by U^1 and ρ, while the demand for the private commodity is smaller by K units of the consumption good. Similarly, $\bar{U}^2$ and $\bar{\rho}$ generate the same labor supply than U^2 and ρ, and a private consumption which is larger by exactly K units. It results that the group's behavior (as defined by the two labor supplies and the aggregate consumption of the private ghood) is the same in the two models.

Now, what about welfare judgments? The basic remark, here, is that the $\bar{U}$ construct has a very simple interpretation in welfare terms. Namely, the $\bar{U}$ are such that the *utility reached by each member, when facing the sharing rule $\bar{\rho}$, is always the same as under the U and ρ.* Under $\bar{\rho}$, 1 always receive K less units of the consumption good (and 2 receives K more), but both achieve the same utility level as initially. It follows, in particular, that any reform that is found to increase 1's welfare under U^1 and ρ will also increase welfare under $\bar{U}^1$ and $\bar{\rho}$, irrespective of the value of K. Or, in other words: the two, observationaly equivalent structural models (U^1, U^2, ρ) and $\left(\bar{U}^1, \bar{U}^2, \bar{\rho}\right)$ are different, but the difference is welfare irrelevant.

As it turns out, this intuition is very general. In a private good setting, exact identifiability never obtains (unless each individual consumption is recorded); indeed, the 'indifference curve shifting' trick can always be applied (and generalized) to generate different but observationally identical structural models. However, exact identifiability is not needed to formulate welfare judgments.

In what follow, we thus proceed to show that exclusivity is sufficient to generate what can be called 'welfare-relevant' identifiability; a precise deifnition of the concept, based on the notion of collective indirect utilities, will be given in the process.

5.3 A negative result

Again, we first consider the case $S = 2$. Then the sharing rule is fully defined by $\rho_1(p)$ (since $\rho_2 = 1 - \rho_1$), and we can use the notation $\rho_1(p) = \rho(p), \rho_2(p) = 1 - \rho(p)$.

From Proposition 11, the question we investigate can thus be stated as follows: when is it possible to uniquely recover individual preferences and the sharing rule from observed demand? As before, we start with a negative result.

Definition 13 *Take any C^2 function $x(p)$, with $p'x(p) = 1$, defined on an open subset $\mathcal{P}$ of $\mathbb{R}_+^N$. We shall say that $x(p)$ is* 2-admissible on $\mathcal{P}$ with private consumptions *if there exists a pair of strictly increasing strictly quasi-concave functions $U^1(x), U^2(x)$ and a sharing rule $\rho(p)$ such that $x(p) =$*

$x_1(p) + x_2(p)$, *where:*

$$x_s(p) = \arg\max \{U^s(x) \mid p'x = \rho_s(p)\}, \ s = 1, 2 \tag{10}$$

In that case, we shall say that (U^1, U^2, ρ) support *the demand function* $x(p)$*.*

Proposition 14 *Assume that all goods are privately consumed and there is no consumption externality. Assume that* $\bar{x}(p)$ *is* 2*-admissible with private consumption, and supported by* $\left(\bar{U}^1, \bar{U}^2, \bar{\rho}\right)$ *on some neighborhood of* $\bar{p}$*. Assume moreover that* $x_s(\bar{p})$ *is a strong maximizer of* $\left(\bar{U}^s, \rho_s\right)$ *at* $\bar{p}$ *, for* $s = 1, 2$*. Then, for every* ε *small enough, there is a family* $(U_\varepsilon^1, U_\varepsilon^2, \rho^\varepsilon)$*, still supporting* $\bar{x}(p)$ *in a neighborhood of* $\bar{p}$*, such that* $(U_0^1, U_0^2, \rho^0) = \left(\bar{U}^1, \bar{U}^2, \bar{\rho}\right)$ *and* $\rho^\varepsilon \neq \bar{\rho}$ *for* $\varepsilon \neq 0$*.*

Proof. Define $V^s(p) = \max_x \left\{\bar{U}^s(x) \mid p'x \leq \bar{\rho}_s(p)\right\}$ for $s = 1, 2$. From the envelope theorem, one gets that

$$DV^s(p) = \lambda_s(p)\left(D\bar{\rho}_s - \bar{x}_s(p)\right) \tag{11}$$

where $\lambda_s(p)$ is the Lagrange multiplier of the budget constraint. Now, consider the partial differential equation in ρ :

$$p' D\rho - \rho = \frac{p' DV^1(p)}{\lambda_1(p)} \tag{12}$$

This is a linear equation which can be solved by the method of characteristics. By (11), $\rho = \bar{\rho}$ is a particular solution, so that the general solution is $\rho(p) = \bar{\rho}(p) + \varphi(p)$, where $\varphi(p)$ solves the homogeneous equation $\varphi - p' D_p \varphi = 0$. But this simply means that φ is a homogeneous function of degree 1 in p.

Choosing such a function $\varphi(p)$, define:

$$\begin{aligned} \rho_1^\varepsilon(p) &= \rho(p) + \varepsilon\varphi(p), \ \rho_2^\varepsilon(p) = 1 - \rho_2^\varepsilon(p) & (13) \\ x_s^\varepsilon(p) &= D\rho_s^\varepsilon - \frac{DV^s(p)}{\lambda_s(p)} & (14) \end{aligned}$$

For $\varepsilon = 0$ we have $x_s^\varepsilon = \bar{x}_s$. Since $x_s(\bar{p})$ is a strong maximizer of $(\bar{U}^s, \rho_s)$ at $\bar{p}$, the demand function $x_s^\varepsilon(p)$ is locally invertible in a neighborhood of $\bar{p}$, provided ε is small enough. Its inverse will be denoted by $p_s^\varepsilon(x)$; it is the inverse demand function. Note that:

$$x_1^{\varepsilon}(p) + x_2^{\varepsilon}(p) = \bar{x}_1(p) + \bar{x}_2(p) = \bar{x}(p) \quad (15)$$

$$p' x_s^{\varepsilon}(p) = \rho_s^{\varepsilon}(p) \quad (16)$$

Now define:

$$U_{\varepsilon}^s(x) = V^s(p_s^{\varepsilon}(x)) \quad (17)$$

This defines a utility function U^s in a neighborhood of $\bar{x}_s(\bar{p})$. Differentiating, we get:

$$\begin{aligned} DU_{\varepsilon}^s(x) &= DV^s(p_s^{\varepsilon}(x))\, Dp_s^{\varepsilon}(x) \\ &= \lambda_s(p_s^{\varepsilon}(x))\,(D\rho_s^{\varepsilon}(p_s^{\varepsilon}(x)) - x_s^{\varepsilon}(p_s^{\varepsilon}(x)))\, Dp_s^{\varepsilon}(x) \end{aligned}$$

On the other hand, differentiating relation (16), we get $D\rho_s^{\varepsilon}(p) = x_s^{\varepsilon}(p) + p' Dx_s^{\varepsilon}(p)$ .Substituting in the above equation, we get:

$$DU_{\varepsilon}^s(x) = \lambda_s(p_s^{\varepsilon}(x))\, p_s^{\varepsilon}(x)$$

Inverting the map $p_s^\varepsilon(x)$, we rewrite this relation as follows:

$$DU_\varepsilon^s(x_s^\varepsilon(p)) = \lambda_s(p)\,p$$

which is the necessary condition for optimality in the problem:

$$\max_x \{U_\varepsilon^s(x) \mid p'x \leq \rho^\varepsilon(p)\}$$

Since $x_s(\bar{p})$ is a strong maximizer of $(\bar{U}^s, \rho_s)$ at $\bar{p}$, the necessary condition is also sufficient, so that $x_s^\varepsilon(p)$ is the solution of the problem. In view of (15) and (16), this proves that $(U_\varepsilon^1, U_\varepsilon^2, \rho^\varepsilon)$ also support $\bar{x}(p)$ near $\bar{p}$, as we claimed.

■

To summarize, the sharing rule can (at best) only be identified up to some one-homogeneous function of prices.

5.4 Identifiability with exclusive goods

As above, we now consider the case of *exclusive goods*, i.e. goods that are consumed by one member only. Throughout this subsection, it will be convenient to abandon the normalization of income to 1. So demand functions will 0-homogeneous in prices and, while sharing rules will be 1-homogeneous.

We start by the simplest case, initially considered by Chiappori (1988, 1992). Namely, we assume there exists three goods, and that good 1 (resp. 2) is consumed exclusively by member 1 (resp. 2). Chiappori's result is the following:

Proposition 15 *Assume that there are three goods, and that good 1 (resp. 2) is consumed exclusively by member 1 (resp. 2). Then, for almost all* 0*-homogeneous,* C^2 *functions* $x(p,y)$ *which satisfy* $p'x(p,y) = y$ *and are* 2*-admissible with private consumptions, the sharing rule can be identified up to a linear function* cp_3*: if* $\rho(p,y)$ *is a sharing rule, so is* $\rho(p,y) + cp_3$ *for small enough* c*. Conversely, if* ρ *and* ρ' *are two sharing rules, then* $\rho - \rho'$ *is of the form* cp_3*. For each value of* c*, there is a unique pair of strictly increasing, strictly concave individual preferences which support* $x(p,y)$*.*

Proof. See Chiappori (1992) for the proof and an explanation of what is meant by "almost all"[13]. ■

The identifiability result, however, is valid only for three goods, a framework that is typically used for models of labor supply but may still seem restrictive. A immediate corollary is the following:

[13] In Chiappori's initial paper, the price p_3 was normalized to be one, so the sharing rule was identified up to an additive constant.

Corollary 16 *Assume that there are n goods, and that good 1 (resp. 2) is consumed exclusively by member 1 (resp. 2). Generically, for any given* 0-*homogeneous function $x(p,y)$, with $p'x(p,y) = y$, the* 1-*homogeneous sharing rule $\rho(p,y)$, with $\rho_1 = \rho$ and $\rho_2 = 1-\rho$, is identifiable up to an additive,* 1-*homogeneous function of $(p_3,...,p_n)$.*

Proof. Choose the vector $(p_3,...,p_n)$ to be proportional to some particular value $(\bar{p}_3,...,\bar{p}_n)$:

$$(p_3,...,p_n) = \lambda(\bar{p}_3,...,\bar{p}_n)$$

Define the functions $\tilde{U}^s$ by

$$\tilde{U}^s(x_s,\xi_s) = \max_{x_s^3,...,x_s^n}\left\{U^s\left(x_s,x_s^3,...,x_s^n\right) \mid \bar{p}_3x_s^3+...+\bar{p}_nx_s^n = \xi_s\right\}$$

Then program (8) become

$$\max\left\{\tilde{U}^s(x_s,\xi_s) \mid p_sx_s+\lambda\xi = \rho_s(p,y)\right\}$$

Proposition 15 applies. Hence ρ is identified up to some linear function of λ. This argument applies for any choice of the vector $(\bar{p}_3,...,\bar{p}_n)$; note, however, that the constant will in general vary with $(\bar{p}_3, ..., \bar{p}_n)$. Hence ρ is identified up to some additive, 1-homogeneous function of $(p_3, ..., p_n)$. ■

In other words, if $\rho(p, y)$ is a sharing rule that is compatible with observed behavior, then any other sharing rule ρ must be of the form

$$\rho(p, y) = \rho(p, y) + \phi(p_3, ..., p_n) \tag{18}$$

5.5 Identification with exclusive goods and a distribution factor.

In the present subsection, we shall assume that there is a distribution factor z, which has a non-zero impact on behaviour, so that the sharing rule ρ_s and the observed demands x_s are functions of (p, y, z), with $\partial x_s / \partial z \neq 0$. We also take $S = 2$, and we assume that there are at least $S + 1$ goods, two of which are exclusive.

In this framework, one can readily generate different structural models generating the same demand. Indeed, take any 1-homogeneous function $\phi(p_3, ..., p_n)$, and define, as above, a new sharing rule by $\hat{\rho}(p, y, z) =$

$\rho(p, y, z) + \phi(p_3, ..., p_n)$. Also, define the new indirect utilities $\hat{V}^i$ by

$$\hat{V}^1(p, y) = V^1(p, y - \phi(p_3, ..., p_n)) \tag{19}$$

$$\hat{V}^2(p, y) = V^2(p, y + \phi(p_3, ..., p_n)) \tag{20}$$

Note that if the V^i are strictly decreasing and quasi-convex in p, so are the $\hat{V}^i$ for small enough.

For $j \geq 3$:

$$\begin{aligned}
\frac{\partial \hat{V}^1(p, y)}{\partial p_j} &= \frac{\partial V^1(p, y - \phi(p_3, ..., p_n))}{\partial p_j} - \frac{\partial V^1(p, y - \phi(p_3, ..., p_n))}{\partial y} \frac{\partial \phi}{\partial p_j} \\
\frac{\partial \hat{V}^1(p, y)}{\partial y} &= \frac{\partial V^1(p, y - \phi(p_3, ..., p_n))}{\partial y}
\end{aligned}$$

and hence, using Roy's identity:

$$\begin{aligned}
\hat{x}_1^j(p, \rho(p, y, z)) &= \frac{\partial \hat{V}^1(p, \rho(p, y, z)) / \partial p_j}{\partial \hat{V}^1(p, \rho(p, y, z)) / \partial y} = \\
&\quad \frac{\partial V^1(p, \rho(p, y, z)) / \partial p_j}{\partial V^1(p, \rho(p, y, z)) / \partial y} - \frac{\partial \phi}{\partial p_j} \\
&= x_1^j(p, \rho(p, y, z)) - \frac{\partial \phi}{\partial p_j} \qquad (21)
\end{aligned}$$

Similarly

$$\hat{x}_2^j\left(p, y-\rho\left(p, y, z\right)\right) = x_2^j\left(p, y-\rho\left(p, y, z\right)\right) + \frac{\partial \phi}{\partial p_j} \tag{22}$$

and

$$\hat{x}_1^j\left(p, \rho\left(p, y, z\right)\right) + \hat{x}_2^j\left(p, y-\rho\left(p, y, z\right)\right) = x_1^j\left(p, \rho\left(p, y, z\right)\right) + x_2^j\left(p, y-\rho\left(p, y, z\right)\right)$$

Finally, for $j = 1, 2$, the partials of ϕ vanish, so

$$\begin{aligned} \hat{x}_1^1\left(p, \rho\left(p, y, z\right)\right) &= x_1^1\left(p, \rho\left(p, y, z\right)\right) \\ \hat{x}_2^2\left(p, y-\rho\left(p, y, z\right)\right) &= x_2^2\left(p, y-\rho\left(p, y, z\right)\right) \end{aligned}$$

and the two frameworks generate the same demand.

In fact, the following result states that this is the *only* possible indeterminacy in the construction of individual demands:

Proposition 17 *Assume that both* $(U^1, U^2, \hat{\rho})$ *and* $\left(\hat{U}^1, \hat{U}^2, \hat{\rho}\right)$ *support the same demand function* $x\left(p, y, z\right)$*, and let* $\left(x_1^j, x_2^j\right), \ j = 1, ..., n$ *and* $\left(\hat{x}_1^j, \hat{x}_2^j\right), \ j = 1, ..., n$ *denote the corresponding individual demands. Suppose that the func-*

tions $\left(x_1^j, x_2^j, \hat{x}_1^j, \hat{x}_2^j\right)$, $j = 1, ..., n$ *are continuously differentiable over some open subset* Ω*, and the partials* $\frac{\partial}{\partial z} x^1 (p, y, z)$*,* $\frac{\partial}{\partial z} x^2 (p, y, z)$*, and* $\frac{\partial}{\partial \rho} x_1^1 (p, \rho)$ *do not vanish on some open, dense subset of* Ω*. Then there exists a function* $\phi (p_3, ..., p_n)$ *defined over* Ω*, such that*

$$\hat{\rho} (p, y, z) = \rho (p, y, z) + \phi (p_3, ..., p_n), \tag{23}$$

and, for all $j = 1, ..., n$*:*

$$\hat{x}_1^j (p, \hat{\rho} (p, y, z)) = x_1^j (p, \rho (p, y, z)) - \frac{\partial \phi}{\partial p_j} (p_3, ..., p_n) \tag{24}$$

$$\hat{x}_2^j (p, y - \hat{\rho} (p, y, z)) = x_2^j (p, y - \rho (p, y, z)) + \frac{\partial \phi}{\partial p_j} (p_3, ..., p_n) \tag{25}$$

Proof. See Appendix 4. ■

Note that equations (24) and (25) are precisely (21) and (22), which follow from (19) and (20). Also, the existence of a distribution factor, although not indispensable, greatly simplifies the proof, and generate identifiability under more general conditions.

5.6 Collective indirect utility

These results are best understood by using the notion of *collective indirect utility*, first introduced by Blundell, Chiappori and Meghir (2005). Let $V^i(p, y)$ denote the indirect utility corresponding to agent i's agent direct utility $U^i(x)$. As usual, V^i only depends on i's preferences; it is agent i's individual indirect utility. When preferences are strictly increasing, $\partial V^i / \partial y > 0$ at each point.

Now, for any prescribed sharing rule $\rho_i(p, y, z)$, we can express V^i directly as a function of prices, income, and the distribution factor:

$$W^i(p, y, z) := V^i(p, \rho_i(p, y, z))$$

In words: for any given sharing rule, W^i describes i's resulting utility when the group is faced with a bundle (p, y, z). It is called the collective indirect utility of agent i, to reflect the fact that the definition of W^i implicitly includes the sharing function ρ_i, hence an outcome of the collective decision process. In particular, in contrast with the individual indirect utility V^i, the collective indirect utility W^i can only be defined in reference to a particular decision process. Whenever normative judgments are at stake, the collective indirect utility is the relevant concept, since it measures the level of utility that will ultimately be reached by each agent, taking into account the redistribution that will take place within the household.

We now can state an important consequence of Proposition 17:

Corollary 18 *In the above setting, the collective indirect preferences of each agent are identifiable: there exists two functions* $F^i(w_i)$, *increasing with respect to* w_i, *such that*

$$\hat{V}^i(p, \hat{\rho}_i(p, y, z)) = F^i\left(V^i(p, \rho_i(p, y, z))\right) \tag{26}$$

Proof. Assume that both (U^1, U^2, ρ) and $\left(\hat{U}^1, \hat{U}^2, \rho\right)$ support $x(p, y, z)$. Denote the corresponding, collective indirect utilities by $W^s(p, y, z)$ and $\hat{W}^s(p, y, z)$. Then, by condition (23) :

$$\hat{\rho}(p, y, z) = \rho(p, y, z) + \phi(p_3, ..., p_n)$$

On the other hand, we have, by the envelope theorem:

$$D_p\left[V^1(p, \rho(p, y, z))\right] = \lambda_1(p, \rho(p, y, z))\left[D_p\rho - x_1(p, \rho(p, y, z))\right]$$

where $\lambda_1(p,\rho)$ is the Lagrange multiplier associated with the budget constraint of the first member, and $x_1(p,\rho)$ her Marshallian demand. Conditions (23) and (24) then tell us that:

$$\begin{aligned}
\frac{1}{\hat{\lambda}_1(p,\hat{\rho})} D_p\left[\hat{V}^1(p,\hat{\rho})\right] &= D_p\hat{\rho} - \hat{x}_1(p,\hat{\rho}(p,y,z)) \\
&= D_p\rho + D_p\phi - x_1(p,\rho(p,y,z)) - D_p\phi \\
&= D_p\rho - x_1(p,\rho(p,y,z))
\end{aligned}$$

so that:

$$\frac{1}{\hat{\lambda}_1(p,\hat{\rho})} D_p\left[\hat{V}^1(p,\hat{\rho})\right] = \frac{1}{\lambda_1(p,\rho)} D_p\left[V^1(p,\rho)\right] \tag{27}$$

Similarly:

$$\frac{1}{\hat{\lambda}_1(p,\hat{\rho})} D_r\left[\hat{V}^1(p,\hat{\rho})\right] = 1 = \frac{1}{\lambda_1(p,\rho)} D_r\left[V^1(p,\rho)\right] \tag{28}$$

Equations (27) and (28) hold for all (p,y,z), after substituting $\rho = \rho(p,y,z)$ and $\hat{\rho} = \hat{\rho}(p,y,z)$, and the Lagrange multipliers are positive. This implies relation (26) for $i=1$, and for $i=2$ it is proved in the same way. ■

In words: while individual demands are identified only up to the one-homogeneous function $\phi(p_3,...,p_n)$, *this indetermination is not welfare rel-*

evant. Indeed, the construct above shows that the level utility reached by each member with the new utilities $\hat{U}^s$ (or $\hat{V}^s$) and the new sharing rule $\hat{\rho}$ is the same as initially.

In the end, the general conclusion is that *the presence of (at least) one exclusive commodity per person allows to identify collective indirect utilities in all cases* (i.e., with both private and public consumptions). In the public goods case, identifying collective indirect utilities is equivalent to identifying direct utilities and the Pareto weights. This equivalence does not hold in the case of private consumption; then direct utilities are not fully identified, although this lack of identification does not hamper welfare judgments.

5.7 Cross-sectional identifiability using distribution factors.

A corollary of Proposition 15 concerns the joint analysis of consumption and labor supply in a cross sectional context. We thus specify the previous model to fit precisely this framework. Assume that leisures L^1, L^2 are privately consumed, and let w_1, w_2 denote the corresponding wages. In addition, there are $n-2$ private goods $x^3, ...x^n$, that are privately (but not exclusively) consumed by the members. We assume there is variation in wages but not in prices (cross-sectional assumption), so that prices can be normalized to 1, and we denote non-labor income by y. Finally, we assume that one can observe a distribution factor z that has a non-zero impact on behavior, in the sense that $\partial L^i/\partial z \neq 0$ for all i. Hence the sharing rule is a function $\rho(w_1, w_2, y, z)$ of wages, non-labor income and the distribution factor.

We now show that, in this context, if $L_z^1/L_y^1 \neq L_z^2/L_y^2$, it is possible:

1. to identify the sharing rule up to an additive constant, and

2. to identify individual demands for commodities $1, ..., n$, as functions of wages and non-labor income, again up to an additive constant.

Although this identifiability result does not allow to recover individual preferences (price variations would be needed for that), it still has an important (and somewhat surprising) implication: in a collective model of consumption and labor supply estimated on cross sectional data, it is possible to recover each person's consumption of each commodity, as well as the income and wage elasticities of individual demands for each good[14].

The proof is in two steps

[14]In particular, this conclusion generalizes the results derived by Browning, Bourguignon, Chiappori and Lechene (1994) under the assumption of fully constrained labor supply.

Step 1: we know that

$$\begin{aligned} L^1(w_1, w_2, y, z) &= \lambda_1(w_1, \rho(w_1, w_2, y, z)) \\ L^2(w_1, w_2, y, z) &= \lambda_2(w_2, y - \rho(w_1, w_2, y, z)) \end{aligned} \tag{29}$$

where λ_s is member s's Marshallian demand for leisure. It follows that:

$$\frac{L_y^1}{L_z^1} = \frac{\rho_y}{\rho_z}, \frac{L_y^2}{L_z^2} = -\frac{1-\rho_y}{\rho_z} \tag{30}$$

and

$$\frac{L_{w_2}^1}{L_z^1} = \frac{\rho_{w_2}}{\rho_z}, \frac{L_{w_1}^2}{L_z^1} = \frac{\rho_{w_1}}{\rho_z} \tag{31}$$

At any point where $L_z^1/L_y^1 \neq L_z^2/L_y^2$, equation (30) allows to recover ρ_y and ρ_z, with $\rho_z \neq 0$; then (31) gives ρ_{w_1}, ρ_{w_2}, and ρ is identifiable up to an additive constant[15]**.** For each value of the constant, λ_1 and λ_2 are exactly identifiable from (29).

Step 2: Now consider the demand for commodity x^i. We have that

[15] Although the identification of ρ is a direct consequence of Proposition 15, the method just presented, initially introduced by Chiappori, Fortin and Lacroix (2002), is specific to the presence of a distribution factor.

$x^i = x_1^i + x_2^i$ with

$$\begin{aligned} x_1^i\left(w_1, w_2, y, z\right) &= \xi_1^i\left(w_1, \rho\left(w_1, w_2, y, z\right)\right) \\ x_2^i\left(w_1, w_2, y, z\right) &= \xi_2^i\left(w_2, y - \rho\left(w_1, w_2, y, z\right)\right) \end{aligned}$$

where $\xi_s^i\left(w_s, \rho\right)$ is member s's Marshallian demand for good i. It follows that

$$\begin{aligned} \frac{\partial x^i}{\partial z} &= \frac{\partial x_1^i}{\partial z} + \frac{\partial x_2^i}{\partial z} = \left(\frac{\partial \xi_1^i}{\partial \rho} - \frac{\partial \xi_2^i}{\partial \rho}\right)\rho_z \\ \frac{\partial x^i}{\partial y} &= \frac{\partial x_1^i}{\partial y} + \frac{\partial x_2^i}{\partial y} = \left(\frac{\partial \xi_1^i}{\partial \rho} - \frac{\partial \xi_2^i}{\partial \rho}\right)\rho_y + \frac{\partial \xi_2^i}{\partial \rho} \end{aligned}$$

where $\frac{\partial \xi_s^i}{\partial \rho}$ denotes s's marginal propensity to consume. These two equations allow to identify the $\frac{\partial \xi_s^i}{\partial \rho}$. Finally, since:

$$\begin{aligned} \frac{\partial x^i}{\partial w_1} &= \frac{\partial x_1^i}{\partial w_1} + \frac{\partial x_2^i}{\partial w_1} = \left(\frac{\partial \xi_1^i}{\partial \rho} - \frac{\partial \xi_2^i}{\partial \rho}\right)\frac{\partial \rho}{\partial w_1} + \frac{\partial \xi_1^i}{\partial w_1} \\ x_{w_1} &= x_{w_1}^1 + x_{w_1}^2 = \xi_w^1 + \left(\xi_\rho^1 - \xi_\rho^2\right)\rho_{w_1} \end{aligned}$$

we identify $\frac{\partial \xi_1^i}{\partial w_1}$, and $\frac{\partial \xi_2^i}{\partial w_2}$ obtains in a similar way. This proves that the ξ_s^i are identifiable up to an additive constant.

5.8 Identifiability with $S > 2$ members.

We now show how the previous results extend to any group size. We start with an extension of Proposition 15.

Proposition 19 *Assume that there are $S+1$ goods, with $S \geq 3$, and that good s is consumed exclusively by member s for $s = 1, ..., S$. Generically, for any given function $x(p)$ that is S-admissible with private consumptions and satisfies $p'x(p) = 1$, the sharing rule $\left(\rho_1(p), ..., \rho_S(p) = 1 - \sum_{i=1}^{S-1} \rho_i(p)\right)$ is identifiable up to $S-1$ additive constants. For each value of the constants, there exist* a unique S-uple *of positive functions $x_1(p), ..., x_S(p)$ and* a unique S-uple *of strictly increasing, strictly convex individual preferences such that $x(p) = \sum_s x_s(p)$ and $x_s(p)$ is the best choice of agent s in the budget set $p'x = \rho_s(p)$*

Proof. For convenience, we change the normalization by posing that the price of the last, non exclusive good $S+1$ is one; then the price vector is $p = (p_1, ..., p_S)$ and income is denoted by y. Note that:

$$x^s(p, y) = \xi_s^s(p, \rho_s(p, y)) \tag{32}$$

implies:

$$\frac{\partial x^s/\partial p_j}{\partial x^s/\partial y} = \frac{\partial \rho_s/\partial p_j}{\partial \rho_s/\partial y} \quad \text{for } j \neq s$$

$$\frac{\partial \rho}{\partial p_j} = f_j \frac{\partial \rho}{\partial y} \text{ for } j \neq s \tag{33}$$

where we have set $\rho = \rho_s$ and $f_j = \frac{\partial x^s/\partial p_j}{\partial x^s/\partial y}$ to simplify notations. Take $i \neq j$, both different from s. It follows from (33) that ρ_s also satisfies the equation:

$$\frac{\partial \rho}{\partial y} \sum_k \left(\frac{\partial f_i}{\partial p_k} - \frac{\partial f_j}{\partial p_k} \right) - \left(\frac{\partial f_i}{\partial y} f_j - \frac{\partial f_j}{\partial y} f_i \right) = 0$$

so that $\frac{\partial \rho_s}{\partial y}$ is identified (unless its coefficient is zero, which does not happen in general), and by (33) again, so are the $\frac{\partial \rho_s}{\partial p_j}$ for $j \neq s$. This last derivative is identified by writing:

$$\frac{\partial \rho_s}{\partial p_j} = 1 - \sum_{s \neq j} \frac{\partial \rho_s}{\partial p_j}$$

■

As before, one can derive an immediate corollary:

Corollary 20 *Assume that there are n goods, $n \geq S + 1$, and that good s is consumed exclusively by member s. Generically, for any given function $x(p)$, with $p'x(p) = 1$, the share of member s, $\rho_s(p)$, is identifiable up to an* *homogeneous function of* $\left(p^{S+1}, ..., p_n\right)$

The generalization of Proposition 18 is straightforward; we just state it, leaving the proof to the reader.

Proposition 21 *In the general private goods setting with* S *agents,* $n \geq S+1$ *goods and at least one distribution factor, assume that good* s *is consumed exclusively by member* s*. Then, for almost all* C^2 *functions* $x(p, y)$ *which satisfy* $p'x(p, y) = y$ *and are 2-admissible with private consumptions, the collective indirect preferences of each agent are exactly identifiable.*

Finally, the result on cross sectional identifiability of individual demands also obtains provided that there are enough distribution factors (technically, at least $S-1$ distribution factors are required). The argument is basically the same as above. We have just seen that the sharing rule is identifiable (up to an additive constant). Now, consider the demand for commodity x^i. We have $x^i = \sum_s x_s^i$ with

$$x_s^i\left(w_1, ..., w_S, y, z_1, ..., z_{S-1}\right) = \xi_s^i\left(w_1, \rho_s\left(w_1, ..., w_S, y, z_1, ..., z_{S-1}\right)\right)$$

where ξ_s^i is member s's Marshallian demand for good i. It follows that

$$\begin{aligned}\frac{\partial x^i}{\partial z_j} &= \sum_s \frac{\partial x_s^i}{\partial z_j} = \sum_s \frac{\partial \xi_s^i}{\partial \rho_s}\frac{\partial \rho_s}{\partial z_j} \\ &= \sum_s \left(\frac{\partial \xi_s^i}{\partial \rho_s} - \frac{\partial \xi_S^i}{\partial \rho_S}\right)\frac{\partial \rho_s}{\partial z_j} \end{aligned} \tag{34}$$

This provides $S-1$ equations. If these equations are independent (in the sense that the matrix $\left(\frac{\partial \rho_s}{\partial z_j}\right)$ is of rank $S-1$), they allow to identify the $S-1$ differences $\left(\frac{\partial \xi_s^i}{\partial \rho_s} - \frac{\partial \xi_S^i}{\partial \rho_S}\right)$. Then:

$$\frac{\partial x^i}{\partial y} = \sum_s \left(\frac{\partial \xi_s^i}{\partial \rho_s} - \frac{\partial \xi_S^i}{\partial \rho_S}\right)\frac{\partial \rho_s}{\partial y} + \frac{\partial \xi_S^i}{\partial \rho_S}$$

identifies $\frac{\partial \xi_S^i}{\partial \rho_S}$, while:

$$\frac{\partial x^i}{\partial w_j} = \sum_s \left(\frac{\partial \xi_s^i}{\partial \rho_s} - \frac{\partial \xi_S^i}{\partial \rho_S}\right)\frac{\partial \rho_s}{\partial w_j} + \frac{\partial \xi_j^i}{\partial w_j}$$

identifies the $\frac{\partial \xi_j^i}{\partial w^j}$. This proves that the ξ_j^i are identifiable up to an additive constant.

6 Public and Private goods

Finally, we consider the case where some goods are privately consumed while others are public within the group. We thus consider the general model defined above, in which (i) externalities are assumed away (preferences are of the egoistic or caring type), (ii) private goods 1 and 2 are exclusive, in the sense that good i is exclusively consumed by consumer i (iii) there is a non-trivial distribution factor z.

The program describing the group's behavior is thus[16]:

$$
\text{(P')}\left\{\begin{array}{c}
\max_{x_1,\ldots x_S,X}\sum_{s=1}^{2}\mu_s\left(p,P,y,z\right)U^s(x^s,x_s^3\ldots,x_s^n,X) \\
p'\left(x_1+x_2\right)+P'X=y
\end{array}\right.
$$

We now proceed to show that under these assumptions, *the collective indirect preferences are identifiable* in general. The approach is in four steps.

1. We first define a *conditional sharing rule*. Its existence stems from the following:

Lemma 22 *Let* (x_1^*, x_2^*, X^*) *denote the solution to program (P'). Then*

[16] For the sake of simplicity, we consider the two agents case. The extension to a larger number is left to the reader.

the vector $x_i^* (p, P, y, z)$ *solves*

$$\max_x \left\{ U^i (x, X^*) \mid p'x = p'x_i^* (p, P, y, z) \right\} \tag{35}$$

Lemma 22 states that no other bundle than an agent's efficient choice of private consumption could provide more utility (at the optimal level of public consumption) without costing more. The proof is straightforward: if a higher utility could be achieved at the same cost (say, for some $\bar{x}_i$), then the maximand in (P') could be increased by replacing x_i^* with $\bar{x}_i$, a contradiction. The conditional sharing rule is then defined by:

$$\rho_i (p, P, y, z) = p'x_i^* (p, P, y, z)$$

and we set as usual $\rho_1 = \rho$ and $\rho_2 = y - P'X^* (p, P) - \rho$. Note that the ρ_i are 1-homogeneous with respect to (p, P, y).

2. Since $P'X^* (p, P)$ is observed, it follows from Propositions 15 that the conditional sharing rule is is identified up to a function of $(p_3, ..., p_n, P)$. More precisely, assume that $(U^1, U^2, \hat{\rho})$ and $\left(\hat{U}^1, \hat{U}^2, \hat{\rho}\right)$ support the same demand function $x (p, P, y, z)$. Define the *conditional individual*

indirect utilities V^i by:

$$V^i(p, r_i, X) = \max\left\{U^i(x_i, X) \mid p'x_i = r_i\right\}$$

Corollary 18 tells us that the conditional collective indirect preferences are identified. More precisely, there are functions $F^i(w_i, X)$ such that

$$V^i(p, \rho_i(p, P, y, z), X) = F^i\left(\hat{V}^i(p, \hat{\rho}_i(p, P, y, z), X), X\right)$$

Note that the function F^i may depend on X in general.

3. We now consider some open subset $\mathcal{O}$ of $\mathbb{R}^N$ such that the Jacobian determinant $D_P X^*$ does not vanish on $\mathcal{O}$. By the implicit function theorem, one can define over $\mathcal{O}$ a function P^* such that the condition $X^*(p, P, y, z) = X$ is equivalent to $P = P^*(p, X, y, z)$.

   We now fix $(p_3, ..., p_n)$ to some given value, and we investigate the demand for the public good X. Assuming an interior solution, the optimal level of consumption is determined by the standard first order conditions of Bowen-Lindahl-Samuelson:

$$\frac{\partial V^1/\partial X^k}{\partial V^1/\partial r_1}(p,\rho_1(p,P,y,z),X)+\frac{\partial V^2/\partial X^k}{\partial V^2/\partial r_2}(p,\rho_2(p,P,y,z),X)=P_k^*(p,X,y,z)$$

or, given the previous computation, for $1 \leq k \leq K$:

$$\frac{1}{\partial\hat{V}^1/\partial r_1}\frac{\partial F^1/\partial X^k}{\partial F^1/\partial w_1}+\frac{1}{\partial\hat{V}^2/\partial r_2}\frac{\partial F^2/\partial X^k}{\partial F^2/\partial w_2}= \tag{36}$$

$$P_k^*(p,X,y,z)-\left(\frac{\partial\hat{V}^1/\partial X^k}{\partial\hat{V}^1/\partial r_1}+\frac{\partial\hat{V}^2/\partial X^k}{\partial\hat{V}^2/\partial r_2}\right) \tag{37}$$

where the $V^i(p,\rho,X)$ are known and the $F^i(w_i,X)$ are unknown. Clearly, only the ratio $\frac{\partial F^i/\partial X^k}{\partial F^i/\partial V^i}$ can (at best) be identifiable, reflecting the fact that F^i is (at best) only identifiable up to some increasing transform. Hence we define

$$\Phi_k^i(w_i,X)=\frac{\partial F^i/\partial X^k}{\partial F^i/\partial w_i}(w_i,X) \tag{38}$$

so that (36) can be rewritten as:

$$\frac{1}{\partial \hat{V}^1/\partial r_1}\Phi_k^1\left(\hat{V}^1\left(p,\hat{\rho}_1,X\right),X\right)+\frac{1}{\partial \hat{V}^2/\partial r_2}\Phi_k^2\left(\hat{V}^2\left(p,\hat{\rho}_2,X\right),X\right) \tag{39}$$

$$P_k^*\left(p,X,y,z\right)-\left(\frac{\partial \hat{V}^1/\partial X^k}{\partial \hat{V}^1/\partial r_1}+\frac{\partial \hat{V}^2/\partial X^k}{\partial \hat{V}^2/\partial r_2}\right)$$

where both sides (including the $\hat{\rho}_i$ and the ϕ_i) have to be evaluated at (p,X,y,z).

4. We now proceed to show that generically (in a sense that will be made precise later), the solution to these equations (if any) is unique. The result is coming from the fact that the unknowns $\Phi^i\left(w_i,z,X\right)$ are functions of $(K+2)$ variables only, while the equations depend in general on the $K+n+2$ variables (p,X,y,z) - in fact, $K+n+1$ because both sides are 1-homogeneous with respect to (p,y). To use this feature, let us first note that the right-hand side (39) is linear in $\Phi^1=(\Phi_1^1,...,\Phi_K^1)$ and $\Phi^2=(\Phi_1^2,...,\Phi_K^2)$. Thus, if there exist two distinct solutions (Φ^1,Φ^2) and $\left(\bar{\Phi}^1,\bar{\Phi}^2\right)$, the differences:

$$\psi^i=\Phi^i-\bar{\Phi}^i$$

must satisfy the homogenous equations:

$$\frac{1}{\partial \hat{V}^1/\partial r_1}\psi_k^1\left(\hat{V}^1\left(p,\hat{\rho}_1,X\right),X\right)+\frac{1}{\partial \hat{V}^2/\partial r_2}\psi_k^2\left(\hat{V}^2\left(p,\hat{\rho}_2,X\right),X\right)=0$$

If ψ_k^1 and ψ_k^2 do not vanish, the functions $\hat{V}^1\left(p,r_1,X\right)$ and $\hat{V}^2\left(p,r_1,X\right)$ must satisfy the relations:

$$\log\left|\frac{\psi_k^1\left(\hat{V}^1\left(p,r_1,X\right),X\right)}{\psi_k^2\left(\hat{V}^2\left(p,r_2,X\right),X\right)}\right|=\log\left|\frac{\frac{\partial \hat{V}^1}{\partial r_1}\left(p,r_1,X\right)}{\frac{\partial \hat{V}^2}{\partial r_2}\left(p,r_2,X\right)}\right|,\quad 1\leq k\leq K$$

where we have to substitute:

$$r_1=\hat{\rho}_i\left(p,P^*\left(p,X,y,z\right),y,z\right)\quad 1\leq i\leq 2$$

Now we change variables, by setting:

$$\hat{V}^i\left(p,\hat{\rho}_i\left(p,P^*\left(p,X,y,z\right),y,z\right),X\right)=v_i,\quad 1\leq i\leq 2$$

By the implicit function theorem, these relations can be inverted, yielding, say:

$$p_i=\sigma_i\left(\tilde{p},X,y,z,v\right),\quad 1\leq i\leq 2$$

for $i = 1, 2$, with $\tilde{p}$ denoting $(p_2, ..., p_n)$ and $v = (v_1, v_2)$ Set

$$\sigma(\tilde{p}, X, y, z, v) = (\sigma_1(\tilde{p}, X, y, z, v), \sigma_2(\tilde{p}, X, y, z, v), \tilde{p})$$

The equation then becomes:

$$\log \left| \frac{\psi_k^1(v_1, X)}{\psi_k^2(v_2, X)} \right| = \log \left| \frac{\frac{\partial \hat{V}^1}{\partial r_1}(\sigma(\tilde{p}, X, y, z, v), r_1, X)}{\frac{\partial \hat{V}^2}{\partial r_2}(\sigma(\tilde{p}, X, y, z, v), r_2, X)} \right| \tag{40}$$

where we have to substitute:

$$\begin{aligned} r_1 &= \hat{\rho}_1(\sigma(\tilde{p}, X, y, z, v), P^*(\sigma(\tilde{p}, X, y, z, v), X, y, z), y, z) \\ r_2 &= \hat{\rho}_2(\sigma(\tilde{p}, X, y, z, v), P^*(\sigma(\tilde{p}, X, y, z, v), X, y, z), y, z) \end{aligned}$$

Equation (40) requires that the right-hand side be the sum of a function of (v_1, X) and a function of (v_2, X). For generic functions $V^i(p, r_i, X)$, this property is not satisfied (note for instance the dependence on z), hence it must be the case that the coefficients ψ_k^1 and ψ_k^2 vanish:

$$\psi_k^1(v_1, X) = \psi_k^2(v_2, X) = 0, \quad k = 1, ..., K$$

almost everywhere. We conclude that, when equations (39) have a solution (Φ^1, Φ^2), the solution is unique, implying that the $\frac{\partial F^i / \partial X^k}{\partial F^i / \partial V^i}$ are uniquely recovered from (38).

This result generalizes a theorem demonstrated in Blundell, Chiappori and Meghir (2005), who consider a collective model of household labor supply with public and private expenditures in a three-commodities framework without price variations.

7 Conclusion

The main goal of the paper is to assess under which conditions the aggregate behavior of a group provides enough information to recover the underlying structure (i.e., preferences and the decision process) even when nothing is known (or observed) about the intra household decision making mechanism beyond efficiency. Although the general version of the model is not identifiable, we show that identifiability may obtain under natural assumptions. Specifically:

- When all commodities are publicly consumed, the existence of an exclusive good for each member buys full identifiability of preferences and the decision process (as summarized by the corresponding Pareto weights).

- In the alternative, polar case where consumption is exclusively private, things are more complex. A first result is that in a three-good setting, the existence of an exclusive good for each member is in general sufficient to guarantee full identifiability, in the sense that the sharing rule (which describes how income is shared across members) and individual preferences are determined up to some additive constant. For an arbitrary number of goods, when a (relevant) distribution factor is available, identifiability still obtains, but only up to an additive function of the prices of the non-exclusive commodities.

  In the latter case, however, the additive function is welfare irrelevant; i.e., the observation of one exclusive commodity for each agent allows to pin down the ordinal representation of indirect collective utilities, which in turn are sufficient to formulate normative evaluations.

- Finally, the conclusion extends to the (more general) case in which some goods are private and others public. Again, one can define conditional indirect utilities, which are identifiable in a welfare-relevant sense. Our general conclusion is thus that, in the variety of cases we consider, *the observation of one exclusive commodity per agent allows in general to fully assess the welfare-relevant concept that summarize preferences and the decision processes.*

Finally, we adopt throughout the paper a 'non parametric' standpoint, in the sense that our results do not rely on specific functional form assumptions. Obviously, the introduction of a particular functional form is likely to considerably facilitate identifiability; that is, it may well be the case that, for models that are not identifiable in the non parametric sense, all parameters of a given functional form can be exactly identifiable, even when the form is quite flexible. In the end, the results above show that not much is needed to formulate normative judgements that take into account the complex nature of collective decision processes; one exclusive commodity par agent is 'generically' sufficient. Now these conclusions are crying out for more empirical applications.

□

References

[1] Arnold, V.I. : *"Mathematical Methods of Classical Mechanics"*, Springer-Verlag, 1978

[2] Bertrand, M. S. Mullainathan and D. Miller (2000), "Public Policy and Extended Families: Evidence from South Africa", *Mimeo*, MIT.

[3] Blundell, R., P.A. Chiappori and C. Meghir (2005), "Collective Labor Supply With Children", *Journal of Political Economy*, forthcoming.

[4] Blundell, R., P.A. Chiappori, T. Magnac and C. Meghir (2000), "Collective Labor Supply: Heterogeneity and Nonparticipation", *Mimeo*, UCL.

[5] Bourguignon, F., M. Browning and P.-A. Chiappori (1995), "The Collective Approach to Household Behaviour", *Working Paper 95-04*, Paris: DELTA.

[6] Browning, M., F. Bourguignon, P.-A. Chiappori and V. Lechene (1994), "Incomes and Outcomes: A Structural Model of Intra–Household Allocation", *Journal of Political Economy*, 102, 1067–1096.

[7] Browning, M. and P.-A. Chiappori (1998), "Efficient intra-household allocations: A general characterization and empirical tests", *Econometrica* 66, 1241-1278.

[8] Bryant, R.L., S.S. Chern, R.B. Gardner, H.L. Goldschmidt and P.A. Griffiths : *"Exterior Differential Systems"*, Springer-Verlag, New York, 1991

[9] Cartan, E. : "*Les systèmes différentiels extérieurs et leurs applications géométriques*", Hermann, Paris, 1945

[10] Chiappori, P.-A. (1988), 'Rational Household Labor Supply', *Econometrica*, 56, 63-89.

[11] Chiappori, P.-A. (1992), ‘Collective Labor Supply and Welfare’, *Journal of Political Economy*, 100, 437-67.

[12] Chiappori, P.-A. (1997), ”Introducing Household Production in Collective Models of Labor Supply”, *Journal of Political Economy*, 105, 191-209

[13] Chiappori, P.A., R., Blundell and C., Meghir (2001), ”Collective Labor Supply with Children”, mimeo.

[14] Chiappori, P.A., and I. Ekeland (1997): ”A Convex Darboux Theorem”, *Annali della Scuola Normale Superiore di Pisa*, 4.25, 287-97

[15] Chiappori, P.A., and I. Ekeland (1999): ”Aggregation and Market Demand : an Exterior Differential Calculus Viewpoint”, *Econometrica*, 67 6, 1435-58

[16] Chiappori, P.-A. and I. Ekeland (2005), ”The Microeconomics of Group Behavior: General Characterization”, *Journal of Economic Theory*, forthcoming.

[17] Chiappori, P.-A., Fortin, B. and G. Lacroix (2002), “Marriage Market, Divorce Legislation and Household Labor Supply”, *Journal of Political Economy*, 110 1, 37-72

[18] Chiappori, P. A. and B. Salanié (2000), ’Empirical Applications of Contract Theory: a survey of some recent work’, invited lecture, World Congress of the Econometric Society; forthcoming in *Advances in Economics*, M. Dewatripont and L. Hansen, ed.

[19] Duflo (2000),“Grandmothers and Granddaughters: Old Age Pension and Intra-household Allocation in South Africa”, *World Bank Economic Review*, vol. 17, no. 1, 2003, pp. 1-25

[20] Ekeland, I., and L. Nirenberg (2000): ”The Convex Darboux Theorem”, to appear, *Methods and Applications of Analysis*

[21] Fong, Y and J. Zhang (2001), ‘The Identifiability of Unobservable Independent and Spousal Leisure’, *Journal of Political Economy*, 109(1), 191-202.

[22] Fortin, Bernard and Lacroix, Guy (1997), A Test of Neoclassical and Collective Models of Household Labor Supply, *Economic Journal*, 107, 933-955.

[23] Galasso, E. (1999): "Intrahousehold Allocation and Child Labor in Indonesia", *Mimeo*, BC.

[24] Koopmans, T. (1949): "Identifiability Problems in Economic Model Construction", *Econometrica*, 17 2, 125-44.

[25] McElroy, Marjorie B. (1990), "The Empirical Content of Nash Bargained Household Behavior", *Journal of Human Resources*, 25(4), 559-83.

[26] Oreffice, S. (2005), "Did The Legalization of Abortion Increase Women's Household Bargaining Power? Evidence from Labor Supply", Mimeo, Clemso University.

[27] Rubalcava, L., and D. Thomas (2000), "Family Bargaining and Welfare", *Mimeo RAND*, UCLA.

[28] Thomas, D. (1990), "Intra–Household Resource Allocation: An Inferential Approach", *Journal of Human Resources*, 25, 635–664.

[29] Thomas, D., Contreras, D. and E. Frankenberg (1997). "Child Health and the Distribution of Household Resources at Marriage." *Mimeo* RAND, UCLA.

[30] Townsend, R. (1994), "Risk and Insurance in Village India", *Econometrica*, 62, 539-591.

APPENDIX

A Proof of Lemma 8

Recall first that, if $X(P)$ arises from a collective demand function in this model, we will have:

$$X(P) = \arg\max \left\{ \mu_1(P) U^1(P) + \mu_2(P) U^2(P) \mid P'X = 1 \right\}$$

for some μ_1, U^1, μ_2, U^2. Setting $\mu = \mu_1/\mu_2$, we have as well:

$$X(P) = \arg\max \left\{ \mu(P) U^1(P) + U^2(P) \mid P'X = 1 \right\}$$

Introducing the indirect utility function:

$$V(P) = \max \left\{ \mu(P) U^1(P) + U^2(P) \mid P'X = 1 \right\}$$

we find, by the envelope theorem:

$$DV(P) = U^1(X(P)) D\mu(P) - \lambda(P) X(P)$$

where $\lambda(P)$ is the Lagrange multiplier, so that:

$$X = -\frac{1}{\lambda} DV + \frac{U^1}{\lambda} D\mu \tag{41}$$

We shall now use the tools of exterior differential calculus, as in Chiappori and Ekeland (1997, 1999); we refer to Bryant et al. (1991) for a treatise on the subject. Introduce the 1-form ω defined by:

$$\omega = \sum_{n=1}^{N} X^n (P) \, dP_n \tag{42}$$

Saying that $X(P)$ satisfies SR(1) means that:

$$\omega \wedge d\omega \wedge d\omega = 0$$

Saying that $X(P)$ does not satisfy SR(0) (that is, the Slutsky matrix is not symmetric) means that:

$$\omega \wedge d\omega \neq 0$$

By the Darboux theorem (see Ekeland and Nirenberg 2000), these two conditions mean that we can find functions f_1, f_2, V^1, V^2 such that:

$$\omega = f_1 dV^1 + f_2 dV^2 \tag{43}$$

in some neighborhood of $\bar{P}$.

The four functions f_1, f_2, V^1, V^2 are not uniquely defined, however the linear span of df_1, dV^1, df_2, dV^2 is. Indeed, introduce the set $\mathcal{A}$ of all differential forms α such that:

$$\alpha \wedge d\omega \wedge d\omega = 0$$

Note that the set $\mathcal{C}$ depends on ω only, and not on any particular choice of f_1, f_2, V^1 or V^2.

Lemma 23 *$\mathcal{A}$ is a differential ideal, and the set $\mathcal{A}_1$ of all 1-forms belonging to $\mathcal{A}$ is a linear subspace of dimension 4.*

Proof. Differentiating the above relation, we get $d\alpha \wedge d\omega \wedge d\omega = 0$. So that if $\alpha \in \mathcal{A}$, then $d\alpha \in \mathcal{A}$. It follows that $\mathcal{A}$ is a differential ideal.

Differentiating relation 43, we get $d\omega = 2df_1 \wedge dV^1 \wedge df_2 \wedge dV^2$,so that the defining relation for $\mathcal{A}$ becomes:

$$\alpha \wedge df_1 \wedge dV^1 \wedge df_2 \wedge dV^2 = 0$$

If α is a 1-form, this means precisely that α belongs to the linear span of df_1, dV^1, df_2 and dV^2 ■

We can go one step further and find a three-dimensional linear subspace $\mathcal{B}_1 \subset \mathcal{A}_1$ which also depends only on ω and not on a particular choice of f_1, f_2, V^1 or V^2. To do this, introduce the set $\mathcal{B}$ of all differential forms $\beta \in \mathcal{A}$ such that:

$$\beta \wedge \omega \wedge d\omega = 0 \tag{44}$$

Lemma 24 *$\mathcal{B}$ is a differential ideal, and the set $\mathcal{B}_1$ of all 1-forms belonging to $\mathcal{B}$ is a linear subspace of dimension 3.*

Proof. Differentiating the above relation, we get:

$$d\beta \wedge \omega \wedge d\omega + \beta \wedge d\omega \wedge d\omega = 0$$

The second term vanishes since $\beta \in \mathcal{A}$ and we are left with the first one, which tells us that $d\beta$ belongs to $\mathcal{B}$ if β does. So $\mathcal{B}$ is a differential ideal. Using the decomposition 43 again, we find:

$$\begin{aligned} \omega \wedge d\omega &= \left(f_1 dV^1 + f_2 dV^2\right) \wedge \left(df_1 \wedge dV^1 + df_2 \wedge dV^2\right) \\ &= (-f_1 df_2 + f_2 df_1) \wedge dV^1 \wedge dV^2 \\ &= (f_2)^2 \, d\left(\frac{f_1}{f_2}\right) \wedge dV^1 \wedge dV^2 \end{aligned}$$

so that formula44 becomes:

$$\beta \wedge d\left(\frac{f_1}{f_2}\right) \wedge dV^1 \wedge dV^2 = 0$$

If β is a 1-form, this means precisely that β belongs to the linear span of $d\left(\frac{f_1}{f_2}\right)$, $dV^1$and dV^2 ■

As a consequence of Lemma 24 , the equations:

$$d\left(\frac{f_1}{f_2}\right) = 0,\ dV^1 = 0,\ dV^2 = 0$$

define a the foliation of R^K by 3-planes, which depends only on ω, and not on the particular choice of f_1, f_2, V^1 or V^2. Equivalently, this foliation can be defined by the equations:

$$\frac{f_1(P)}{f_2(P)} = h_1,\ V^1(P) = h_2,\ V^2(P) = h_3$$

In the particular case when ω is given by relation (42), relation (41) becomes:

$$\omega = -\frac{1}{\lambda}dV + \frac{U^1}{\lambda}d\mu$$

so that the family of submanifolds defined by the three equations:

$$U^1(X(P)) = h_1,\ V(P) = h_2,\ \mu(P) = h_3$$

is identifiable. But $U^2(X(P)) = V(P) - \mu(P)U^1(X(P))$, so the result follows.

B Proof of Proposition 9

Consider the program

$$\max_X \left\{\mu(P)U^1(X) + (1-\mu(P))U^2(X) \mid P'X = 1\right\}$$

The first order conditions imply the standard Bowen-Lindahl-Samuelson conditions:

$$P_1 \frac{\partial U^1/\partial X^k}{\partial U^1/\partial X^1} + P_2 \frac{\partial U^2/\partial X^k}{\partial U^2/\partial X^2} = P_k \quad \forall k \geq 3 \tag{45}$$

or equivalently:

$$\frac{P_1(X)}{P_k(X)} \frac{\partial U^1/\partial X^k}{\partial U^1/\partial X^1} + \frac{P_2(X)}{P_k(X)} \frac{\partial U^2/\partial X^k}{\partial U^2/\partial X^2} = 1 \quad \forall k \geq 3 \tag{46}$$

where $P(X)$ is the observed inverse demand function. Note that the same computation yields:

$$\frac{\mu(P(X))}{1-\mu(P(X))} \frac{\partial U^2/\partial X^2}{\partial U^1/\partial X^1} = \frac{P_1(X)}{P_2(X)} \tag{47}$$

Our goal is to show that, generically in μ, the solution to (46), seen as an equation in U^1 and U^2, is unique up to an increasing transform. Let $\widetilde{U}^1$ and $\widetilde{U}^2$ be another solution. By linearity, we have that:

$$\frac{P_1(X)}{P_k(X)} \left(\frac{\partial U^1/\partial X^k}{\partial U^1/\partial X^1} - \frac{\partial \widetilde{U}^1/\partial X^k}{\partial \widetilde{U}^1/\partial X^1} \right) + \frac{P_2(X)}{P_k(X)} \left(\frac{\partial U^2/\partial X^k}{\partial U^2/\partial X^2} - \frac{\partial \widetilde{U}^2/\partial X^k}{\partial \widetilde{U}^2/\partial X^2} \right) = 0$$

The parenthesis are both zero or both non-zero. If they are both zero, the first alternative holds. If they are both non-zero, we have:

$$\log \left| \frac{\partial U^1/\partial X^k}{\partial U^1/\partial X^1} - \frac{\partial \widetilde{U}^1/\partial X^k}{\partial \widetilde{U}^1/\partial X^1} \right| - \log \left| \frac{\partial U^2/\partial X^k}{\partial U^2/\partial X^2} - \frac{\partial \widetilde{U}^2/\partial X^k}{\partial \widetilde{U}^2/\partial X^2} \right| = \log \left(\frac{P_2(X)}{P_1(X)} \right)$$

Using relation (47), we find that:

$$\begin{aligned}\log\mu\left(P\left(\mu,X\right)\right)-\log(1-\mu\left(P\left(\mu,X\right)\right) &= \log\left|\frac{\partial U^1}{\partial X^1}\right|-\log\left|\frac{\partial U^1/\partial X^k}{\partial U^1/\partial X^1}-\frac{\partial\widetilde{U}^1/\partial X^k}{\partial\widetilde{U}^1/\partial X^1}\right| \\ &\quad -\log\left|\frac{\partial U^2}{\partial X^2}\right|+\log\left|\frac{\partial U^2/\partial X^k}{\partial U^2/\partial X^2}-\frac{\partial\widetilde{U}^2/\partial X^k}{\partial\widetilde{U}^2/\partial X^2}\right|\end{aligned}$$

is the sum of a function of $(X_1, X_3, ...)$ and a function of $(X_2, X_3, ...)$. The second alternative follows immediately.

C Proof of Proposition 19

Proceeding as in the case where $S = 2$, we find that the U_i must satisfy the $(K - S)$ equations:

$$\sum_{i=1}^{S}\frac{P_i}{P_k}\frac{\partial U^i/\partial X^k}{\partial U^i/\partial X^i}=1,\ S+1\leq k\leq K$$

where $P(X)$ is the observed collective demand function. By the same calculation, we have:

$$\frac{P_i}{P_j}=\frac{\mu_i}{\mu_j}\frac{\partial U^i/\partial X^i}{\partial U^j/\partial X^j},\quad 1\leq i,j\leq K \tag{48}$$

Suppose there is another set $\left(\tilde{\mu}_i, \tilde{U}_i\right), 1 \leq i \leq S$, corresponding to the same collective demand function, and hence to the same inverse demand $P(X)$. We get by substraction:

$$\sum_{i=1}^{S}P_i\left(\frac{\partial U^i/\partial X^k}{\partial U^i/\partial X^i}-\frac{\partial\tilde{U}^i/\partial X^k}{\partial\tilde{U}^i/\partial X^i}\right)=0,\ S+1\leq k\leq K$$

Taking into account equation (48), this becomes:

$$\sum_{i=1}^{S} A_k^i \mu_i = 0,\ S+1 \leq k \leq K \tag{ck1}$$

where the functions:

$$A_k^i = \frac{\partial U^i}{\partial X^i}\left(\frac{\partial U^i/\partial X^k}{\partial U^i/\partial X^i} - \frac{\partial \tilde{U}^i/\partial X^k}{\partial \tilde{U}^i/\partial X^i}\right)$$

have the property that the A_k^j, for $j \neq i$, do not depend on X^i.

Differentiate each of the $(K-S)$ equations (**??**). More precisely, fix a number $R > 1$, and let us consider all partial derivatives of order up to and including R, that is, all operators of the form:

$$\frac{\partial^r}{\left(\partial X^1\right)^{r_1} ... \left(\partial X^K\right)^{r_K}}$$

with $r_1 + ... + r_K = r \leq R$. There are $C_K^{K+r-1} = C_{r-1}^{K+r-1}$ partial derivatives of order r in K variables, and hence:

$$N\left(R\right) = \sum_{r=1}^{R} C_K^{K+r-1} = \sum_{r=1}^{R} \frac{(K+r-1)!}{K!\,(r-1)!}$$

partial derivatives of order up to and including R. Applying each of these partial derivatives to the equations (**??**) gives us a total of $(K-S)\left(1+N\left(R\right)\right)$ linear equations (including the original ones) relating the partial derivatives of the A_k^i.

But the A_k^i are functions of $(K-1)$ variables only, so the number of partial derivatives up to and including order R is:

$$n\left(R\right) = \sum_{r=1}^{R} C_{K-1}^{K+r-2} = \sum_{r=1}^{R} \frac{(K+r-2)!}{(K-1)!\,(r-1)!}$$

We have a system of $(K-S)(1+N(R))$ linear equations for $S(K-S)(1+n(R))$ unknowns (the partial derivatives of the A_k^i). But:

$$\begin{aligned} N(R)-n(R) &> \frac{(K+R-1)!}{K!\,(R-1)!}-\frac{(K+R-2)!}{(K-1)!\,(R-1)!} \\ &= \frac{(K+R-2)!}{(K-1)!\,(R-1)!}\left[\frac{(K+R-1)}{K}-1\right] \\ &\to \infty \text{ when } R\to\infty \end{aligned}$$

so that the number of equations eventually exceeds the number of unknowns. Set:

$$\bar{R}=\inf\{R \mid (K-S)(1+N(R))>S(K-S)(1+n(R))\}$$

For $R=\bar{R}$, the equations in the system must satisfy certain compatibility conditions (certain determinants must vanish), which take the form of polynomial relations between the coefficients. But these coefficients are partial derivatives of the μ_i of order up to and including $\bar{R}$. In other words, there are polynomials $\Pi_1,...,\Pi_m$ such that

$$\Pi_m\left(\mu_s,\frac{\partial\mu_s}{\partial X^k},....,\frac{\partial^r\mu_s}{(\partial X^1)^{r_1}...(\partial X^K)^{r_K}}\right)=0,\ 1\leq m\leq M, \tag{49}$$

If equations (**??**) do not hold, then the system has the trivial solution only. In particular, $A_k^i=0$ for all i,k, leading to:

$$\frac{\partial U^i/\partial X^k}{\partial U^i/\partial X^i}=\frac{\partial\tilde{U}^i/\partial X^k}{\partial\tilde{U}^i/\partial X^i},\ 1\leq i\leq S,\ S+1\leq k\leq K$$

and hence $\tilde{U}^s=\varphi_s(U^s)$ for all s.

D Proof of Proposition 17.

D.1 Condition (23)

This is an adaptation of Chiappori (1988, 1992), and can be found in Chiappori, Fortin and Lacroix (2002). We give the proof for the sake of completeness.

Denote by $\xi_1(p,\rho)$ and $\xi_2(p,\rho)$ the Marshallian demands. Exclusivity implies that:

$$x^1(p,y,z) = \xi_1^1(p,\rho(p,y,z)) \tag{50}$$
$$x^2(p,y,z) = \xi_2^2(p,y-\rho(p,y,z)) \tag{51}$$

Differentiating these equations, we get:

$$\begin{aligned}\frac{\partial x^1(p,y,z)}{\partial y} &= \frac{\partial \xi_1^1(p,\rho)}{\partial \rho}\frac{\partial \rho}{\partial y}\\ \frac{\partial x^1(p,y,z)}{\partial z} &= \frac{\partial \xi_1^1(p,\rho)}{\partial \rho}\frac{\partial \rho}{\partial z}\end{aligned}$$

and hence:

$$\frac{\partial x^1(p,y,z)}{\partial y}\frac{\partial \rho}{\partial z} = \frac{\partial x^1(p,y,z)}{\partial z}\frac{\partial \rho}{\partial y} \tag{52}$$

Similarly, we have:

$$\frac{\partial x^2(p,y,z)}{\partial y}\frac{\partial \rho}{\partial z} = \frac{\partial x^2(p,y,z)}{\partial z}\left(\frac{\partial \rho}{\partial y}-1\right) \tag{53}$$

Since $\frac{\partial x^1}{\partial z}$ and $\frac{\partial x^2}{\partial z}$ do not vanish, equations (52) and (53) identify the derivatives $\frac{\partial \rho}{\partial y}$ and $\frac{\partial \rho}{\partial z}$, the latter being non-zero.

Now differentiating equations (50) and (51) with respect to p_1 and p_2, we get:

$$\begin{aligned}
\frac{\partial x^1(p,y,z)}{\partial p_2}\frac{\partial \rho}{\partial z} &= \frac{\partial x^1(p,y,z)}{\partial z}\frac{\partial \rho}{\partial p_2} \\
\frac{\partial x^2(p,y,z)}{\partial p_1}\frac{\partial \rho}{\partial z} &= \frac{\partial x^2(p,y,z)}{\partial z}\frac{\partial \rho}{\partial p_1}
\end{aligned}$$

which identifies $\frac{\partial \rho}{\partial p_1}$ and $\frac{\partial \rho}{\partial p_2}$. The result follows.

D.2 Conditions (24) and (25)

Exclusivity implies that

$$\hat{\xi}_1^1(p,\hat{\rho}(p,y,z)) = \xi_1^1(p,\rho(p,y,z)) \tag{54}$$

$$\hat{\xi}_2^2(p,y-\hat{\rho}(p,y,z)) = \xi_2^2(p,y-\rho(p,y,z)) \tag{55}$$

Consider now some $j \geq 3$. The argument goes in 4 steps:

1. Differentiating the identity:

$$\begin{aligned}
x^j(p,y,z) &= \xi_1^j(p,\rho(p,y,z)) + \xi_2^j(p,y-\rho(p,y,z)) \\
&= \hat{\xi}_1^j(p,\hat{\rho}(p,y,z)) + \hat{\xi}_2^j(p,y-\hat{\rho}(p,y,z))
\end{aligned}$$

   we get:

$$\begin{aligned}
\frac{\partial x^j(p,y,z)}{\partial z} &= \left(\frac{\partial \xi_1^j(p,\rho)}{\partial \rho} - \frac{\partial \xi_2^j(p,y-\rho)}{\partial \rho}\right)\frac{\partial \rho}{\partial z} \\
&= \left(\frac{\partial \hat{\xi}_1^j(p,\hat{\rho})}{\partial \hat{\rho}} - \frac{\partial \hat{\xi}_2^j(p,y-\hat{\rho})}{\partial \hat{\rho}}\right)\frac{\partial \hat{\rho}}{\partial z}
\end{aligned}$$

Condition (23) then implies that:

$$\frac{\partial \xi_1^j (p, \rho)}{\partial \rho} - \frac{\partial \xi_2^j (p, y - \rho)}{\partial \rho} = \frac{\partial \hat{\xi}_1^j (p, \hat{\rho})}{\partial \hat{\rho}} - \frac{\partial \hat{\xi}_2^j (p, y - \hat{\rho})}{\partial \hat{\rho}}$$

Similarly:

$$\begin{aligned} \frac{\partial x^j (p, y, z)}{\partial y} &= \frac{\partial \xi_2^j (p, y - \rho)}{\partial \rho} + \left(\frac{\partial \xi_1^j (p, \rho)}{\partial \rho} - \frac{\partial \xi_2^j (p, y - \rho)}{\partial \rho} \right) \frac{\partial \rho}{\partial y} \\ &= \frac{\partial \hat{\xi}_2^j (p, y - \hat{\rho})}{\partial \hat{\rho}} + \left(\frac{\partial \hat{\xi}_1^j (p, \hat{\rho})}{\partial \hat{\rho}} - \frac{\partial \hat{\xi}_2^j (p, y - \hat{\rho})}{\partial \hat{\rho}} \right) \frac{\partial \hat{\rho}}{\partial y} \end{aligned}$$

We conclude that:

$$\begin{aligned} \frac{\partial \xi_1^j (p, \rho)}{\partial \rho} &= \frac{\partial \hat{\xi}_1^j (p, \hat{\rho})}{\partial \hat{\rho}} \\ \frac{\partial \xi_2^j (p, y - \rho)}{\partial \rho} &= \frac{\partial \hat{\xi}_2^j (p, y - \hat{\rho})}{\partial \hat{\rho}} \end{aligned}$$

2. In the same way, since good 1 is exclusively consumed by agent 1:

$$\begin{aligned} \frac{\partial x^j (p, y, z)}{\partial p_1} &= \frac{\partial \xi_1^j (p, \rho)}{\partial p_1} + \left(\frac{\partial \xi_1^j (p, \rho)}{\partial \rho} - \frac{\partial \xi_2^j (p, y - \rho)}{\partial \rho} \right) \frac{\partial \rho}{\partial p_1} \\ &= \frac{\partial \hat{\xi}_1^j (p, \rho)}{\partial p_1} + \left(\frac{\partial \hat{\xi}_1^j (p, \hat{\rho})}{\partial \hat{\rho}} - \frac{\partial \hat{\xi}_2^j (p, y - \hat{\rho})}{\partial \hat{\rho}} \right) \frac{\partial \hat{\rho}}{\partial p_1} \end{aligned}$$

implying that

$$\frac{\partial \xi_1^j (p, \rho)}{\partial p_1} = \frac{\partial \hat{\xi}_1^j (p, \hat{\rho})}{\partial p_1}$$

and similarly, since good 2 is exclusively consumed by agent 2:

$$\frac{\partial \xi_2^j (p, y-\rho)}{\partial p_2} = \frac{\partial \hat{\xi}_2^j (p, y-\hat{\rho})}{\partial p_2}$$

3. Since both ξ_1 and $\hat{\xi}_2$ satisfy Slutsky symmetry:

$$\frac{\partial \xi_1^1 (p, \rho)}{\partial p_j} + \xi_1^j (p, \rho) \frac{\partial \xi_1^1 (p, \rho)}{\partial \rho} = \frac{\partial \xi_1^j (p, \rho)}{\partial p_1} + \xi_1^1 (p, \rho) \frac{\partial \xi_1^j (p, \rho)}{\partial \rho} \quad (56)$$

$$\frac{\partial \hat{\xi}_1^1 (p, \hat{\rho})}{\partial p_j} + \hat{\xi}_1^j (p, \hat{\rho}) \frac{\partial \hat{\xi}_1^1 (p, \hat{\rho})}{\partial \hat{\rho}} = \frac{\partial \hat{\xi}_1^j (p, \hat{\rho})}{\partial p_1} + \hat{\xi}_1^1 (p, \hat{\rho}) \frac{\partial \hat{\xi}_1^j (p, \hat{\rho})}{\partial \hat{\rho}} \quad (57)$$

Differentiating (54):

$$\frac{\partial \hat{\xi}_1^1 (p, \hat{\rho})}{\partial p_j} + \frac{\partial \hat{\xi}_1^1 (p, \hat{\rho})}{\partial \hat{\rho}} \frac{\partial \hat{\rho}}{\partial p_j} = \frac{\partial \xi_1^1 (p, \rho)}{\partial p_j} + \frac{\partial \xi_1^1 (p, \rho)}{\partial \rho} \frac{\partial \rho}{\partial p_j}$$

and hence, using step 1:

$$\begin{aligned} \frac{\partial \hat{\xi}_1^1 (p, \hat{\rho})}{\partial p_j} &= \frac{\partial \xi_1^1 (p, \rho)}{\partial p_j} + \frac{\partial \xi_1^1 (p, \rho)}{\partial \rho} \left(\frac{\partial \rho}{\partial p_j} - \frac{\partial \hat{\rho}}{\partial p_j} \right) \\ &= \frac{\partial \xi_1^1 (p, \rho)}{\partial p_j} + \frac{\partial \xi_1^1 (p, \rho)}{\partial \rho} \frac{\partial \phi}{\partial p_j} \end{aligned}$$

4. Plugging this into equation (57), and using step 2, we get:

$$\frac{\partial \xi_1^1 (p, \rho)}{\partial p_j} + \frac{\partial \xi_1^1 (p, \rho)}{\partial \rho} \frac{\partial \phi}{\partial p_j} + \hat{\xi}_1^j (p, \hat{\rho}) \frac{\partial \xi_1^1 (p, \rho)}{\partial \rho} = \frac{\partial \xi_1^j (p, \rho)}{\partial p_1} + \xi_1^1 (p, \rho) \frac{\partial \xi_1^j (p, \rho)}{\partial \rho}$$

Substracting from (56) gives:

$$\frac{\partial \xi_1^1 (p, \rho)}{\partial \rho} \left(\xi_1^j (p, \rho) - \left(\frac{\partial \phi}{\partial p_j} + \hat{\xi}_1^j (p, \hat{\rho}) \right) \right) = 0$$

On the open, dense subset of K on which $\partial \xi_1^1 (p, \rho) / d\rho$ does not vanish, we find that:

$$\hat{\xi}_1^j (p, \hat{\rho}) = \xi_1^j (p, \rho) - \frac{\partial \phi}{\partial p_j}$$

and the conclusion obtains by continuity.